Deriving the eikonal approximation: a scattering diagram tutorial

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ABSTRACT

"Diagram analysis" is a way of parsing, classifying, and manipulating the non-linear terms of inverse scattering, ultimately aiding in the derivation of new seismic processing algorithms. Scattering diagrams originate in forward as opposed to inverse scattering, and so an introduction to these topological organizational devices is best accomplished by considering the forward problem. We will use them to help derive a familiar expression in wave theory. The eikonal approximation, a relative of the WKBJ approximation, can be arrived at in several ways, for instance by direct integration of the Lippmann-Schwinger equation. In this paper we will demonstrate that, by retaining only Born series terms that correspond to a certain class of scattering diagrams, the same approximation can be recovered. In fact the diagram derivation could be argued to achieve its goal in a less roundabout way than the older approach.

INTRODUCTION

The eikonal approximation is an expression for modeling scalar wave propagation (Morse and Feshbach, 1953), which, together with its relative the WKBJ approximation, is relevant to seismic exploration (e.g., Clayton and Stolt, 1981; Amundsen et al., 2005). Non-linear scattering theory too is highly relevant to seismic exploration—inverse scattering diagram analysis has led to powerful algorithms for the removal of multiple reflections from seismic data, and is the subject of current research into processing and inversion of primaries (for a review and discussion, see Weglein et al., 2003). Here we link the two, providing a simple derivation of the eikonal approximation from a scattering-diagram analysis of the Born series. We compare it with the derivation of Morse and Feshbach, which is based on a truncation of the integral in the Lippmann-Schwinger equation.

Consider two simple 1D configurations of actual and reference media, $c(z)$ and c_0 respectively, actual and reference wavefields, G and G_0 respectively, and source depth z_s and observation depth z_g. The two fields, in the space-frequency domain, satisfy

$$\begin{aligned}\left[\frac{d^2}{dz_g^2}+\frac{\omega^2}{c^2(z_g)}\right]G(z_g,z_s)&=\delta(z_g-z_s),\\ \left[\frac{d^2}{dz_g^2}+\frac{\omega^2}{c_0^2}\right]G_0(z_g,z_s)&=\delta(z_g-z_s).\end{aligned} \tag{1}$$

Scattering theory is an expression of actual media and actual fields as series expansions about reference media and reference fields. Defining $\alpha(z)=1-c_0^2/c^2(z)$, the Lippmann-Schwinger equation is

$$G(z_g,z_s)=G_0(z_g,z_s)+\int_{-\infty}^{\infty}G_0(z_g,z')k^2\alpha(z')G(z',z_s)dz', \tag{2}$$

where $k = \omega/c_0$*. The Born series then arises through iterative back-substitution of equation (2) into itself, generating

$$G(z_g, z_s) = G_0(z_g, z_s) + G_1(z_g, z_s) + G_2(z_g, z_s) + ..., \tag{3}$$

where G_n is n'th order in α. For instance,

$$G_1(z_g, z_s) = \int_{-\infty}^{\infty} dz' G_0(z_g, z') k^2 \alpha(z') G_0(z', z_s), \tag{4}$$

$$G_2(z_g, z_s) = \int_{-\infty}^{\infty} dz' G_0(z_g, z') k^2 \alpha(z') \int_{-\infty}^{\infty} dz'' G_0(z', z'') k^2 \alpha(z'') G_0(z'', z_s), \tag{5}$$

etc. Assuming convergence, summation of a large number of these terms produces an expression for the full wavefield. Each term contains propagations and interactions strung together in a chain. For instance, G_2 in equation (5) involves, reading right to left, reference propagation from z_s to z'', at which location an interaction of strength $k^2\alpha$ occurs, then a further reference propagation from z'' to z', another interaction of strength $k^2\alpha$, and a final propagation from z' to z_g. The term G_n involves n interactions with the perturbation α and $n+1$ propagations in the reference medium. To discuss "scattering geometry" is to discuss the characteristic path in z that all or part of G_n takes during the course of its n interactions.

A SCATTERING DIAGRAM DERIVATION OF THE EIKONAL APPROXIMATION

Consider a source plane embedded in a homogeneous 1D reference medium, above a measurement plane at depth, and assume the reference and the actual medium to be in agreement at and above the source, but different above and below the measurement point. In Figure 1 some of the components of the resulting wavefield are illustrated. The eikonal approximation is an expression for the "direct" component, (A) in Figure 1, which dominates when medium variations are smooth. Equations (1)–(5) reflect this arrangement if z_s is placed above the depth support of the perturbation, embedded in the homogeneous reference medium, and z_g is placed below or within the perturbation.

Scattering-diagrams arise because of the absolute value operation within the reference Green's function (De Santo, 1992)

$$G_0(z_g, z_s) = \frac{e^{ik|z_g - z_s|}}{i2k}. \tag{6}$$

When G_0 is substituted into the terms in equation (3), and each term is broken up into cases based on the absolute values, each broken up bit has a characteristic scattering geometry. For instance, in equation (5) there are 4 possible cases: (A) $z_g > z'$, $z' > z''$; (B) $z_g >$

*To derive the LS equation, replace $c(z_g)$ in the actual equation in (1) with α and c_0 using the definition of α. Bring the α term to the right hand side of the equation. Notice that the resulting expression looks exactly like the reference equation in (1) except with a more complicated source. From standard PDE theory, we know we can solve for G then by multiplying the complicated source by the Green's function G_0 and integrating over all space. The result is equation (2).

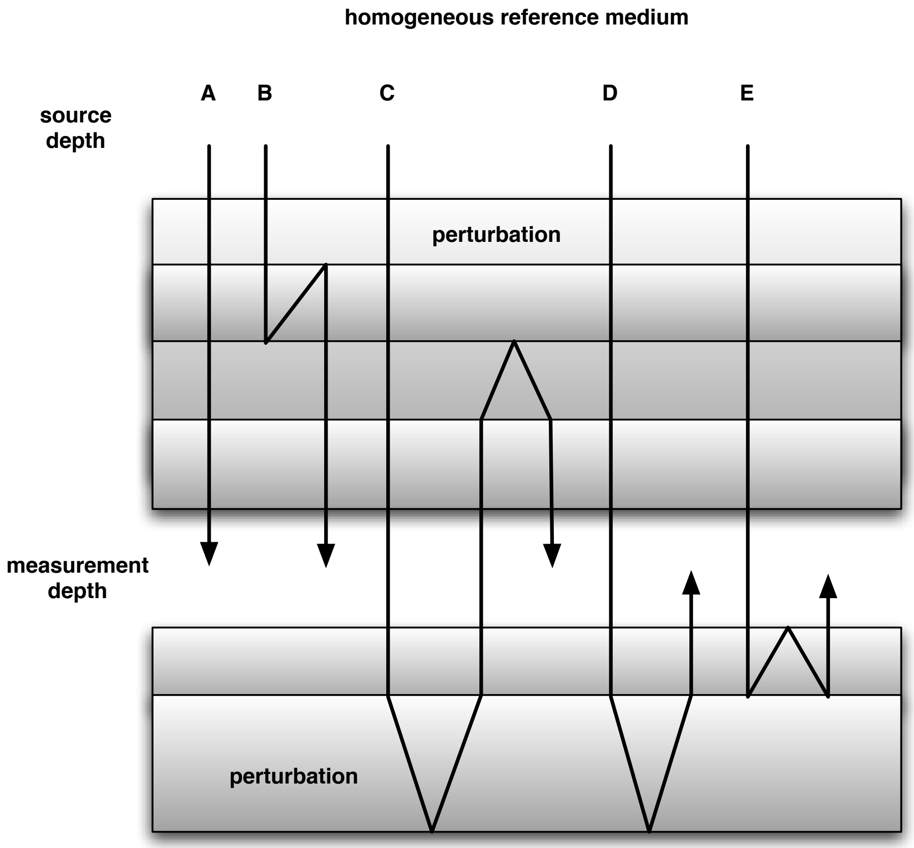

FIG. 1. A schematic illustration of a 1D wavespeed profile $c(z)$, with source plane (z_s) above all layer structure and receiver plane (z_g) embedded within it, and the paths in the medium of some of the wave events measured at z_g, (A)–(E). The eikonal approximation is an expression for event (A), which dominates the full solution when the profile $c(z)$ is smooth.

z', $z' < z''$; (C) $z_g < z'$, $z' > z''$; and (D) $z_g < z'$, $z' < z''$. These are represented schematically by scattering diagrams (Figure 2).

Let us allow the geometry of these diagrams to suggest an approach for deriving certain types of wave solution. We know that the eikonal approximation corresponds to the direct part of the wave; no reflections, or changes in direction with respect to the z axis occur as this part of the wave propagates. So, let us see what happens if, instead of summing together all terms in equation (3), we reject from the summation any contribution whose diagram involves a change in direction (B–D). At first order, rejecting scattering interactions taking place below z_g leaves a portion of the full wavefield we call T_1:

$$T_1(z_g, z_s) = \frac{e^{ik(z_g - z_s)}}{i2k}\left(-\frac{ik}{2}\int_{z_s}^{z_g} \alpha(z')dz'\right), \quad z_g > z_s. \tag{7}$$

At second order, 3 of the 4 contributing terms involve a change in reference propagation direction. Our program retains only one contribution (as described above and in Figure 2), which we call T_2:

$$T_2(z_g, z_s) = -\frac{e^{ik(z_g - z_s)}}{i2k}\frac{k^2}{4}\int_{z_s}^{z_g}\alpha(z')\int_{z_s}^{z'}\alpha(z'')dz''dz' \\ = \frac{e^{ik(z_g - z_s)}}{i2k}\frac{1}{2}\left(-\frac{ik}{2}\int_{z_s}^{z_g}\alpha(z')dz'\right)^2, \quad z_g > z_s. \tag{8}$$

By repeating this retention/rejection of scattering diagrams over several orders, we discern

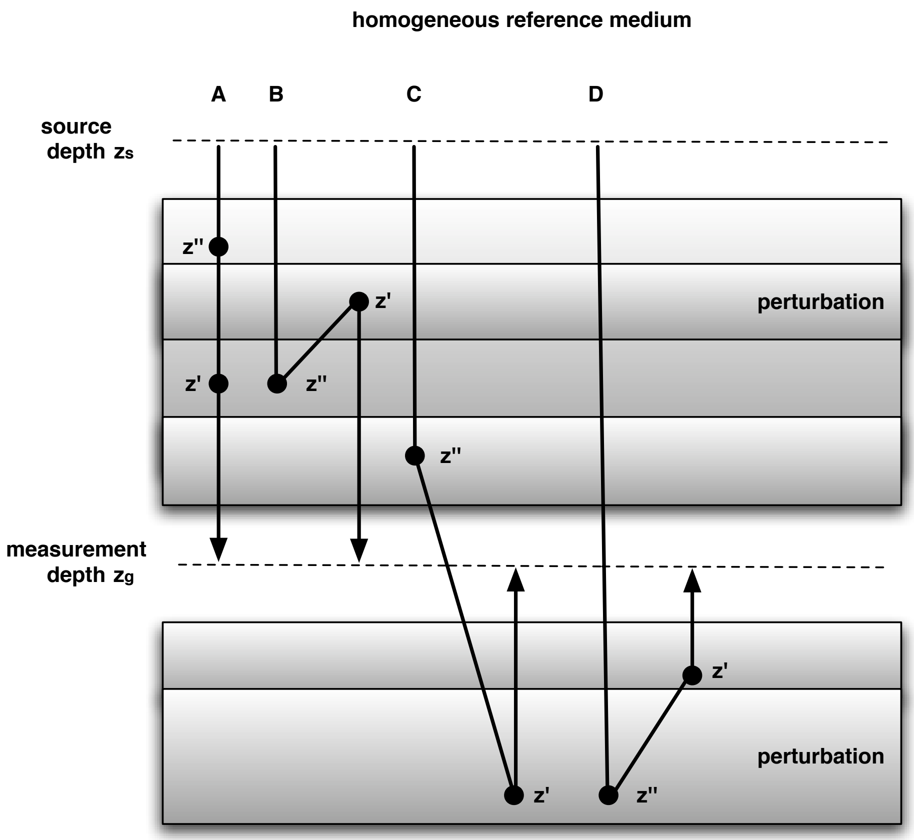

FIG. 2. Scattering diagrams at second order, overlain on the wavespeed profile of Figure 1. The reference medium is chosen to be a homogeneous wholespace, meaning all of the layer structure is part of the perturbation. In contrast to the scheme in Figure 1, all of the wave paths drawn here represent propagation in the reference medium. Dots represent interactions with the perturbation. The eikonal approximation is derived by retaining only diagrams like (A) at all orders.

a pattern in the retained terms, and use the pattern to collect the desired contributions together at all orders. Calling the result T, we have

$$T(z_g, z_s) = \sum_{n=0}^{\infty} T_n(z_g, z_s) \\ = \frac{e^{ik(z_g - z_s)}}{i2k}\sum_{n=0}^{\infty}\frac{1}{n!}\left(-\frac{ik}{2}\int_{z_s}^{z_g}\alpha(z')dz'\right)^n, \quad z_g > z_s. \tag{9}$$

The series in equation (9) is recognizable as a Taylor's series expansion. Summing it, we produce

$$T(z_g, z_s) = \frac{e^{ik\left[z_g - z_s - \frac{1}{2}\int_{z_s}^{z_g} \alpha(z')dz'\right]}}{i2k}, \quad z_g > z_s. \tag{10}$$

Equation (10) is substantially the same as the eikonal approximation presented by Morse and Feshbach (1953, pg. 1095), where it is referred to not by that name but rather in terms of a low-order expansion of the phase of the WKBJ solution.

INTEGRATING THE LIPPMANN-SCHWINGER EQUATION

Consider again the 1D Lippmann-Schwinger equation:

$$G(z_g, z_s) = G_0(z_g, z_s) + \int_{-\infty}^{\infty} G_0(z_g, z')k^2\alpha(z')G(z', z_s)dz'. \tag{11}$$

The derivation of the eikonal approximation presented by Morse and Feshbach (1953), which we will deviate from only slightly, is obtained by altering the upper limit of the integral in the Lippmann-Schwinger equation to coincide with the observation point z_g. Let us confirm that this produces the same result that we got in equation (10). Remembering that $z_g > z_s$ and z_s is smaller than all z for which $\alpha \neq 0$, equation (11) turns into

$$T(z_g, z_s) = G_0(z_g, z_s) + \int_{-\infty}^{z_g} G_0(z_g, z')k^2\alpha(z')T(z', z_s)dz', \tag{12}$$

which upon substitution of the Green's function in equation (6) becomes

$$T(z_g, z_s) = \frac{e^{ik(z_g - z_s)}}{i2k} - \frac{ik}{2}e^{ikz_g}\int_{-\infty}^{z_g} e^{-ikz'}\alpha(z')T(z', z_s)dz'. \tag{13}$$

This expression is next transformed into a differential equation and integrated. We multiply through by e^{-ikz_g}, and take the derivative with respect to z_g, producing:

$$\frac{dT(z_g, z_s)}{dz_g}e^{-ikz_g} - ikT(z_g, z_s)e^{-ikz_g} = -\frac{ik}{2}e^{-ikz_g}\alpha(z_g)T(z_g, z_s). \tag{14}$$

Collecting the two terms in T we integrate to obtain

$$T(z_g, z_s) = Ce^{ik\left(z_g - \frac{1}{2}\int_{z_s}^{z_g} \alpha(z')dz'\right)}, \tag{15}$$

where C is the integration constant. The z_s lower limit in the integral is allowable since $\alpha = 0$ at z_s and below. We determine C by placing ourselves, the observers, at the origin $z_g = 0$ and considering z_s to be slightly negative. We furthermore have that $T(z_g, z_s)|_{z_g=0} = C$. We then assume that the only non-negligible contribution to T for this z_g, z_s pair is the reference wave, which implies that the perturbation is too smooth to have reflected any observable wave energy. This leads to an integration constant of

$$C = T(z_g, z_s)|_{z_g=0} = G_0(z_g, z_s)|_{z_g=0} = \frac{e^{-ikz_s}}{i2k}, \tag{16}$$

in which case we recover the same form for the eikonal approximation,

$$T(z_g, z_s) = \frac{e^{ik\left[z_g - z_s - \frac{1}{2}\int_{z_s}^{z_g} \alpha(z')dz'\right]}}{i2k}, \tag{17}$$

as that due to the scattering-diagram retention/rejection program.

COMPARING THE TWO DERIVATIONS

At first blush the above equivalence might seem strange. The diagram version of the derivation looks like it is "throwing away" much more of the wave than Morse and Feshbach do in their approach. Comparing equations (5) and (8), we see that at second order only 1/4 of the diagrams are retained, and at every order higher, retention is dramatically decreased (to be precise, only 1 of a total of 2^n diagrams contribute at n'th order), whereas Morse and Feshbach merely throw away contributions from below z_g. Surely a large part of the actual field comes from things happening above z_g. Why would we get the same result from both?

The fact is, M&F reject more of the wave field by changing the upper limit than one might think. To see this, let's look again at the altered Lippmann-Schwinger equation:

$$T(z_g, z_s) = G_0(z_g, z_s) + \int_{-\infty}^{z_g} G_0(z_g, z')k^2\alpha(z')T(z', z_s)dz'. \tag{18}$$

This time, rather than manipulating it as an integral equation, we will expand equation (18) in series through back-substitution, just as if we were deriving the Born series. Taking care in particular with the variable z_g, we have

$$\begin{aligned} T(z_g, z_s) =& G_0(z_g, z_s) + \int_{-\infty}^{z_g} dz' G_0(z_g, z')k^2\alpha(z') \\ &\times \left\{ G_0(z', z_s) + \int_{-\infty}^{z'} dz'' G_0(z', z'')k^2\alpha(z'') \right. \\ &\times \left. \left[G_0(z'', z_s) + \int_{-\infty}^{z''} dz''' G_0(z'', z''')k^2\alpha(z''') \left[G_0(z''', z_s) + ... \right] \right] \right\}, \end{aligned} \tag{19}$$

or,

$$T(z_g, z_s) = T_0(z_g, z_s) + T_1(z_g, z_s) + T_2(z_g, z_s) + ..., \tag{20}$$

where

$$T_0(z_g, z_s) = G_0(z_g, z_s), \tag{21}$$

$$T_1(z_g, z_s) = \int_{-\infty}^{z_g} dz' G_0(z_g, z')k^2\alpha(z')G_0(z', z_s), \tag{22}$$

$$T_2(z_g, z_s) = \int_{-\infty}^{z_g} dz' G_0(z_g, z')k^2\alpha(z') \int_{-\infty}^{z'} dz'' G_0(z', z'')k^2\alpha(z'')G_0(z'', z_s), \tag{23}$$

etc.

Now what is going on is a bit clearer. The z_g in the upper limit of equation (18) *looks* benign and would appear to have a straightforward effect on the field, making it amenable to sweeping interpretive statements like the one at the beginning of this section. But, equation (18) is not a solution, rather it is an integral equation for T, and its ingredients can affect the actual solution in ways that are difficult to guess. When we do solve it, by expanding T as a modified Born series, we see that z_g does not affect the character of the solution in a simple way at all. It gets forced to play the role of an integration variable again and again in the solution, stopping and starting wave contributions not at one fixed position, z_g, but everywhere, and repeatedly.

Especially if you follow along going through the math, you can see that with the particular brand of stopping and starting that Morse and Feshbach enforce, each nested integral in equation (20) is interrupted exactly where it begins to incorporate scattering interactions that involve a change of propagation direction. In other words, this is the same rejection of wavefield components that underlies the scattering-diagram approach. Substitution of the appropriate Green's functions confirms that the T_i in equation (20) are identical to those of equation (9).

DISCUSSION

As a tutorial and introduction to the origin and use of scattering diagrams, a scattering-diagram based derivation of the eikonal approximation for transmitted wave fields is discussed. The derivation is designed to produce a form substantially the same as that presented by Morse and Feshbach (1953), so that through comparison scattering diagrams and their retention and rejection can be understood.

Since equations (9) and (20) are exactly the same, it appears that the two derivations differ only procedurally, with the new one simply going about its business using scattering diagrams. However, there is a slight additional difference. The two approaches make fundamentally the same choices about retention of wavefield components, but because in the truncated Lippmann-Schwinger integral approach the equation was differentiated, we were forced to additionally argue for the form of C. Since it is based on a series, the scattering diagram derivation lacks a certain expediency, but it clarifies that the original integral limitation alone is sufficient to obtain the eikonal approximation in that form. From our comparison we can now see that, in carrying out the older approach, it was necessary to at first *throw away* information critical to the solution, and later *return it again* by imposing a (in the grand scheme of things) redundant boundary condition. With diagrams we were able to avoid all that.

It is finally worth emphasizing that making direct waves with a scattering-diagram approach can and has been generalized to multidimensional fields and perturbations (Innanen, 2009), while the truncated Lippmann-Schwinger approach seems to be fundamentally restricted to 1D media.

ACKNOWLEDGMENTS

The author is grateful to CREWES sponsors and personnel.

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