

## Differential operators 3: The square-root derivative

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### ABSTRACT

Differential operators are used in many seismic data processes such as triangle filters to reduce aliasing, finite difference solutions to the wave equation, and wavelet correction when modelling with diffractions or migrating with Kirchhoff algorithms. Short operators may be quite accurate when the data are restricted to low order polynomials, but may be inaccurate in other applications.

This is the third of three papers on differential operators and deals specifically with the square-root derivative or rho filter. The first paper deals with the first derivative and the second paper deals with the second derivative.

The purpose of this paper is to evaluate visually the short operators that approximate a rho filter.

### APPLICATIONS

The first paper in this series presents a number of applications for the different derivatives. They are:

1. The rho filter that applies corrections to the wavelet after diffraction modelling or Kirchhoff migration.
2. Fast filtering in the time domain that differentiates the filter operator to delta functions and is then applied to a trace that has been integrated. Convolution with the delta functions is equivalent to summing a few samples.
3. Finite difference solutions to the wave equation

### Rho filter (copy from first paper)

Modelling and migration should produce no change in a horizontal event, however some of these processes distort the wavelet. This distortion is illustrated in Figure 1 that shows in (a) a portion of a horizontal event created with a zero-phase wavelet. After modelling with diffractions this horizontal event should remain the same as the input, but there is a phase distortion as illustrated in (b). When the event in (a) is migrated with a 2D Kirchhoff algorithm, the event becomes that illustrated in (c). These wavelet distortions are corrected using a rho filter (operator) that corrects the shape of the wavelet back to the zero phase shape of (a). The rho filter modifies the phase and applies a taper to the amplitude spectrum.

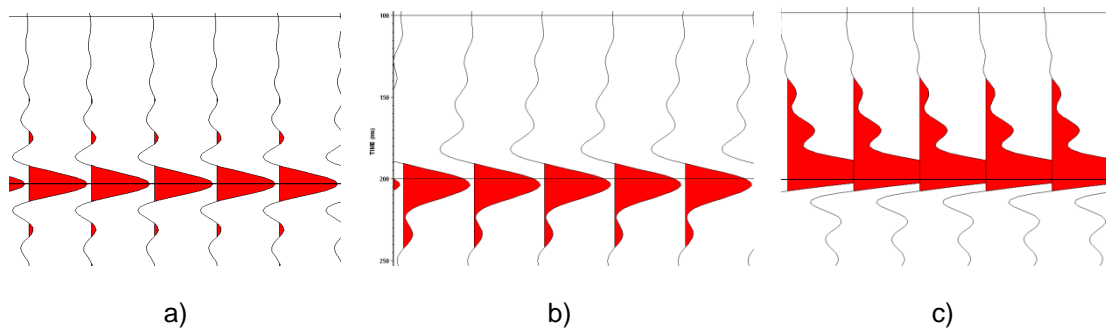


FIG. 1. An input event in a) is shown in b) after modelling with diffractions, and in c) after 2D Kirchhoff migration.

## ASSUMPTIONS

I assume an array of data  $f_n$ , that I call a trace. The trace can be in either time or distance transforming into the frequency or wavenumber domains. I will assume the trace to be in the time domain and refer to the transform domain parameters as frequency.

The displayed results are created using MATLAB with code  
`\2008-Matlab\DifferentialOperatorSqrt.m`

## INTRODUCTION

Early Kirchhoff migration introduced a phase error into the wavelets. A number of methods were used to correct the distortion such as a 45 degree phase shift filter, or by using a least squares approach to match the migrated wavelet with the input wavelet. Schneider (1978) provided the industry with the integral solution to the wave equation that required the input data to be filtered with differential operators. For 3D, the data required a differential operator, while 2D data required a square-root differential operator. This square-root derivative operator is known as the rho filter and is typically applied after a 2D Kirchhoff migration.

Differentiators are filters or operators. It is easy to describe the square of an operator by applying the initial operator twice. We have seen that it is possible to approximate a second derivative by applying the first derivative twice. The square-root of the second derivative is the original derivative. Our objective is to find an operator, that when applied twice, will produce a derivative.

In the frequency domain, the rho filter is a trivial task. Convolution becomes a product, the squaring of an operator becomes a squaring process, and the square-root of an operator becomes a square root. Defining the derivative to be  $j\omega$  the square-root operator becomes  $\sqrt{j\omega}$ . We then get our time domain operator from the inverse Fourier transform. However the time domain operator is quite complicated and choosing a suitable window is difficult. For many applications, it may be better to perform the operation in the frequency domain.

In the frequency domain we apply the operator twice with multiplication, i.e.

$$\sqrt{j\omega} \times \sqrt{j\omega} = j\omega. \tag{1}$$

In the time domain, I represent the rho operator as  $r(t)$ , and compute it from the inverse Fourier transform, i.e.,

$$r(t) = \text{IF}\{\sqrt{j\omega}\}. \tag{2}$$

We apply  $r(t)$  twice to be equivalent of the derivative  $d(t)$ , i.e.,

$$r(t) * [r(t) * f(t)] \equiv \frac{d f(t)}{dt} = d(t) * f(t),$$

or

$$r(t) * r(t) = d(t).$$

Therefore we can now understand that given an operator  $d(t)$ , the square-root of that operator  $r(t)$  needs to be convolved twice to get the original operator .

### THE RHO FILTER IN THE FREQUENCY DOMAIN

The derivative defined in the frequency domain  $j\omega$  is illustrated in Figure 1 and the square-root derivative  $\sqrt{j\omega}$  in Figure 2. Both spectrums have large values near the Nyquist frequency that cause ringing in the time domain response.

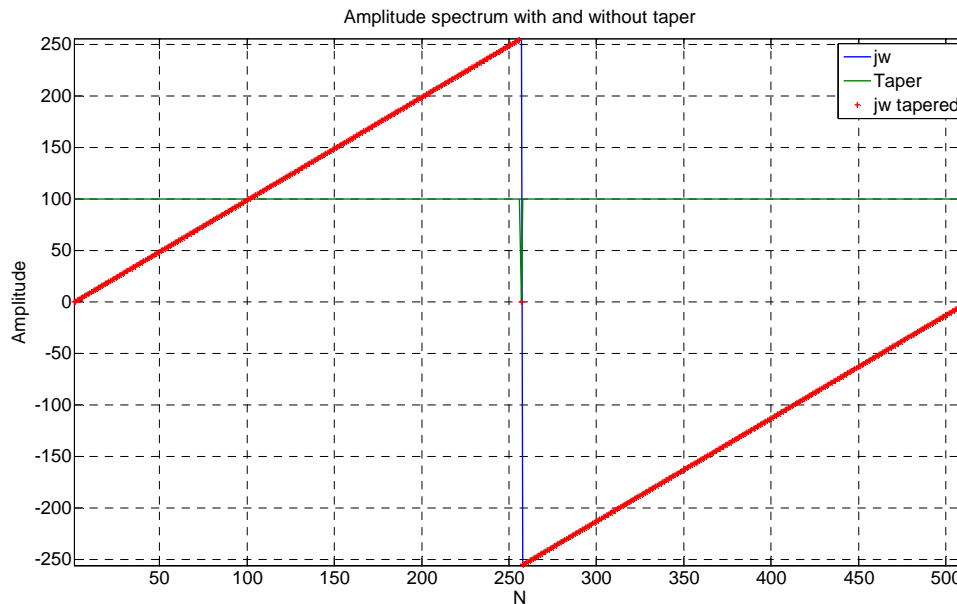


FIG. 1. Spectrum of the derivative.

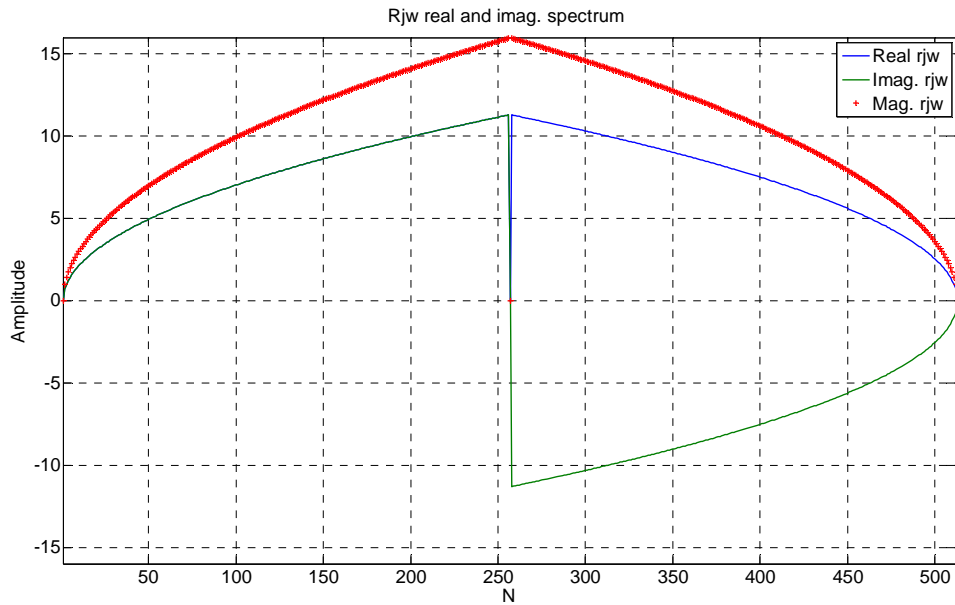


FIG. 2. Spectrum of the square-root derivative.

In the first half of the spectrum, the real and imaginary components are equal, but they are opposite in the negative frequencies, which occupy the right side of the spectrum. As with the derivative, the discontinuity at the Nyquist frequency causes problems in the time domain that is displayed in Figure 3. This figure was created with 512 frequency domain samples, but only the central 50 samples are displayed..

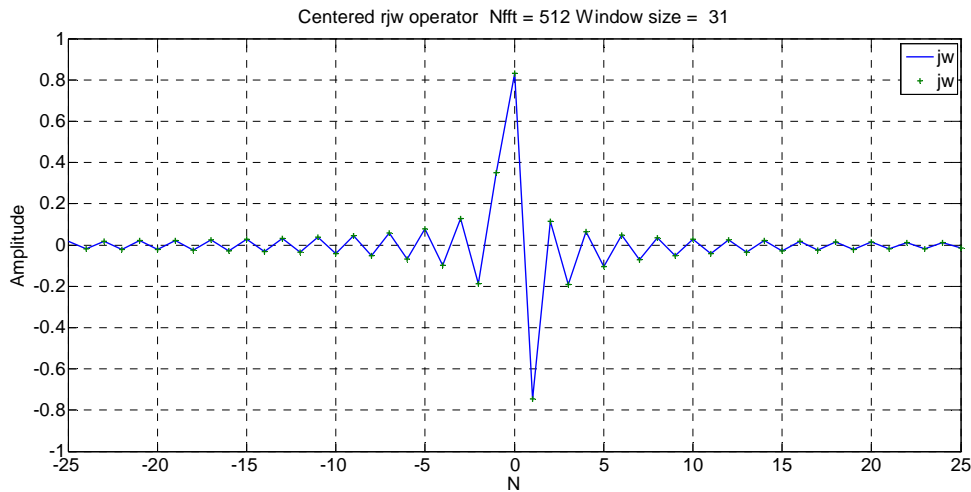


FIG. 3 Time domain RSD operator.

Note the oscillations due to the discontinuity at the Nyquist frequency.

This operator was convolved with itself to produce the derivative operator. The values close to time zero follow and demonstrate the rho filter is functioning correctly.

-0.2500 0.3333 -0.5000 1.0000 -0.0000 -0.9999 0.4999 -0.3332 0.2499

The oscillations can be reduced by including a taper in the frequency domain as illustrated in the next figures. A cosine taper was applied in the frequency domain that started at 80% of the Nyquist frequency as illustrated in Figure 4a which produces a smoother operator in (b).

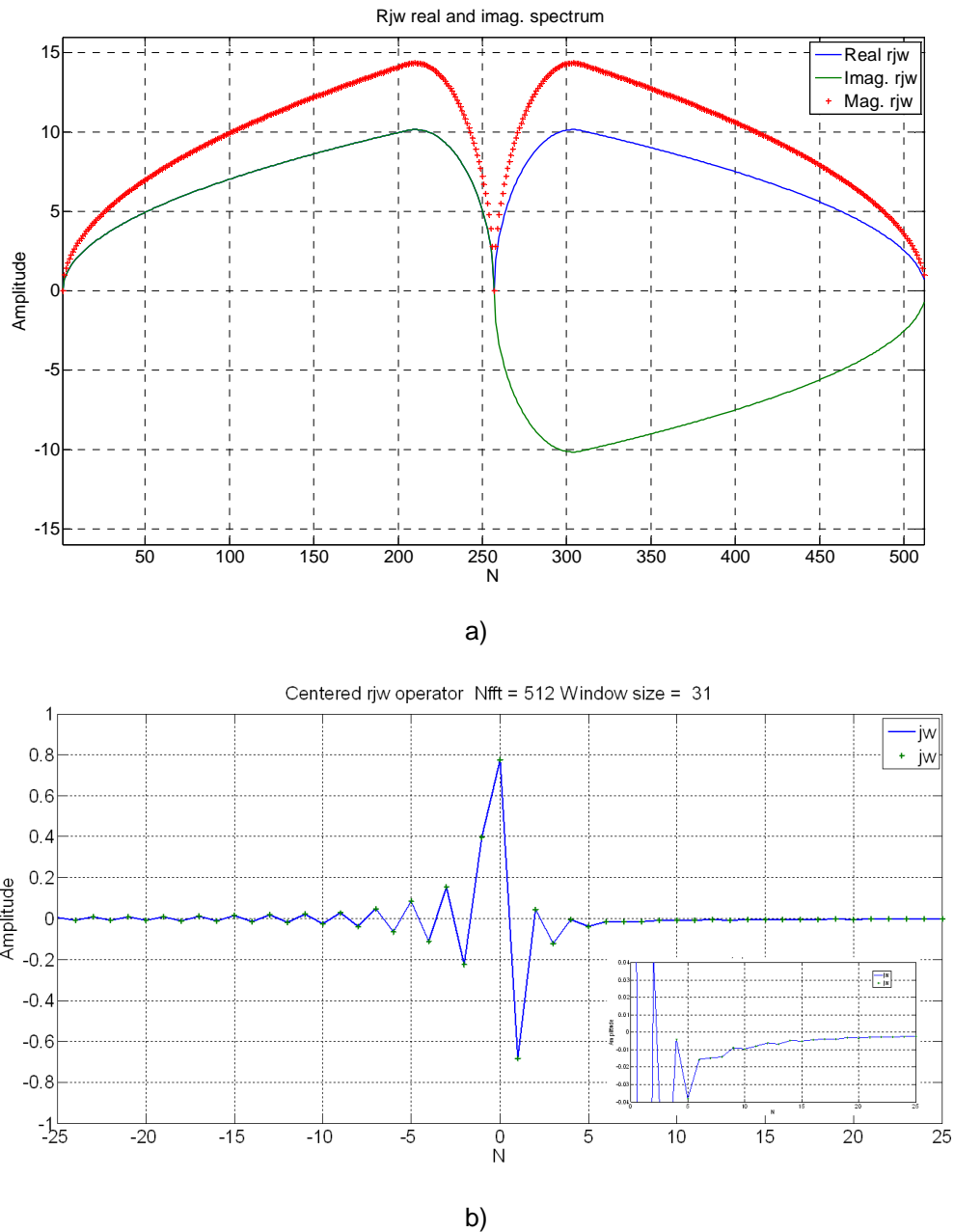


FIG. 4. The amplitude spectrum a) with a taper and b) the corresponding wavelet operator.

Note the negative low frequency trend to the right of the operator and emphasized in its zoom. It is this low frequency trend that creates problems in designing a small sample number operator.

## ORIGINAL FILTER

I developed a rho filter many years ago. It has 15 negative values and 5 positive values for a total of 21 points. The amplitudes are:

```
-0.0010 -0.0030 -0.0066 -0.0085 -0.0060 -0.0083 -0.0107 -0.0164 -0.0103 -0.0194
-0.0221 -0.0705 0.0395 -0.2161 -0.3831 0.5451 0.4775 -0.1570 0.0130 0.0321
-0.0129
```

This data is plotted in Figure 4. This shape of this data is time reversed from the previous figure as it is ready for convolution.

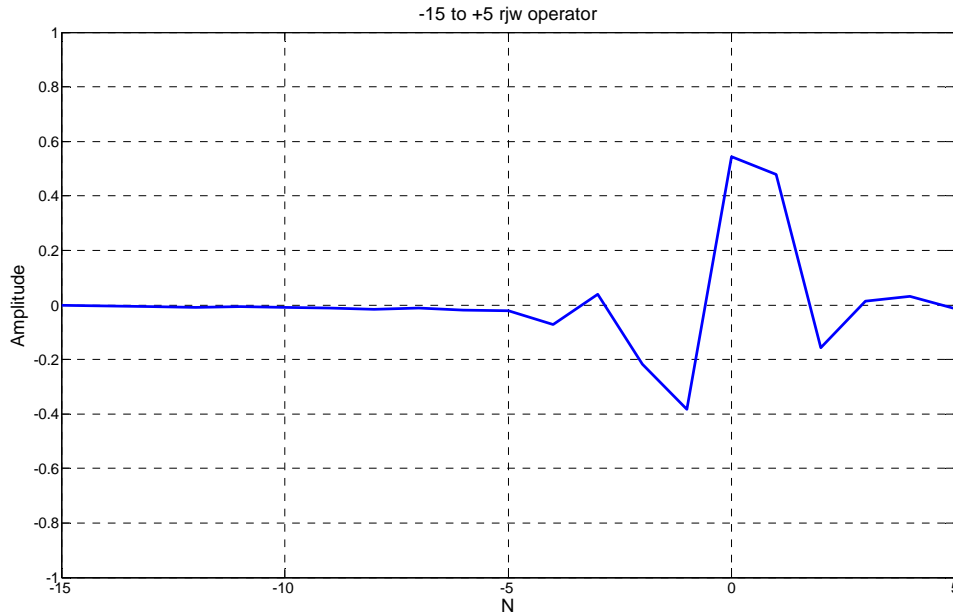
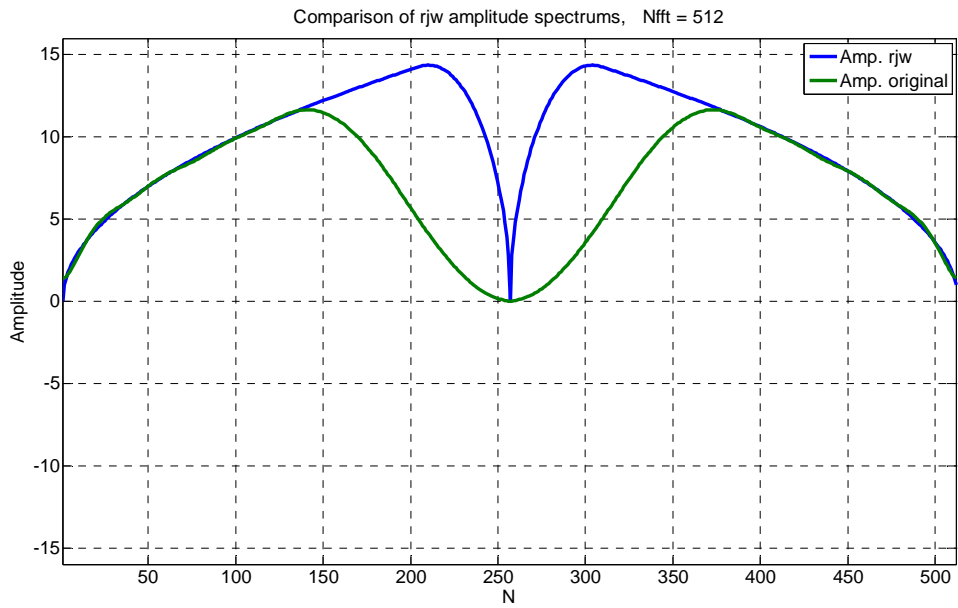
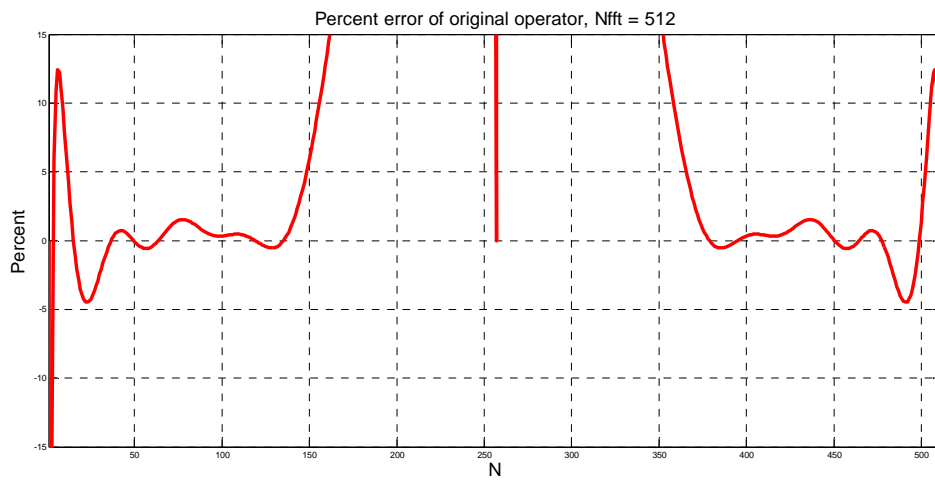


FIG. 5. Old rho filter with samples from -15 to +5.

The amplitude spectrum of this “old” operator is compared with a tapered amplitude spectrum in Figure 6a and the percent difference shown in (b).



a)



b)

FIG. 6. The amplitude spectrum of the original rho filter a) and its percent error.

This filter is still quite acceptable for seismic data as the low frequency error is below that of the data. The high frequencies also extend to 150 hz.

The corresponding real, imaginary, and phase are plotted in Figure 7, with the phase amplitude scaled by ten. The phase value should be  $\pi / 4 = 0.7854$ .

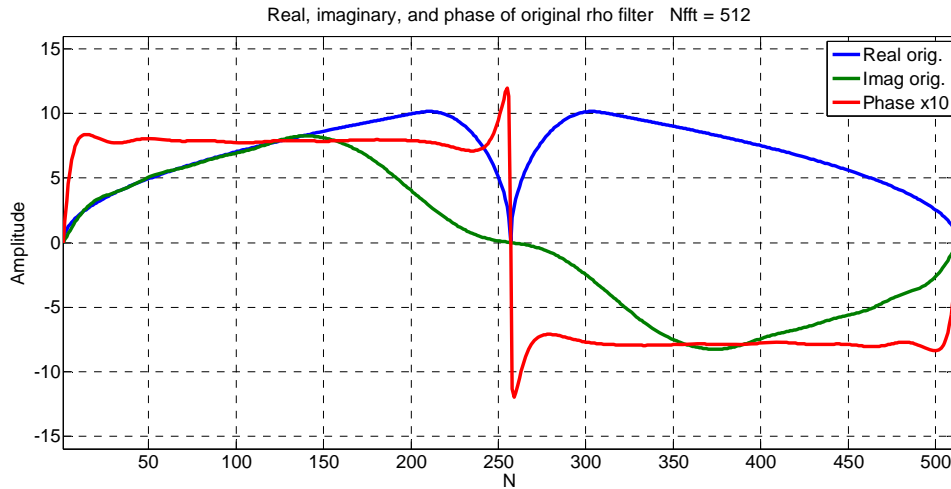


FIG. 7. Real, imaginary, and phase of the original rho filter.

### ALTERNATE ESTIMATES

Many attempts have been made to improve on the original filter. In my latest attempt, I subtracted off (approximately) the negative low frequency amplitude, and windowed the data with a 21 point window as shown in Figure 8. The plan is to use the windowed operator as a convolution, and replace the low frequencies with a recursive filter. The results of this approach are shown in Figure 9 that displays the amplitude spectrum, and its percentage error.

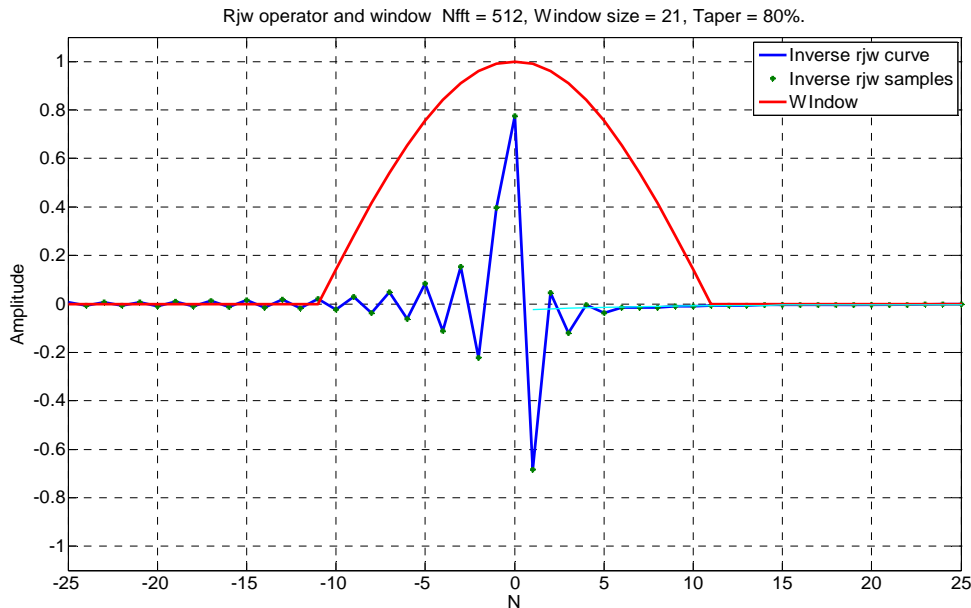
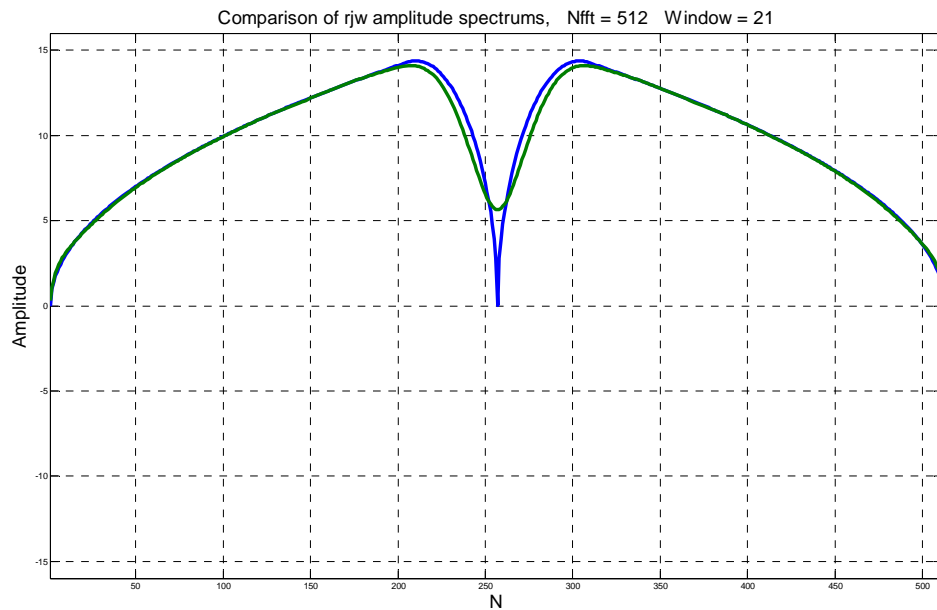
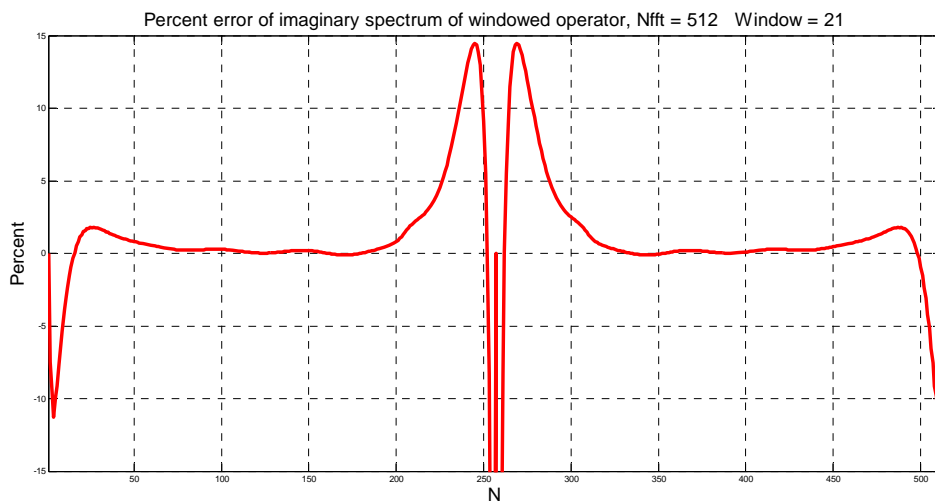


FIG. 8. New rho filter with the windowed data.





a)



b)

FIG. 9. New rho filter a) the amplitude spectrum, and b) the percentage error.

The error in the main seismic band is quite reasonable, but there still remain a problem with the low frequencies.

## CONCLUSIONS

Filters for the rho or square-root derivative were presented. The best solution is still the original filter developed many years ago. A new version has two parts; a convolution for the main part of the wavelet, and a recursive part for the low frequency. This new version is still under development.

## **REFERENCES**

- Bancroft, J. C., and Geiger, H., 2003, Analysis and design of filters for differentiation, CREWES Research Report, **9**.
- Schneider, W.A., 1978, Integral Formulation for Migration in Two and Three Dimensions, *Geophysics*, **43**, p 49-76.