A linearized group velocity approach for two point *qP* ray tracing in a layered orthorhombic medium

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ABSTRACT

Using a linearized approximation for the quasi-compressional phase velocity, v_{qP} in an orthorhombic anisotropic medium, which is a subset of the related quasi-compressional (qP) wave propagation in a general 21 parameter anisotropic medium, a linearized compressional group velocity may be derived as a function of group angles only. In addition, linearized analytic expressions for the components of the slowness vector in terms of group velocities and angles are also obtained. These expressions are used to define two nonlinear equations which are a generalization of Snell's Law. The solutions of these are used to determine the propagation directions of the reflected and transmitted rays due to an incident ray at an interface between two orthorhombic media. The axes of anisotropy in both media are, in general, not aligned with the interface separating them. Computer code has been written to consider ray tracing in media defined by a type of large scale $3D$ finite element blocking (blocky structures). However, a plane parallel layered will be used in preliminary investigations. Additionally, in each of these elements (layers) the anisotropic parameters are initially assumed to be constant, although provisions for at least minimal spatial variations of the anisotropic parameters have been considered.

INTRODUCTION

In the geophysical literature related to wave propagation in anisotropic media, specifically quasi-compressional (qP) waves in a medium displaying orthorhombic symmetry, a linearized approach to derive an approximate phase velocity expression for quasi-compressional (qP) wave propagation has been presented in Backus (1965) and more recently by Pšenčík and Farra (2005). These and other methods, such a perturbation theory, have been employed to formulate expressions for quantities related to the propagation of qP waves as well as for the two shear wave modes, qS_1 and qS_2, in a general 21 parameter anisotropic medium (Every, 1980, Every and Sachse, 1992, Jech and Pšenčík, 1989, Pšenčík and Gajewski, 1998, Song and Every, 2000, Song, Every, and Wright, 2001, Pšenčík. and Farra, 2008). Once linearized phase velocity approximations have been obtained, eikonal equations with comparable accuracy may be written. From these, using the method of characteristics, (Courant and Hilbert, 1962, Cerveny, 2001) the formulae for the vector components of group velocity may be obtained.

Employing an extension of this method of approximation, Daley and Krebes (2005) obtained expressions for the qP group velocities in an orthorhombic anisotropic medium, expressed in terms of group angles. It was further shown that for a weakly anisotropic medium the average deviation of this approximation from the exact expression for the qP group velocity, in an orthorhombic medium, over a range of polar angles for a number of azimuths, Φ, was of the order of a few percent , for media that could be classified as *weakly* anisotropic. The expression for the qP group velocity was also compared with the linearized qP group velocity expressions obtained using the First Order Ray Tracing (FORT) method presented in Pšenčík and Farra (2005), with similar accuracy, for an orthorhombic medium. For this reason it was chosen for use in preliminary ray tracing procedures where speed, together with reasonable accuracy, was required.

It will be convenient to first consider the case where the anisotropic parameters and orientation of anisotropy are assumed constant within any finite element. What is required to be determined is an analogue of Snell's Law at the plane interface between two adjacent elements. Within the context of "two-point" ray tracing this topic will be treated in what follows for an orthorhombic anisotropic medium.

THEOREORETICAL BACKGROUND

The linearized quasi-compressional (qP) phase velocity, $v_{qP}(n_k)$, in an orthorhombic medium, may be written in Voigt notation, where the A_{ij} have the dimensions of velocity squared, as

$$v_{qP}^2(n_k) = A_{11}n_1^2 + A_{22}n_2^2 + A_{33}n_3^2 + E_{12}n_1^2n_2^2 + E_{13}n_1^2n_3^2 + E_{23}n_2^2n_3^2 \tag{1}$$

$$\mathbf{n} = (n_1, n_2, n_3) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) \tag{2}$$

where (θ,ϕ) are the polar and azimuthal angles associated with the phase or wavefront normal velocity, v_{qP}. The E_{ij} are the anellipsoidal terms, specifying the deviation of the slowness or ray surface from the ellipsoid, and are defined as

$$E_{12} = 2(A_{12} + 2A_{66}) - (A_{11} + A_{22}). \tag{3}$$

$$E_{13} = 2(A_{13} + 2A_{55}) - (A_{11} + A_{33}). \tag{4}$$

$$E_{23} = 2(A_{23} + 2A_{44}) - (A_{22} + A_{33}). \tag{5}$$

These expressions could be compared to those given for a *mildly* anisotropic orthorhombic medium presented in Gassmann (1964) or Schoenberg and Helbig (1996) as an indication of how linearization simplifies the phase velocity expression.

The $3D$ phase velocity propagation direction vector, $\mathbf{n}$, is defined by equation (2). The related slowness vector in terms of these quantities is of the form

$$\mathbf{p} = (p_1, p_2, p_3) = \left(\frac{n_1}{v_{qP}(n_k)}, \frac{n_2}{v_{qP}(n_k)}, \frac{n_3}{v_{qP}(n_k)}\right) \tag{6}$$

It was shown in earlier works that the slowness vector, $\mathbf{p}$, for the qP case may be further approximated for the linearized case and written in terms of group velocity quantities, those being the angles Φ and Θ, and velocity, $V_{qP}(\Theta,\Phi)$, as

$$\mathbf{p} = (p_1, p_2, p_3) = \left(\frac{N_1 V_{qP}(N_k)}{A_{11}}, \frac{N_2 V_{qP}(N_k)}{A_{22}}, \frac{N_3 V_{qP}(N_k)}{A_{33}} \right) \tag{7}$$

where $\mathbf{N}$ is the orthonormal group (ray) vector defined as

$$\mathbf{N} = (N_1, N_2, N_3) = (\sin\Theta\cos\Phi, \sin\Theta\sin\Phi, \cos\Theta). \tag{8}$$

It is along this ray that energy propagates from one point in a medium to another. The general group velocity expression for a *qP* ray in an orthorhombic medium is given by

$$\frac{1}{V_{qP}^2(N_k)} = \frac{N_1^2}{A_{11}} + \frac{N_2^2}{A_{22}} + \frac{N_3^2}{A_{33}} - \frac{E_{12}N_2^2N_3^2}{A_{11}A_{22}} - \frac{E_{13}N_1^2N_3^2}{A_{11}A_{33}} - \frac{E_{23}N_1^2N_3^2}{A_{22}A_{33}} \tag{9}$$

where the E_{ij} were previously defined in equations (3) – (5). Equation (9) may be expressed in terms of (Θ, Φ) using equation (8).

As the slowness vector $\mathbf{p} = (p_1, p_2, p_3)$ is assumed to be known, the sines of the group angles $(\xi_1 = \sin\Theta, \xi_2 = \sin\Phi)$ will be used as an alternate parameterization of group quantities. In the ellipsoidal case $(E_{12} = E_{13} = E_{23} = 0)$ ξ_1 and ξ_2 may be obtained analytically

$$\xi_1 = \left(\frac{p_r}{p_3}\right) \frac{\left[\cos^2\Phi / A_{11}^2 + \sin^2\Phi / A_{22}^2\right]^{-1/2}}{A_{33}} \left\{ 1 + \left(\frac{p_r}{p_3}\right)^2 \frac{\left[\cos^2\Phi / A_{11}^2 + \sin^2\Phi / A_{22}^2\right]^{-1}}{A_{33}^2} \right\}^{-1} \tag{10}$$

$$\xi_2 = \frac{\left(A_{22}p_2 / A_{11}p_1\right)}{\left[1 + \left(A_{22}p_2 / A_{11}p_1\right)^2\right]^{1/2}} \tag{11}$$

where p_r, the radial component of the slowness vector, is defined as

$$p_r = \left(p_1^2 + p_2^2\right)^{1/2} \tag{12}$$

The relationship between the phase angles (θ, ϕ) and the group angles (Θ, Φ) in the ellipsoidal case are calculated by equating individual components in equations (6) and (8) which are used to derive equations (10) – (12).

SNELL'S LAW

Given a three dimensional slowness surface that may be rotated to any orientation with respect to the model axis system, it would be expected that Euler Angles be introduced. However, as one of the motivations for this work is to introduce 3*D* ray tracing in an orthorhombic anisotropic medium in the simplest possible manner, this will not be done. When more complex ray tracing methods (comparatively) are addressed, such as FORT, this will be necessary. As the slowness vector in terms of phase angles and velocities is not used here, one level of theoretical and numerical complexity has been removed, allowing for the subsequent formulation.

Two orthorhombic media in welded contact are separated by a plane boundary with the axes of anisotropy in both the upper and lower medium aligned with some Cartesian model coordinates system within which the plane boundary is parallel to the horizontal (x_1, x_2) plane. As previously mentioned, this is a very simplistic problem type, but once solved may be extended to more complex geometries with minimal effort.

Given that the incident ray angles (Θ_i, Φ_i) are known, it is required to determine the ray angle and magnitude of the ray velocity for either the reflected ray in the upper medium or the transmitted ray in the lower medium. Specifically, the two horizontal components of the slowness vector, $\mathbf{p}_h = (p_1, p_2)$ at the point of incidence at an interface, measured with respect to the model coordinates, are required to be determined. It has been assumed that the plane interface is aligned with the model coordinate axes, as are the qP slowness surfaces in both media. As a consequence of these assumptions, the projections of the slowness surface onto the $p_1 p_3$ and $p_2 p_3$ planes may be realized. From equation (7) the continuity of the horizontal components of the slowness vector may be stated in terms of (ξ_1, ξ_2) by the two equations

$$F_1(\xi_1, \xi_2) = p_1 A_{11} - \xi_1 \left(1 - \xi_2^2\right)^{1/2} V_{qP}(\xi_1, \xi_2) = 0 \tag{13}$$

$$F_2(\xi_1, \xi_2) = p_2 A_{22} - V_{qP}(\xi_1, \xi_2)\,\xi_1 \xi_2 = 0 \tag{14}$$

These coupled nonlinear equations are required to be solved numerically. For coupled equations of this type Newton's Method, or some variant, would be a reasonable choice as a solution method, having the standard formulation

$$\boldsymbol{\xi}_{k+1} = \boldsymbol{\xi}_k - \mathbf{F}_k(\xi_1, \xi_2)\left[\Im_k(\xi_1, \xi_2)\right]^{-1} \tag{15}$$

where $\boldsymbol{\xi}_k = (\xi_1, \xi_2)_k^T$, $\mathbf{F}_k(\xi_1, \xi_2) = \left[F_1(\xi_1, \xi_2), F_2(\xi_1, \xi_2)\right]_k^T$ and the Jacobian, $\Im_k(\xi_1, \xi_2)$, is defined as

$$\Im_k(\xi_1, \xi_2) = \begin{bmatrix} \partial F_1/\partial\xi_1 & \partial F_1/\partial\xi_2 \\ \partial F_2/\partial\xi_1 & \partial F_2/\partial\xi_2 \end{bmatrix}_k \tag{16}$$

with its inverse given by

$$\left[\Im_k(\xi_1, \xi_2)\right]^{-1} = \frac{1}{\mathcal{D}_k}\begin{bmatrix} \partial F_2/\partial\xi_2 & -\partial F_1/\partial\xi_2 \\ -\partial F_2/\partial\xi_1 & \partial F_1/\partial\xi_1 \end{bmatrix}_k \tag{17}$$

$$\mathcal{D}_k = \left[(\partial F_1/\partial\xi_1)(\partial F_2/\partial\xi_2) - (\partial F_2/\partial\xi_1)(\partial F_1/\partial\xi_2)\right]_k . \tag{18}$$

where the superscripts "T" and "-1" indicate transpose and inverse, respectively.

The partial derivatives of $F_j(g_1, g_2), (j = 1,2)$ with respect to ξ_1 and ξ_2, and analytic expressions for the partial derivatives of $V_{qP}(\xi_1, \xi_2)$ with respect to ξ_1 and ξ_2, are given in the Appendix. The initial "guesses" for the two components of the quantity $\boldsymbol{\xi}_0$ may be found using the ellipsoidal (degenerate) equations (10) and (11). Once $\boldsymbol{\xi} = (\xi_1, \xi_2)$ has been determined to within a specified tolerance, the group velocity, and as a result, the three components of the slowness vector in the medium of reflection or transmission, may be determined employing equations (7).

SNELL'S LAW AT A PLANE INTERFACE WHERE THE AXES OF ANISOTROPY ARE NOT ALIGNED WITH THE INTERFACE

For this problem, it is assumed that the slowness vector, $\mathbf{p}$, relative to the model coordinates is known. To solve the problem, that is to determine the polar ray angles (Θ, Φ) relative to the model coordinates, the angles in the rotated system $(\Theta' = \Theta + \gamma, \Phi' = \Phi + \zeta)$ must be determined, under the assumption that the polar and azimuthal rotation angles (γ, ζ) are also known. The primed angles define the slowness vector, $\mathbf{p}'$, in the rotated coordinate system, and the angle pair (γ, ζ) specifies the relationship between (Θ, Φ) and (Θ', Φ').

With $\mathbf{p}' = (p_1', p_2')$ known, the following system of nonlinear equations in terms of ξ_1' and ξ_2' is required to be determined where

$$\xi_1' = \sin(\Theta + \gamma) = \sin\Theta' \tag{19}$$

and

$$\xi_2' = \sin(\Phi + \zeta) = \sin\Phi' . \tag{20}$$

This following nonlinear system is, apart from the "primed" notation, the same as that for the unrotated ("unprimed") case and may be solved in a similar manner.

$$F_1(\xi_1', \xi_2') = p_1' A_{11} - V_{qP}(\xi_1', \xi_2')\xi_1'(1 - \xi_2'^2)^{1/2} = 0 \tag{21}$$

$$F_2(\xi_1', \xi_2') = p_2' A_{22} - V_{qP}(\xi_1', \xi_2')\xi_1'\xi_2' = 0 \tag{22}$$

What is required to solve this system using numerical methods is initial estimates for ξ_1' and ξ_2'. This, as previously, may be done by assuming the ellipsoidal case and solving for $(\xi_1')_{ellip.}$ and $(\xi_2')_{ellip.}$ using the sequence of equations (10) – (12) in the "primed" system. As the system of nonlinear equations above is essentially the same as the unrotated problem, the solution in this case follows.

Once (ξ_1', ξ_2') has been determined, the three components of the slowness vector in the primed system may be obtained. Given that the vector $\mathbf{p}'$ is known, $\mathbf{p}$ may be obtained using the inverse of the rotational transforms given by equations (19) and (20).

It should be noted that the orientation of the local vertical axis in both slowness space and ray space must be taken in the proper sense as a reflected and transmitted ray and slowness vertical component of each individual vector have different signs. It is often better to do away with the signs and measure the angles as acute with respect to either the positive or negative direction of the vertical axis and one of the horizontal axes in model coordinates.

RAY TRACING METHOD

Assume a medium composed of N plane parallel homogeneous anisotropic layers where both the source and receiver are located at the surface. The source is situated at the origin of a Cartesian system aligned with the plane layered medium, with the vertical axis, x_3, chosen to be positive downwards. A description of unconverted P ray propagation is given by the two coupled nonlinear equations in terms of ξ_1 and ξ_2 as

$$F_1(\xi_1,\xi_2) = r\left(1-(\xi_2^2)_{surf}^{\uparrow}\right)^{1/2} - \sum_{j=1}^{M} \frac{n_j^{\uparrow} h_j\,(\xi_1)_j^{\uparrow}\left(1-(\xi_2^2)_j^{\uparrow}\right)}{\left(1-(\xi_1^2)_j^{\uparrow}\right)^{1/2}} - \sum_{j=1}^{M} \frac{n_j^{\downarrow} h_j\,(\xi_1)_j^{\downarrow}\left(1-(\xi_2^2)_j^{\downarrow}\right)^{1/2}}{\left(1-(\xi_1^2)_j^{\downarrow}\right)^{1/2}} = 0 \tag{23}$$

$$F_2(\xi_1,\xi_2) = r(\xi_2)_{surf}^{\uparrow} - \sum_{j=1}^{M} \frac{n_j^{\uparrow} h_j\,(\xi_1)_j^{\uparrow}(\xi_2)_j^{\uparrow}}{\left(1-(\xi_1^2)_j^{\uparrow}\right)^{1/2}} - \sum_{j=1}^{M} \frac{n_j^{\downarrow} h_j\,(\xi_1)_j^{\downarrow}(\xi_2)_j^{\downarrow}}{\left(1-(\xi_1^2)_j^{\downarrow}\right)^{1/2}} = 0 \tag{24}$$

where $M\,(M \le N)$ is the deepest layer traversed by the ray. The integers n_j are the number of P ray segments in the j^{th} layer, with the superscripts "$\uparrow$" and "$\downarrow$" indicating whether the ray segment(s) are upward or downward propagating. As upward and downward propagating ray segments within a layer generally do not make the same angle with vertical or horizontal axes they are considered separately. The thickness of the j^{th} layer is given as h_j. What should be noted in equations (23) – (24) is that there are no "primes" on the $\xi_i\ (i=1,2)$. However, they are implied by the use of the "$\uparrow$" and "$\downarrow$" notation. The angle $\Phi_{surf}\left[(\xi_2)_{surf}^{\uparrow},\left(1-(\xi_2^2)_{surf}^{\uparrow}\right)^{1/2}\right]$ is the angle measured in a positive manner from the x_1 axis from the source to the receiver on the surface at $\mathbf{r}=\left(x_1^R, x_2^R\right)$, with $r=|\mathbf{r}|$.

Embedded in these two equations are the two coupled nonlinear equations for the analogue of Snell's Law discussed in earlier sections. Thus, this "two-point" ray tracing scheme is quite computationally intensive. The method or methods used in the solution of these numerical problems include those of Newton among others. The technique chosen is usually based on previous experience. It should, however, be capable of dealing with more complex situations, a detail to be kept in mind when devising the ray tracing code. In the next section examples will be presented.

Finally, the travel time along the ray may also be written in terms of the sines $\left((\xi_i)_j^{\uparrow\downarrow}\ (i=1,2), j=1,M \right)$ of the angles $\left(\Theta_j^{\uparrow\downarrow}, \Phi_j^{\uparrow\downarrow}, j=1,M \right)$ as

$$\tau\left((\xi_1)_j^{\uparrow\downarrow}, (\xi_2)_j^{\uparrow\downarrow} \right) = \sum_{j=1}^{M} \frac{n_j^{\uparrow} h_j}{\left(1 - \left[(\xi_1)_j^{(qP\uparrow)} \right]^2 \right)^{1/2} V\left((\xi_1)_j^{\uparrow}, (\xi_2)_j^{\uparrow} \right)} + \sum_{j=1}^{M} \frac{n_j^{\downarrow} h_j}{\left(1 - \left[(\xi_1)_j^{(qP\downarrow)} \right]^2 \right)^{1/2} V\left((\xi_1)_j^{\downarrow}, (\xi_2)_j^{\downarrow} \right)} \tag{25}$$

COMMENTS ON THE SOLUTION AND NUMERICAL RESULTS

The problem discussed here may be approached in a number of ways, depending on what the ray tracing code is being used for. It will be assumed that ray tracing for the purposes of Born-Kirchhoff migration or some related task is being considered.

The $3D$ ray tracing algorithm for qP rays in an orthorhombic anisotropic medium is reasonably complex. However, for it to be used in a productive manner, a reasonably sophisticated model building program is an essential addition. In $2D$, a grid with all of the anisotropic parameters, together with other required quantities, could be specified at each node of this grid. At present, a realistic $3D$ grid is not an option unless top level computer hardware is available. Few persons would have access to this current generation of "super computers". A reasonably fast lap top with $2-4$ gigabytes of memory should be adequate if the model is specified as a "blocky type structure", composed of slabs or wedges or other similar types of constituents, separated by interfaces. For some applications these interfaces may be smoothed, but the ray tracing program should be written to handle jump discontinuities in the anisotropic parameters defining each of the blocks in the model. Although these types of model building programs exist, obtaining a *freeware* type is not as easy as might be expected. Those which are in the public domain usually require more than a minimum of effort to be of use for an existing ray tracing program. It is highly advisable to choose and become acquainted with one of these and then proceed to write the ray tracing code. It should not have to be mentioned that the possibility of errors in these types of codes is high. Additionally, the known $3D$ model building software requires at least one, usually two, licensed numerical software packages, which introduces more problems. If replacements in the public domain are used to replace the equivalent of the licensed packages, integrating these into the model building code can produce another source of error as well as an increased development time. This leaves the preferable option of spending the time to write one's own (not an insignificant undertaking).

The ray tracing described here is computationally fast. Most of the time required is iterating to a solution, which increases if bad initial guesses are used. Consequently, shooting the initial ray of a sequence at near vertical incidence is done followed by a line of receivers increasing in distance at polar angle increments from the origin at a near constant azimuth. This is *not* "two-point" ray tracing so its speed is significantly enhanced as one level of solving two coupled nonlinear equations is removed. This is applicable to both offset or shooting to some reference depth. If the coverage is not sufficient at increasing distances from the source, interspersed lines may be shot between the lines already computed, using nearest neighbours as initial guesses. A recommended alternative is to tessellate the receiver surface using a program such as Shewchuk's (1996), *Triangle*. The decision to shoot a ray to a point near the centre of a given triangle is based on the area of the triangle. If it is above some preset tolerance, a ray is shot using the weighted average of the "attributes" of the three bounding rays as input. This procedure may be repeated a number of times until a sufficient coverage is attained.

In the first example considered here, a plane layered three layer model is used. The surface layer is chosen to be isotropic. Layer two has an orthorhombic type structure similar to olivine, with the third layer being an orthorhombic sandstone type medium. In layer two the symmetry axes are not aligned with "model coordinates" but rather rotated about the vertical axis by an angle "$\Phi_{rot.}$=30°", measured from the positive x_1 axis. In the third layer, a rotation of the ray (slowness) surface(s) of "$\Theta_{rot.}$=15°" about the x_2 axis is introduced.

Two point ray tracing is undertaken that varies somewhat from that described in the text. Rather than solve for $\Theta_j^{\uparrow\downarrow}$ and $\Phi_j^{\uparrow\downarrow}$, ($j = 1,2,3$), the parameters used are $\Theta_j^{\uparrow\downarrow}$ ($j = 1,2,3$) and offset. This was done so that the $\Phi_j^{\uparrow\downarrow}$ ($j = 1,2,3$) could "float" providing an indication of how the azimuthal rotation in the second layer affects the arrival at the surface receivers, which are a fixed distance apart. Three views of this are given in Figure (1). The scale varies along all axes in the figure to produce plots that are easily viewed. It should be noted that proper ray $3D$ plotting software can significantly enhance the presentation of results. Three views of the reflected qP primary ray are shown in Figure (2). A similar plot for a number of azimuths is given in Figure (3). The anisotropic parameters used are given in Tables (1) and (2).

The second example, Figure (4), uses the same model as in the previous case and the modified "two-point" ray tracing method is also used. Here a number of azimuths are chosen and the rays are computed from a source located at the surface down to some reference depth. If the coverage is *sufficient* the computed data may be used in migration programs. Usually, the coverage must be enhanced. This may be done using the program *Triangle* discussed above, or by shooting another set of azimuthal lines which bisect the original lines shot, and start at some distance from the source using nearest neighbours from adjacent lines for initial data. A faster way of doing this is to bisect the original lines and shoot from the origin, and retain only those arrivals that are greater than a specified distance from the source.

CONCLUSIONS

The basic formulae and solution method for tracing quasi-compressional (qP) rays in a plane layered orthorhombic anisotropic medium where the axes of anisotropy (polar and azimuthal) in any of the layers may be rotated with respect to the general Cartesian coordinate system have been presented. A linearized form of the qP ray velocity is used for this. Proceeding in this manner removes one "level" of nonlinear equations. The theory for this alternate case is presented in another report in this issue of the CREWES Annual Report. Included in this discussion is a presentation of an analogue of Snell's Law for reflected and transmitted rays at an interface separating two of the layers. Thus, for "two point" the problem consists of two nonlinear equations (Snell's Law) embedded in two nonlinear (qP) ray tracing equations. Writing the shear ray velocities as is done for the qP group velocity does not present itself in a manner as that for the qP case. For that matter, group velocity expressions for the two possible shear wave modes is also a fairly complex undertaking.

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REFERENCES

Backus, G.E., 1965, Possible forms of seismic anisotropy of the uppermost mantle under oceans, Journal of Geophysical Research, 70, 3429-3439.

Courant, R. and Hilbert, D., 1962, Methods of Mathematical Physics, Vol. II: Partial Differential Equations, Interscience Publications, New York.

Červený, V., 2001, Seismic Ray Theory, Cambridge University Press, Cambridge.

Daley, P.F. and Krebes, E.S., 2006, Quasi-compressional group velocity approximations in a weakly anisotropic orthorhombic medium, Journal of Seismic Exploration, 14, 319-334.

Every, A.G, 1980, General closed-form expressions for acoustic waves in elastically anisotropic solids, Physics Review B, 22 1746-1760

Every, A.G. and Sachse, W., 1992, Sensitivity of inversion algorithms for recovering elastic constants of anisotropic solids from longitudinal wavespeed data, Ultrasonics, 30, 43-48.

Farra, V. and Pšenčík, I., 2008, First-order ray computations of coupled S waves in inhomogeneous weakly anisotropic media, Geophysical Journal International, 173, 979-989.

Gassmann, F., 1964. Introduction to seismic travel time methods in anisotropic media: Pure and Applied Geophysics, 58, 63-112.

Jech, J. and Pšenčík, I., 1989, First order perturbation method for anisotropic media, Geophysical Journal International, 99, 369-376.

Pšenčík, I. and Gajewski, D., 1998, Polarization, phase velocity and NMO velocity of qP waves in arbitrary weakly anisotropic media, Geophysics, 63, 1754-1766.

Pšenčík, I. and Farra, V., 2005, First-order ray tracing for qP waves in inhomogeneous weakly anisotropic media, Geophysics, 70, D65-D75.

Schoenberg, M. and Helbig, K., 1996, Orthorhombic media: Modeling elastic wave behavior in a vertically fractured earth: Geophysics, 62, 1954-1974.

Song, L-P. and Every, A.G., 2000, Approximate formulae for acoustic wave group slowness in weakly orthorhombic media, Journal of Physics D: Applied Physics, 33, L81-L85.

Song, L-P. Every, A. G. and Wright, C., 2001, Linearized approximations for phase velocities of elastic waves in weakly anisotropic media, Journal of Physics D: Applied Physics, 34, 2052-2062.

APPENDIX: PARTIAL DERIVATIVES OF THE GROUP VELOCITY

The partial derivatives of the functions $F_1(\xi_1,\xi_2)$ and $F_2(\xi_1,\xi_2)$ with respect to ξ_1 and ξ_2 are given by

$$\frac{\partial F_1(\xi_1,\xi_2)}{\partial \xi_1} = -\left[\left(1-\xi_2^2\right)^{1/2} V_{qP}(\xi_1,\xi_2) + \xi_1\left(1-\xi_2^2\right)^{1/2}\left(\partial V_{qP}(\xi_1,\xi_2)/\partial \xi_1\right)\right] \quad (A.1)$$

$$\frac{\partial F_1(\xi_1,\xi_2)}{\partial \xi_2} = \left[\left(\xi_1\xi_2\big/\left(1-\xi_2^2\right)^{1/2}\right)V_{qP}(\xi_1,\xi_2) - \xi_1\left(1-\xi_2^2\right)^{1/2}\left(\partial V_{qP}(\xi_1,\xi_2)/\partial \xi_2\right)\right] \quad (A.2)$$

$$\frac{\partial F_2(\xi_1,\xi_2)}{\partial \xi_1} = -\left[\xi_2 V_{qP}(\xi_1,\xi_2) + \xi_1\xi_2\left(\partial V_{qP}(\xi_1,\xi_2)/\partial \xi_1\right)\right] \quad (A.3)$$

$$\frac{\partial F_2(\xi_1,\xi_2)}{\partial \xi_2} = -\left[\xi_1 V_{qP}(\xi_1,\xi_2) + \xi_1\xi_2\left(\partial V_{qP}(\xi_1,\xi_2)/\partial \xi_2\right)\right]. \quad (A.4)$$

The partial derivatives of the group velocity with respect to ξ_1 and ξ_2 are obtained using the following sequence of steps:

$$\frac{\partial}{\partial \xi_i}\left[\frac{1}{V_{qP}^2(\xi_1,\xi_2)}\right] = -\frac{2V_{qP}(\xi_1,\xi_2)\,\partial V_{qP}(\xi_1,\xi_2)/\partial x_i}{V_{qP}^4(\xi_1,\xi_2)} = -\frac{2\,\partial V_{qP}(\xi_1,\xi_2)/\partial \xi_i}{V_{qP}^3(\xi_1,\xi_2)} \quad (A.5)$$

$$\frac{\partial V_{qP}(\xi_1,\xi_2)}{\partial \xi_i} = -\frac{V_{qP}^3(\xi_1,\xi_2)}{2}\frac{\partial}{\partial \xi_i}\left[\frac{1}{V_{qP}^2(\xi_1,\xi_2)}\right] \quad (A.6)$$

$$\frac{\partial}{\partial \xi_1}\left[\frac{1}{V_{qP}^2(\xi_1,\xi_2)}\right] = 2\xi_1\left\{\frac{\left(1-\xi_2^2\right)}{A_{11}} + \frac{\xi_2^2}{A_{22}} - \frac{1}{A_{33}} - \frac{2E_{12}\xi_1^2\xi_2^2\left(1-\xi_2^2\right)}{A_{11}A_{22}} - \right.$$
$$\left. \frac{E_{13}\left(1-\xi_2^2\right)\left(1-2\xi_1^2\right)}{A_{11}A_{33}} - \frac{E_{23}\xi_2^2\left(1-2\xi_1^2\right)}{A_{22}A_{33}}\right\} \quad (A.7)$$

$$\frac{\partial}{\partial \xi_2}\left[\frac{1}{V_{qP}^2(\xi_1,\xi_2)}\right] = 2\xi_2\left\{-\frac{\xi_1^2}{A_{11}} + \frac{\xi_1^2}{A_{22}} - \right.$$
$$\left. \frac{E_{12}\xi_1^4\left(1-\xi_2^2\right)}{A_{11}A_{22}} + \frac{E_{12}\xi_1^4\xi_2^2}{A_{11}A_{22}} + \frac{E_{13}\xi_1^2\left(1-\xi_1^2\right)}{A_{11}A_{33}} - \frac{E_{23}\xi_1^2\left(1-\xi_1^2\right)}{A_{22}A_{33}}\right\} \quad (A.8)$$

A_{11}	A_{22}	A_{33}	A_{44}	A_{55}	A_{66}	A_{12}	A_{13}	A_{23}
7.00	*7.00*	*7.00*	*2.50*	*2.50*	*2.50*	*2.00*	*2.00*	*2.00*
9.900	*6.023*	*7.093*	*1.964*	*2.448*	*2.438*	*1.926*	*2.074*	*2.225*
10.00	*9.84*	*5.94*	*2.00*	*1.60*	*2.18*	*3.60*	*2.25*	*2.40*

Table 1. The anisotropic parameters in the three layers in Voigt notation. The A_{ij} have the dimensions of velocity squared $\left(km^2/s^2 \right)$.

E12	E13	E23
0.0	*0.0*	*0.0*
10.89	*10.41*	*11.08*
-3.92	*-5.04*	*-2.98*

Table 2. The anellipsoidal parameters in the three layers. The E_{ij} have the dimensions of velocity squared $\left(km^2/s^2 \right)$ and are defined in terms of the A_{ij} using equations (3) – (5) in the text.

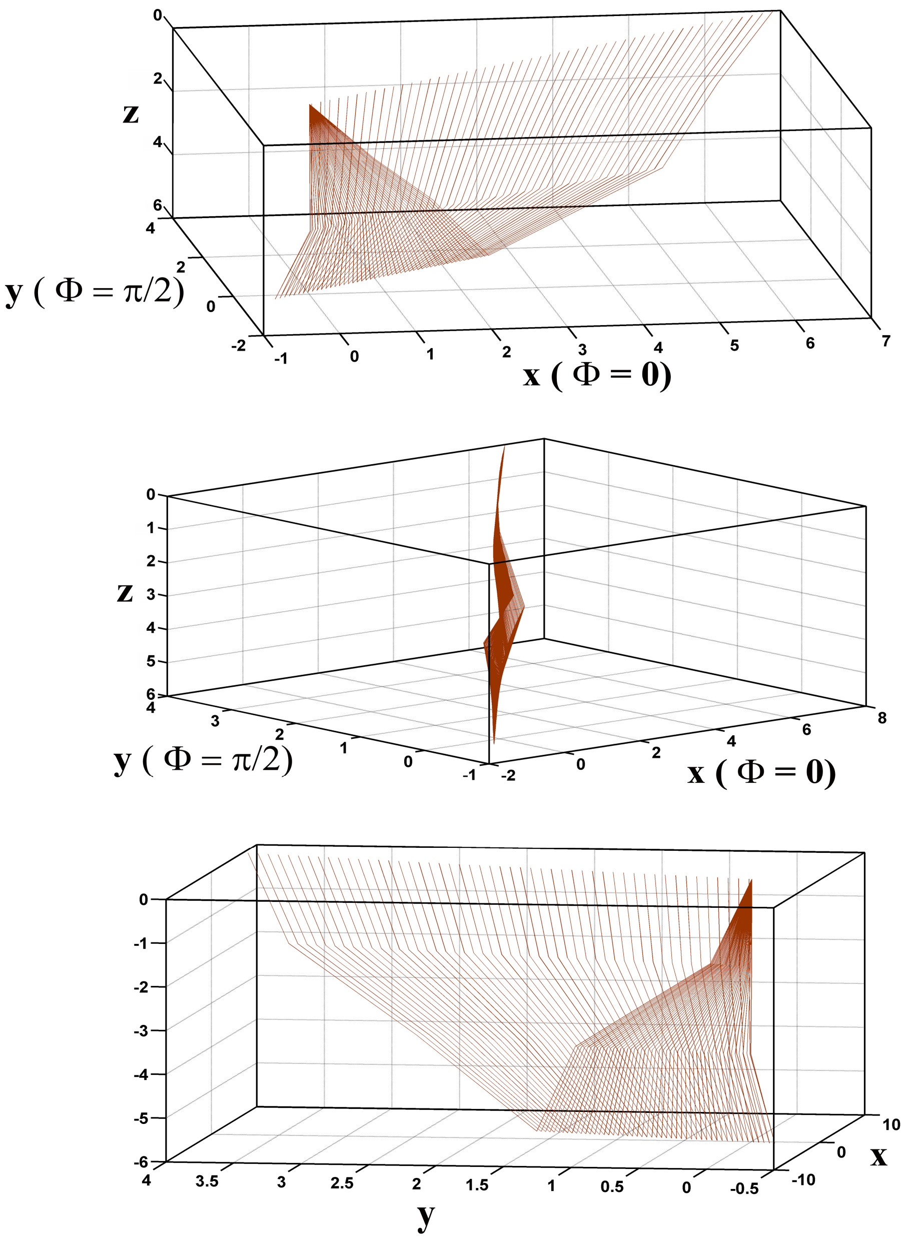

Fig. 1. Three layer model. The first layer is isotropic, the second layer is orthorhombic with a rotation in the $x-y$ plane about the $z-\text{axis}$ and the third layer is also orthorhombic with a rotation in the $x-z$ plane. The line shot is very close to two point ray tracing, requiring some minor tweaking. The analogue of Snell's Law at an interface is a system of two coupled nonlinear equation in two unknown angles (sines of the transmitted or reflected angles). The x and y axes in the three panels have different scaling to enhance the effects of the rotations of the orthorhombic ray surfaces. The line is shot at an angle of 70 degrees with respect to the $x-\text{axis}$. The modified two point ray tracing algorithm consists of two coupled nonlinear equations in two unknown angles per layer in which is embedded the other nonlinear equation set for Snell's Law. Different scaling parameters are used on three panels to improve visibility.

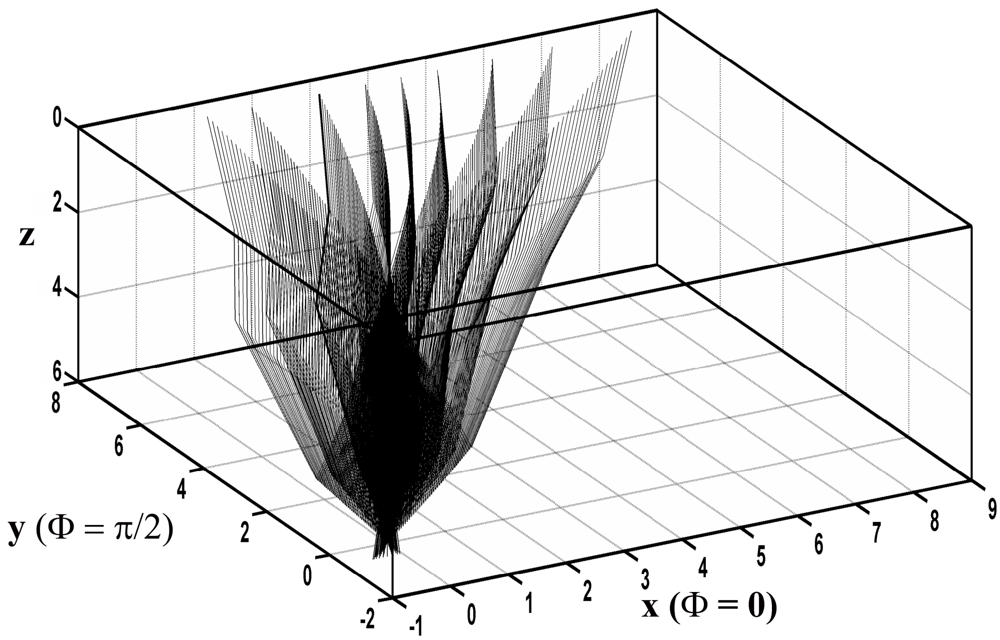

Fig. 2. Three layer model. The model used is the same as that in Figure (1), except that a number of quasi-azimuthal lines are displayed. The modified ray tracing procedure using the polar angle Θ and a fixed distance between adjacent receivers are again used as the dependent variables.

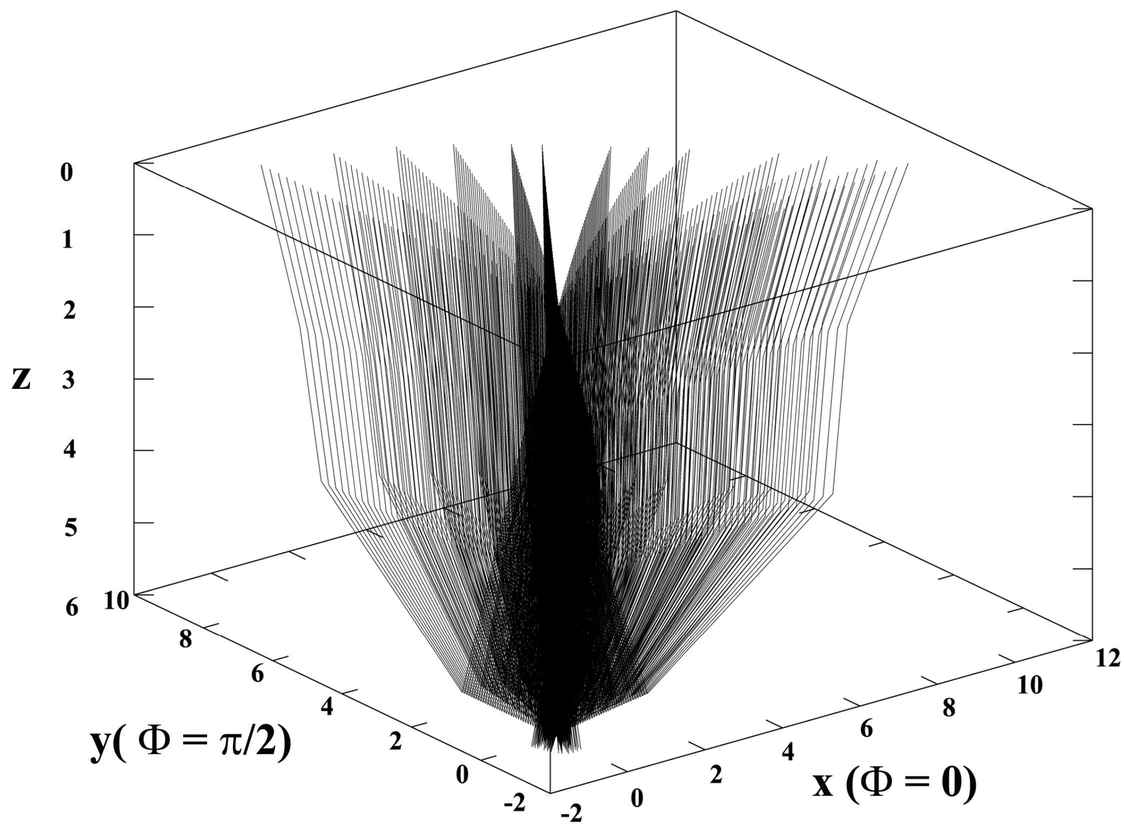

Fig. 3. Three layer model. The model used is the same as that in Figure (2), except that the polar angle Θ and azimuthal angle Φ are used as the dependent variables. The surface receivers for some azimuthal line are now in a straight line at the surface.

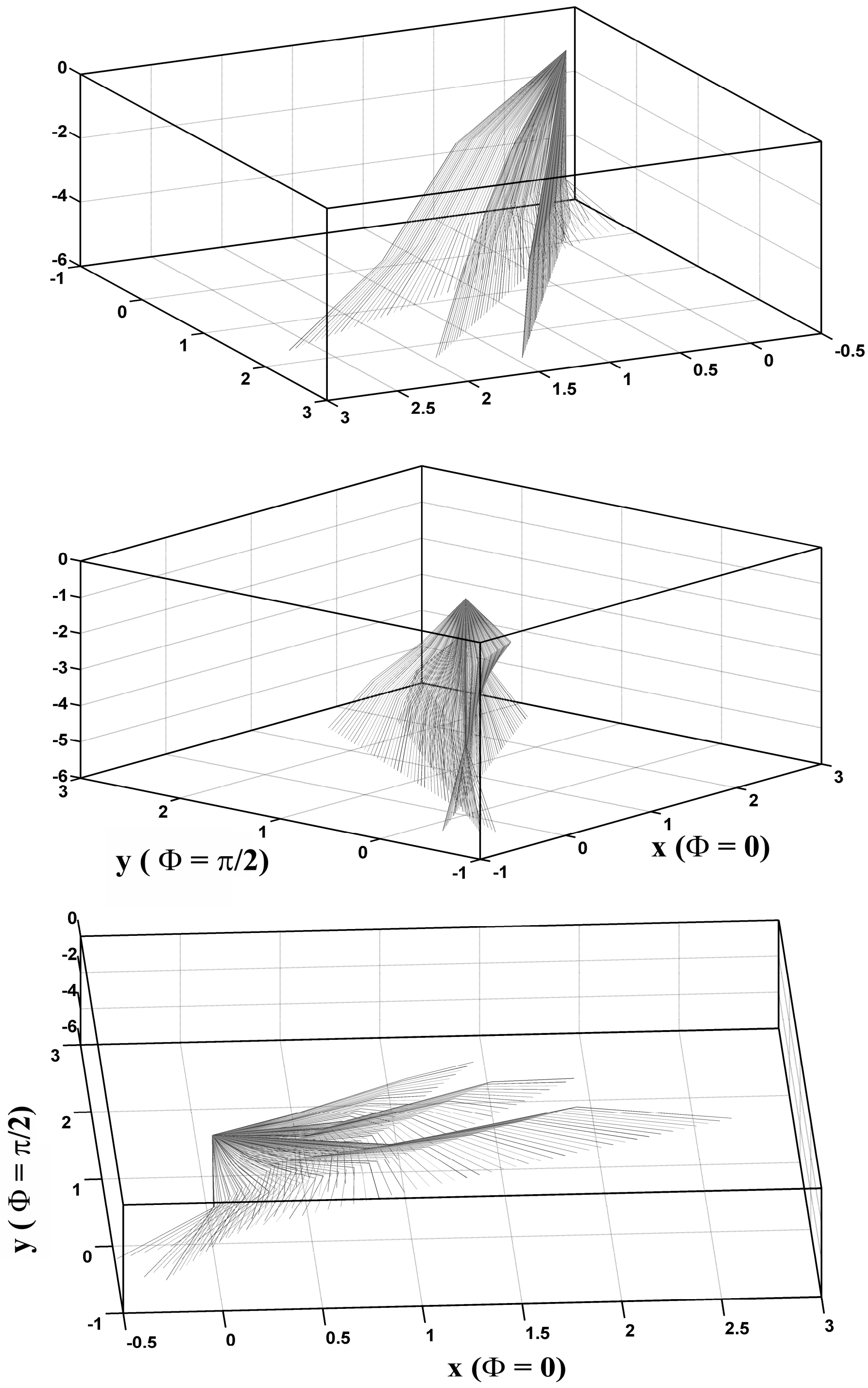

Fig. 4. Three layer model. In this case what is wanted is time to some reference depth. A sampling of three azimuths are shown. The anisotropic parameters describing the layers are given in Tables (1) and (2).