Investigating reflections from smooth transition boundaries

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ABSTRACT

Reflections from impedance boundaries are well understood in many scientific applications including geophysics. These boundaries are typically assumed to be a step like function that reflects the energy of all frequencies, and are consistent with geological processes. Reflections from boundaries with a smooth transition do not reflect all frequencies.

The purpose of this study is to create a numerical modelling environment that accurately propagates a waveform in a variable velocity medium, with the purpose of evaluating the reflection energy from interfaces that have various transition shapes.

INTRODUCTION

Reflection energy from an interface between two media is defined by Zoeppritz's equations. When the incident direction is normal to the interface, the reflection and transmission coefficients are defined by the simple relationships:

$$R = \frac{Z_t - Z_i}{Z_t + Z_i}, \quad T = \frac{2Z_i}{Z_t + Z_i} \tag{1}$$

where Z_i and Z_t are the acoustic impedance of the incident "*i*" and transmitted "*t*" media. The impedance is the product of density and velocity. The amplitude of the reflected energy is the incident waveform, scaled by $-R$, and the transmitted energy scaled by T.

The reflection and transmission coefficients are only valid when the transition zone between the two media is very small relative to the wavelengths of the incident waveform. When the transmission zone is much larger than the wavelength of the incident waveform, there is no reflection. All the energy is transmitted and the amplitude of the transmitted energy is scaled by the square-root of the acoustic impedances,

$$T_{smooth} = \sqrt{\frac{Z_i}{Z_t}}. \tag{2}$$

When the acoustic impedance increases with depth, typically with increasing velocities, the amplitude of a vertical plane wave will decrease in amplitude.

When the wavelengths of the incident waveform are similar to the size of the transition zone, the reflected and transmitted waveforms will differ. Lower frequencies will be reflected more than higher frequencies, and more higher frequencies will be transmitted than lower frequencies.

The traditional wave-equation in one spatial dimension is

$$\frac{\partial^2 P}{\partial z^2} = \frac{1}{Z^2}\frac{\partial^2 P}{\partial t^2}. \tag{3}$$

This equation is only valid when the impedance is uniform. It cannot be used when there is a change in impedance (or velocity). When the impedance changes, the following wave-equation is required:

$$\frac{\partial^2 P}{\partial t^2} = \frac{\partial}{\partial z}\left(Z^2 \frac{\partial P}{\partial z}\right) = Z^2 \frac{\partial^2 P}{\partial z^2} + \frac{\partial Z^2}{\partial z}\frac{\partial P}{\partial z}. \tag{4}$$

IMPLEMENTATION

The wave-equation is typically implemented using finite differences. The finite difference method is only an approximation to the wave equation and care must be taken to ensure the desired accuracy. The errors in the finite difference approximation accumulate, and even though they may appear minor, their accumulated error can be significant and will distort the amplitudes and phase of the waveforms. These errors are often visible in the time and space domain as low energy waves that are often referred to as grid dispersion. In the frequency domain, the errors for a finite difference approximation to a derivative are identified as deviations from the 90 degree phase shift, and from the linear amplitudes that should be proportional to the frequency to the Nyquist frequency. The errors are largest at higher frequencies (Bancroft 2008) as illustrated by the figures in the appendix.

Sampling

To reduce the error of the finite difference approximations, over-sampling in time and space domains is used. This keeps the energy in the lower part of the frequency domain, where the approximations are more accurate.

Finite difference approximations

In addition to oversampling, more accurate finite difference equation should be used. A typical approximation to the first derivative is the forward difference

$$\frac{dP}{dx} \approx \frac{P(z+\delta z) - P(z)}{\delta z}. \tag{5}$$

This approximation is taken to be valid at the point ($x + \delta x / 2$), or mid way between the two samples. This introduces a phase error. The following equation eliminates the phase error, but increases the length of the operator, i.e.,

$$\frac{dP}{dx} \approx \frac{P(z+\delta z) - P(z-\delta z)}{2\delta z}. \tag{6}$$

Higher order approximations perform much better, as described in a previous CREWES paper (Bancroft 2008). I will use a six-point approximation for the first derivative and use a subscript to identify the sample number, i.e.,

$$\frac{dP}{dx} \approx \frac{-P_{-3} + 9P_{-2} - 45P_{-1} + 45P_1 - 9P_2 + P_3}{60\delta x}, \tag{7}$$

and seven points for the second derivative

$$\frac{d^2P}{dx^2} \approx \frac{2P_{-3} - 27P_{-2} + 270P_{-1} - 490P_0 + 270P_1 - 27P_2 + 2P_3}{60\delta x}. \tag{8}$$

These higher order approximations do require additional computation time.

The high order approximations also require the impedance to be smooth. A step type change in impedance will be aliased and/or have large amounts of energy at the Nyquist frequency. These higher frequencies will contribute to the dispersion noise. I will use a four or five point smoother to ensure that my desired accuracy is maintained, which is that the waveform shape remains similar to its initial form as it propagates through an impedance change.

Building the chirps

Chirps or sweeps are straight forward to design, but care must be taken to achieve the desired frequencies. The actual frequency at a portion of a sweep is the instantaneous frequency. It is not a simple value of the frequency defined in the sine or cosine. We know that for a constant frequency, $f(t) = \sin(2\pi F\, t)$, where F is the desired frequency. For a linear sweep $F_l(t)$ from F_1 to F_2, i.e.,

$$F_l(t) = F_1 + \frac{(F_2 - F_1)t}{T_{\max}}, \tag{9}$$

and we may try $f_F(t) = \cos(2\pi \times t \times F_l(t))$. The frequency f_F will not be the desired linear sweep. From Carroll (1971) the desired sweep should be

$$Sweep(t) = \sin\left\{2\pi\left[tF_1 + \frac{(F_2 - F_1)t^2}{2T_{\max}}\right]\right\}, \tag{10}$$

or

$$Sweep(t) = \sin\left\{2\pi t\left[F_1 + \frac{(F_2 - F_1)t}{2T_{\max}}\right]\right\}. \tag{11}$$

Equation (10) contains the phase within the square brackets, and equation (11) contains the frequency in the square brackets.

Where does the "2" come from in the denominator of equation (11)? It differs from the linear frequencies of equation (9).

The instantaneous frequencies are the time derivative of phase, i.e.,

$$\frac{d}{dt}\left[tF_1 + \frac{(F_2 - F_1)t^2}{2T_{\max}}\right] = F_1 + \frac{(F_2 - F_1)t}{T_{\max}}. \tag{12}$$

To build a more complex waveform such as a modulated chirp, I first build the desired time-frequency model, and then integrate it to be used as the phase.

EXAMPLES

I will use a very simple model where the velocity (impedance) changes from 1000 m/s to 2000 m/s. The time sampling is 0.25 ms, and the depth sampling is 1 m. I create a chirp with a linear sweep from 20 to 100 Hz over a time of 400 ms. The over-sampling requires two different initiating wavelets, one at time sample 1 and the other at time sample 2 to propagate energy to the right. The chirp moves spatially to the right at each time increment. The red line in the following figures represents the shape of the transition zone. Figure 1a shows the starting waveform, and the remaining figures show the transmitted and reflected waveforms. Note the time reversed direction of the reflected waveform relative to the starting waveform. The reflection amplitude should be 1/3, and the transmitted amplitude 2/3. The transmitted amplitude of Figure 1g should be $1/\sqrt{2}$ = 0.707. The details of Figure 1 are listed below with incorrect equations and transition steps in red, correct equations in green, and the correct transition zones in blue:

(a) The chirp located at the starting time. Note the square shape of the chirp.

(b) WE (3), a step transition, FD equation (6), RT = 8.5 s.
Note the poor waveform shape and incorrect amplitudes.
The polarity of the reflected waveform is the same as the incident in (a)
The polarity of the reflected waveform is spatially reversed.

(c) WE (3), a step transition, FD equation (8), RT = 12 s.
Better quality of waveform, incorrect amplitudes.

(d) WE (4), a step transition, FD equation (5) and (6), RT = 12 s.
Polarity of the reflected wave is correct but poor quality, amplitudes still wrong.

(e) WE (4), 5 m transition, FD equation (5) and (6), RT = 17 s.
Amplitudes have improved, almost correct.

(f) WE (4), 5 m transition, FD equations (7) and (8), RT = 25 s.
Amplitudes and shape correct.

(g) WE (4), 200 m transition, FD equation (7) and (8), RT = 25 s.
All energy is preserved in the transmitted waveform.
The transmitted amplitude is 0.707.

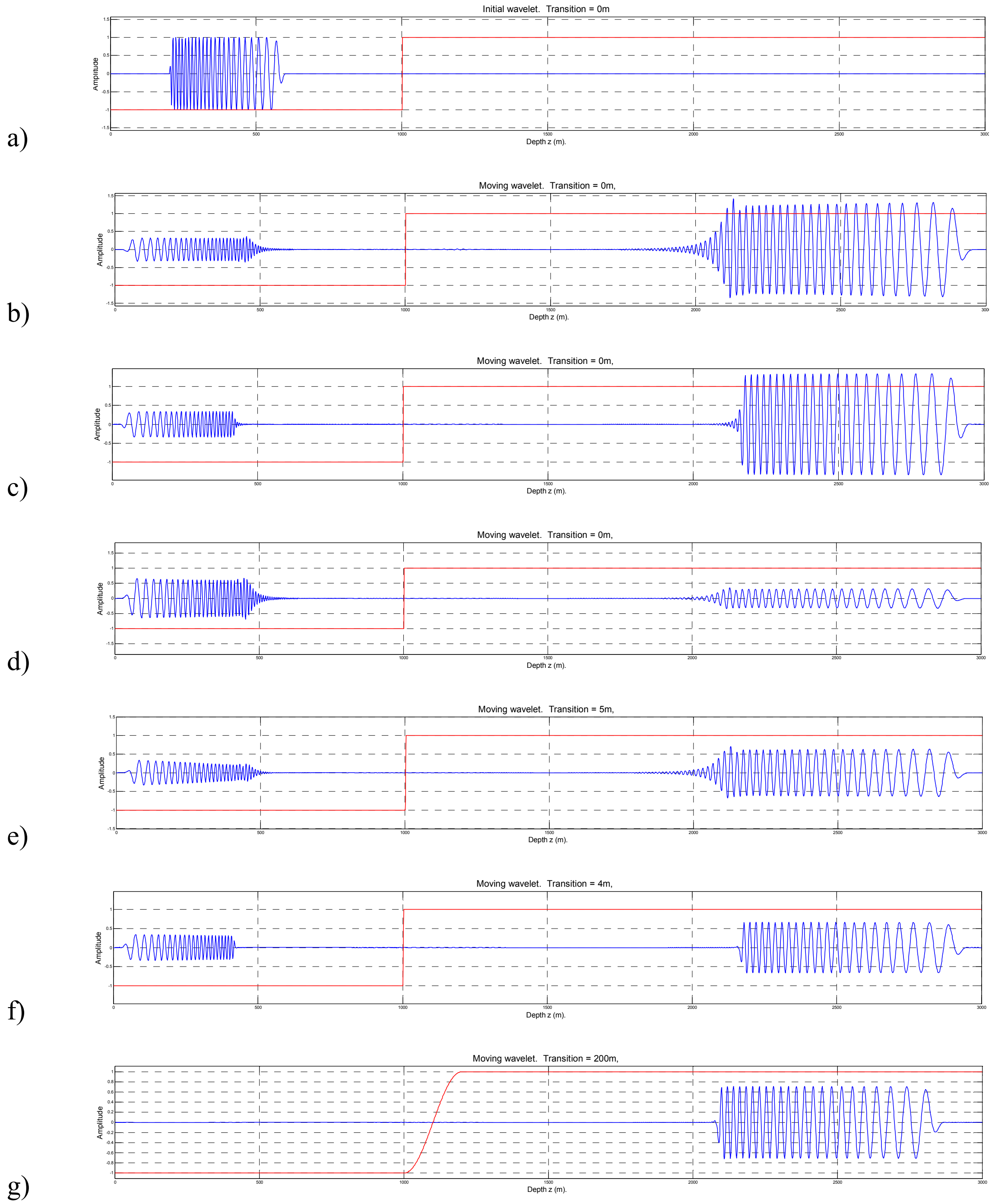

FIG. 1

a) The chirp located at the starting time
b) WE (3), a step transition, FD equation (6), RT = 8.5 sec.
c) WE (3), a step transition, FD equation (8), RT = 12 sec.
d) WE (4), a step transition, FD equation (5) and (6), RT = 12 sec.
e) WE (4), 5 m transition, FD equation (5) and (6), RT = 17 sec.
f) WE (4), 5 m transition, FD equation (7) and (8), RT = 25 sec.
g) WE (4), 200 m transition, FD equation(7) and (8), RT = 25 sec.

The following figure shows a modulated waveform with a carrier frequency of 40 Hz, and with modulated frequencies of ±30 Hz. The instantaneous frequencies start at 40 Hz, with linear transitions to 70 Hz, then to 70 Hz and back to 40 Hz. Note the frequency dependence on the amplitudes in the reflected and transmitted waveforms.

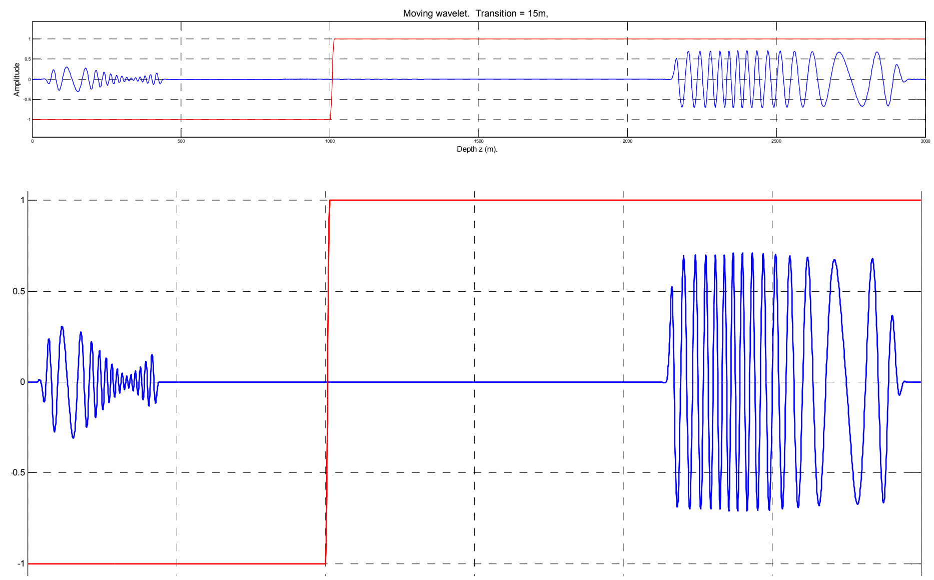

FIG. 2 Modulated chirp at 40 ± 30 Hz with a) at regular size, and b) in expanded form.

CONCLUSIONS

A high quality 1D modelling capability was described for media with varying impedances. The method used over-sampling and high-order finite difference approximations to the wave equation. The model also required smooth transitions at the impedance boundaries to produce the correct amplitudes.

SOFTWARE

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2010-Matlab\ChiprReflections\ChirpVerticalHighOrder.m
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ACKNOWLEDGEMENTS

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REFERENCES

Bancroft, J. C., 2008, Differential operators 1: the first derivative, CREWES Report.

Bancroft, J. C., 2008, Differential operators 2: the second derivative, CREWES Report.

Carroll, P., 1971, A note on synthetic vibroseis sweep generation, Possibly an CSEG publication. (See http://www.cseg.ca/publications/journal/1971_11/1971_Carroll_P_synth_vibroseis_sweep.pdf

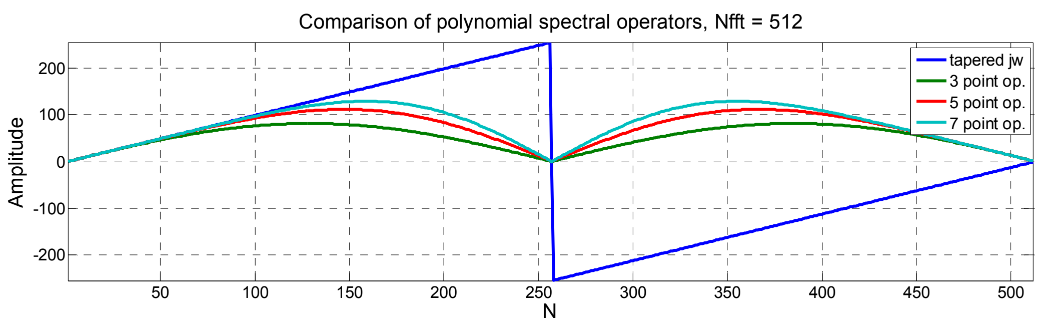
APPENDIX :COMPARISON OF AMPLITUDE SPECTRA FOR FIRST AND SECOND DERIVATIVES

The following figures illustrate the accuracy of finite difference approximations to the first and second derivatives. The figure shows the spectral amplitudes, which should be linear for a derivative, i.e. proportional to the frequency $(j\omega)$ and quadratic for the second derivative, i.e. proportional to the square of the frequencies $(-\omega^2)$.

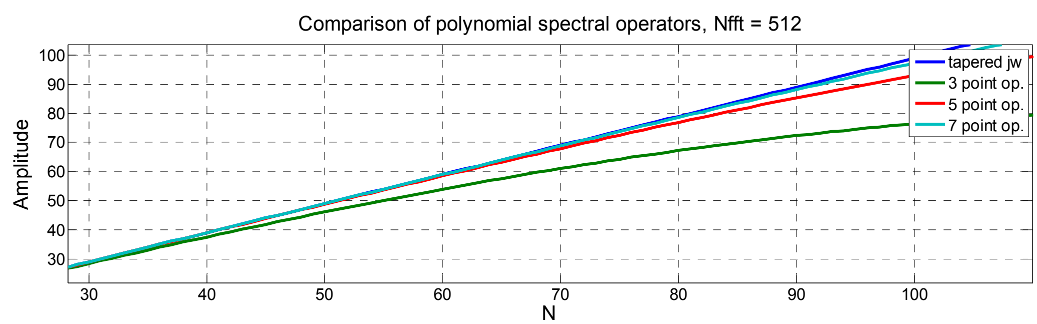
The following figures are from Bancroft (2008).

First derivative

Figure 3a shows the two sided amplitude spectrum of the desired and finite difference approximations to the first derivative. The number across the bottom (N) can be used to approximately represent the frequencies for data samples at 0.002 s, or a sampling frequency of 500 Hz. Part (b) is a close up of part (a).



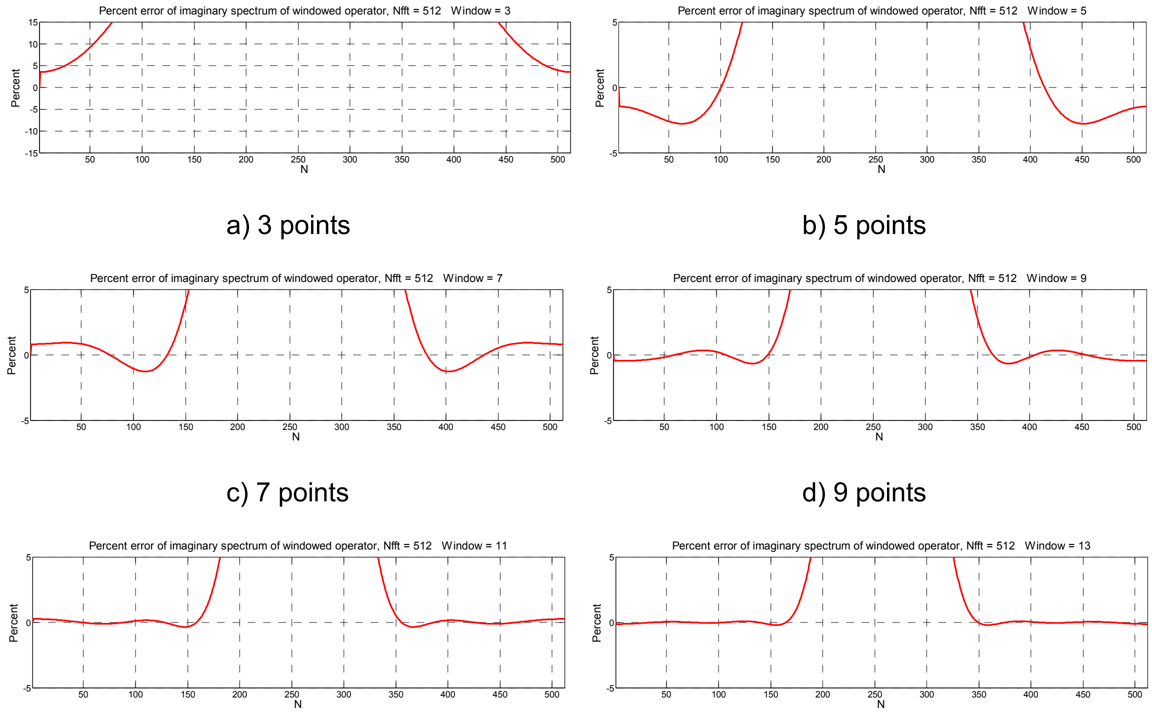
a)



b)

FIG. 3 Comparison of a) the amplitude spectrums for operator sizes 3, 5, and 7 points and b) a zoomed display. The horizontal numbers approximate the frequencies for 2 ms sampled data.

The following images show the error in the amplitude spectrum for 3 to 13 point finite difference operators. These operators were defined using spectral methods.



a) 3 points

b) 5 points

c) 7 points

d) 9 points

e) 11 points

f) 13 points

FIG. 4 A visual comparison of the percent errors of spectrally derived operators for size that range from 3 to 13 points as annotated.

Continued on the following page.

Second Derivative

The following images represent the finite difference solutions for the second derivative for 3, 5, 7, and 9 points.

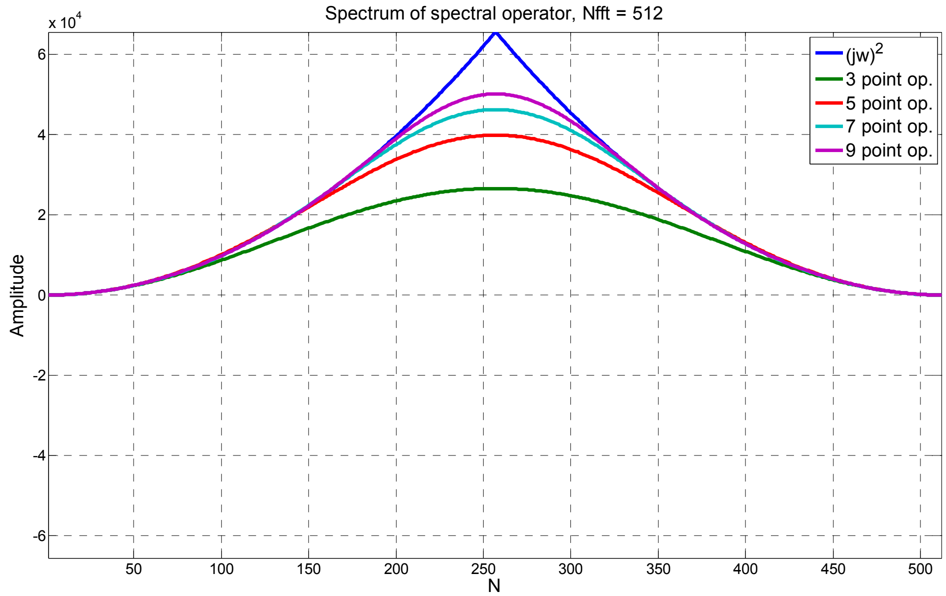


Figure 5 Amplitude spectrum for operators with 3 to 9 points.

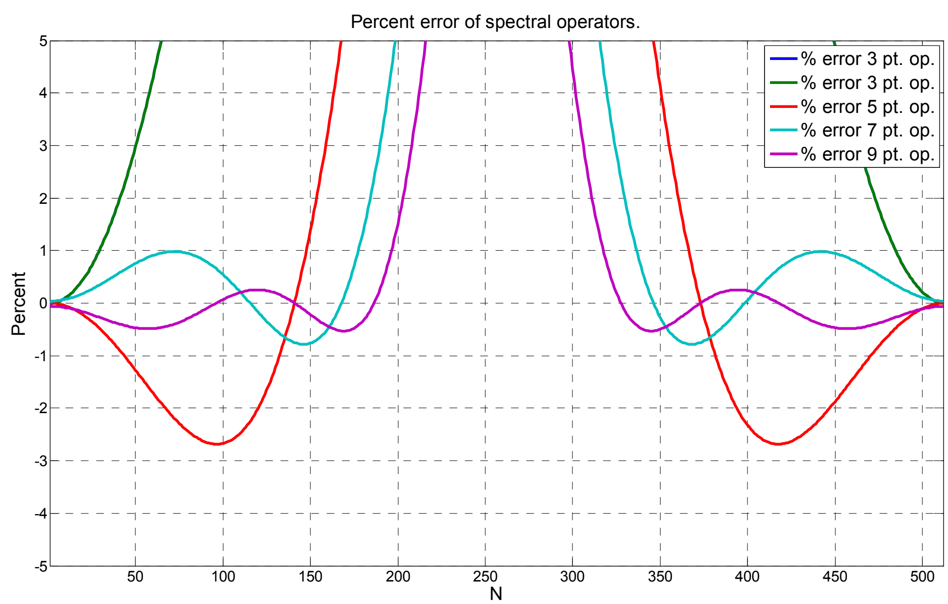


FIG. 6 Percent errors for the finite difference approximations to the second derivative.