P-S$_V$ wave propagation in a radially symmetric vertically inhomogeneous TI medium: Finite difference hybrid method

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ABSTRACT

The hybrid finite difference – finite integral transform method is considered for the coupled $P-S_V$ wave propagation problem in a radially symmetric vertically inhomogeneous (plane layered) transversely isotropic medium. Apart from the development of the equations of motion, a number of numerical considerations are addressed. As in most problems where numerical methods are employed in the solution, there are several areas that are given special attention to indicate how to improve run times and accuracy.

INTRODUCTION

This method is most often referred to as the pseudo-spectral method, but due to the extensive work done in this area by B.G. Mikhailenko and A.S. Alekseev it is sometimes referred to, in seismic applications, as the Alekseev-Mikhailenko Method (AMM), (Alekseev and Mikhailenko, 1980). It falls within the genetic class of pseudo-spectral methods (Gazdag, 1973, 1981, Kosloff and Baysal, 1982) but is possibly more formal and rigorous in its development. However, much of their work is relatively physically inaccessible and a considerable number of the more significant contributions are in Russian.

One numerical advantage of applying finite integral transforms is that the resultant finite difference problem is in one spatial variable and time and there are no cross derivative terms. These are differentials of the form $\partial/\partial x_i\left[c\left(x_1,x_2,x_3\right)\partial u_k/\partial x_j\right]$ $i,j,k=1,2,3:$ $i\neq j$. Several approaches for dealing with these in a finite difference context may be found in Zahradník et al. (1993).

THEORETICAL DEVELOPMENT

General Theory

Consider the problem of coupled $P-S_V$ wave propagation in a radially symmetric (no lateral inhomogeneities), vertically inhomogeneous transversely isotropic half space.

The equations of motion are defined by the elastodynamic equations (Martynov and Mikhailenko, 1984 or Mikhailenko, 1985)

$$\rho\frac{\partial^2 U}{\partial t^2}=c_{11}\left[\frac{\partial^2 U}{\partial r^2}+\frac{1}{r}\frac{\partial U}{\partial r}-\frac{U}{r^2}\right]+c_{13}\frac{\partial^2 V}{\partial r\partial z}+\frac{\partial}{\partial z}\left[c_{55}\left(\frac{\partial U}{\partial z}+\frac{\partial V}{\partial r}\right)\right]+\rho F_r \tag{1}$$

$$\rho\frac{\partial^2 V}{\partial t^2}=c_{55}\left[\frac{\partial}{\partial r}\left(\frac{\partial U}{\partial z}+\frac{\partial V}{\partial r}\right)+\frac{1}{r}\left(\frac{\partial U}{\partial z}+\frac{\partial V}{\partial r}\right)\right]+\frac{\partial}{\partial z}\left[c_{13}\left(\frac{\partial U}{\partial r}+\frac{U}{r}\right)+c_{33}\frac{\partial V}{\partial z}\right]+\rho F_z \tag{2}$$

where the particle displacement vector $\mathbf{u}$ is of the form

$$\mathbf{u} \equiv \mathbf{u}(r,z,t) = \left(U(r,z,t), V(r,z,t)\right). \tag{2}$$

Here $U(r,z,t)$ and $V(r,z,t)$ are the radial (horizontal) and vertical components of vector particle displacement, the azimuthal component of displacement being zero for the coupled $P - S_V$ problem. The coordinates r and z are the radial and vertical coordinates in a cylindrical coordinate system, respectively, t is time In Voigt notation, the c_{ij} are the stiffness parameters of the medium and ρ is the density, all of which may be dependent on the vertical (z) coordinate. The density normalized anisotropic parameters, $a_{ij} = c_{ij}/\rho$, having dimensions of velocity squared, may also be used at some points within this report.

The problem is solved subject to the initial conditions

$$\mathbf{u}\big|_{t=0} = \frac{\partial \mathbf{u}}{\partial t}\bigg|_{t=0} = 0 \tag{3}$$

and the free surface boundary conditions that are required to be satisfied are

$$\sigma_{zz}\big|_{z=0} = 0, \qquad \text{and} \qquad \sigma_{rz}\big|_{z=0} = 0. \tag{4}$$

That is, the normal stress and shear stress are zero at the free surface. These will be defined shortly for a transversely isotropic medium.

Two typical types of point sources, $\mathbf{F}(r,z,t)$, used in seismic applications are (Mikhailenko, 1980):

1. Vertical point force :

$$\mathbf{F}(x,y,z,t) = \delta(x-x_s)\delta(y-y_s)\delta(z-z_s) f(t)\mathbf{n_z}. \tag{5}$$

   where $\mathbf{n_z}$ is a unit vector in the *z* (vertical downwards) direction.

2. Explosive point source of *P* waves:

$$\mathbf{F}(x,y,z,t) = \nabla\left[\delta(x-x_s)\delta(y-y_s)\delta(z-z_s)\right] f(t). \tag{6}$$

In the above, $\delta(\xi)$ is the Dirac delta function and $f(t)$ is some band limited source wavelet, about which more will be said later. In what follows, an explosive point source of *P* waves is assumed. The Green's function solution for this problem would require that, $f(t) = \delta(t - t_0)$ such that $(0 \le t_0 < t_{\max})$ for some finite time $t_{\max}$.

In terms of $U(r,z,t)$, $V(r,z,t)$ and the anisotropic stiffness coefficients, c_{ij}, the expressions for the normal and shear stresses at the free surface are given by

$$\sigma_{zz}\big|_{z=0} = \left[c_{13}\left(\frac{\partial U}{\partial r} - \frac{V}{r} \right) + c_{55}\frac{\partial V}{\partial z} \right] = 0 \tag{7}$$

$$\sigma_{rz}\big|_{z=0} = c_{55}\left(\frac{\partial U}{\partial z} - \frac{\partial V}{\partial r} \right) = 0 \tag{8}$$

Introducing the finite Hankel integral transforms and the vector designation $\mathbf{G}\left(\tilde{k}_i, k_i, z, t\right) = \left(S\left(\tilde{k}_i, z, t\right), R\left(k_i, z, t\right)\right)$ yields

$$S\left(k_i, z, t\right) = \int_0^a U\left(r, z, t\right) J_1\left(k_i r\right) r\, dr \tag{9}$$

$$R\left(\tilde{k}_i, z, t\right) = \int_0^a V\left(r, z, t\right) J_0\left(\tilde{k}_i r\right) r\, dr \tag{10}$$

where the k_i and $\tilde{k}_i$ are the roots of the transcendental equations

$$J_0\left(\tilde{k}_i r\right) = 0 \tag{11}$$

and

$$J_1\left(k_i r\right) = 0\,, \tag{12}$$

respectively. Using the two formulations of the Hankel transforms discussed in Appendix A, it may be shown that both of the inverse series summations may be accomplished using only the roots of one of the Bessel function transcendental equation, $J_1\left(k_i r\right) = 0$, so that the inverse transforms are defined by

$$U\left(r, z, t\right) = \frac{2}{a^2}\sum_{i=1}^{\infty} \frac{S\left(k_i, z, t\right) J_1\left(k_i r\right)}{\left[J_0\left(k_i a\right)\right]^2} \tag{13}$$

$$V\left(r, z, t\right) = \frac{2}{a^2}\sum_{i=1}^{\infty} \frac{R\left(k_i, z, t\right) J_0\left(k_i r\right)}{\left[J_0\left(k_i a\right)\right]^2} \tag{14}$$

Thus both inverse series summations may be taken over the roots of one rather than two transcendental equations and as a consequence, $\mathbf{G}\left(k_i, z, t\right) = \left(S\left(k_i, z, t\right), R\left(k_i, z, t\right)\right)$. The matter of what, numerically, constitutes an infinite number of terms in the inverse series summations is addressed in Appendix C. It is shown there that an earlier assumption that the source wavelet be band limited is significant in this determination. As the only spatial direction in which a finite difference is used is the z direction the most economical manner to introduce a damping conditions at the lower z boundary, i.e., $\gamma\left(z\right)\partial R/\partial t$ and $\gamma\left(z\right)\partial S/\partial t$. A safe estimate for the length of this damping region is of the order of 1 wavelength *(WL)* but $2WL$ are commonly used (B.G. Mikhailenko, 1980).

Applying the appropriate Hankel transforms to equations (1) and (2) results in

$$\rho\frac{\partial^2 S}{\partial t^2}=\frac{\partial}{\partial z}\left(c_{55}\frac{\partial S}{\partial z}-k_i c_{55}R\right)-k_i c_{13}\frac{\partial R}{\partial z}-k_i^2 c_{11}S+\rho\hat{F}_r \tag{15}$$

$$\rho\frac{\partial^2 R}{\partial t^2}=\frac{\partial}{\partial z}\left(c_{33}\frac{\partial R}{\partial z}+k_i c_{13}S\right)+k_i c_{55}\frac{\partial S}{\partial z}-k_i^2 c_{55}R+\rho\hat{F}_r \tag{16}$$

while the transforms of the shear and normal stresses at the free surface, which is assumed to be planar, have the form

$$\left[c_{55}\left(\frac{\partial S}{\partial z}-k_i R\right)\right]_{z=0}=0 \tag{17}$$

$$\left[c_{33}\frac{\partial R}{\partial z}+k_i c_{13}S\right]_{z=0}=0 \tag{18}$$

The transformed initial conditions at $t=0$ are

$$S(z,k_i,t)\big|_{t=0}=0\,;\quad R(z,k_i,t)\big|_{t=0}=0\,;\quad \frac{\partial S}{\partial t}\bigg|_{t=0}=\frac{\partial R}{\partial t}\bigg|_{t=0}=0. \tag{19}$$

Here k_i are the roots of the transcendental equation $J_1(k_i a)=0$ which requires additional boundary conditions at $r=a$ (pseudo boundary such that)

$$U\big|_{r=a}=\frac{\partial V}{\partial r}\bigg|_{r=a}=0 \tag{20}$$

The pseudo boundary is placed at some distance $r=a$ so that no spurious reflections from this boundary are present in the synthetic traces. Care is required in choosing this distance, as the number of terms in the inverse series summation depends on it in a linear fashion. More on this may be found in Appendix C.

Development of a finite difference analogue for terms of the form

$$\frac{\partial}{\partial z}\left[\zeta(z)\frac{\partial\psi(k_i,z,t)}{\partial z}\right] \tag{21}$$

is given in Appendix B, as is a staggered grid analogue for

$$\frac{\partial}{\partial z}\left[\zeta(z)\psi(k_i,z,t)\right] \tag{22}$$

If it is assumed that the anisotropic parameters (stiffness coefficients) are spatially independent of the Hankel transformed equations and take on the simplified forms given below. For convenience, it is assumed that the first two grid points in z, at the free surface are of this form so that equations (15) and (16) may be written there as

$$\frac{\partial^2 S}{\partial t^2} = a_{55}\frac{\partial^2 S}{\partial z^2} - k_i\left(a_{13}+a_{55}\right)\frac{\partial R}{\partial z} - k_i^2 a_{11} S \tag{23}$$

$$\frac{\partial^2 R}{\partial t^2} = a_{33}\frac{\partial^2 R}{\partial z^2} + k_i\left(a_{13}+a_{55}\right)\frac{\partial S}{\partial z} - k_i^2 a_{55} R \tag{24}$$

The Hankel transformed shear and normal stresses required at the free surface as boundary conditions have been given in equations (17) and (18).

Free Surface Finite Difference Analogues

Introducing the transformed stress equations, (17) and (18), into equations (23) and (24) results in the finite difference analogues at the free surface having the form

$$\begin{aligned} S_0^{m+1} = {} & \left[2 - \frac{2a_{55}\delta^2}{h^2} + k_i^2\delta^2\left[\left(a_{13}+a_{55}\right)\left(\frac{a_{13}}{a_{33}}\right) - a_{11}\right]\right]S_0^m - \\ & S_0^{m-1} + \frac{2a_{55}\delta^2}{h^2}S_1^m - \frac{2k_i a_{55}\delta^2}{h}R_0^m \end{aligned} \tag{25}$$

$$\begin{aligned} R_0^{m+1} = {} & \left[2 - 2\frac{a_{33}\delta^2}{h^2} - k_i^2\delta^2 a_{13}\right]R_0^m \\ & + \frac{2a_{33}\delta^2}{h^2}R_1^m - \frac{2a_{33}\delta^2 k_i}{h}\frac{c_{13}}{c_{33}}S_0^m - R_0^{m-1} \end{aligned} \tag{26}$$

where $\delta - \text{time step}$, $h - \text{spatial step}$ and all quantities with the subscript "0" are to be evaluated at the surface $\left(z=0\right)$. The time point "m" corresponds to time along the synthetic trace of $t_m = m\delta t\left(m = 0,1,\ldots,M\right)$, where $M\delta t$ is the length in time of the trace and k_i has been previously defined.

General Point Finite Difference Analogues

Horizontal component:

$$S_n^{m+1} = \left[2 - \frac{\delta^2}{\rho_n h^2}\left((\bar{c}_{55})_{n+1/2} + (\bar{c}_{55})_{n-1/2} \right) - \frac{k_i^2 \delta^2 (\bar{c}_{11})_n}{\rho_n} \right] S_n^m +$$
$$\frac{\delta^2}{\rho_n h^2}\left[(\bar{c}_{55})_{n+1/2} S_{n+1}^m + (\bar{c}_{55})_{n-1/2} S_{n-1}^m \right] -$$
$$\frac{k_i \delta^2}{2h\rho_n}\left[(\bar{c}_{55})_{n+1/2} + (\bar{c}_{13})_{n+1/2} \right] R_{n+1}^m + \frac{k_i \delta^2}{2h\rho_n}\left[(\bar{c}_{55})_{n-1/2} + (\bar{c}_{13})_{n-1/2} \right] R_{n-1}^m - \tag{27}$$
$$\left. \left(1 - \frac{\gamma_n (\Delta t)}{2\rho_n} \right) S_n^{m-1} \right\} \Big/ \left(1 + \frac{\gamma_n (\Delta t)}{2\rho_n} \right) + \delta^2 \hat{F}_r$$

Vertical component:

$$R_n^{m+1} = \left[2 - \frac{\delta^2}{\rho_n h^2}\left[(\bar{c}_{33})_{n+1/2} + (\bar{c}_{33})_{n-1/2} \right] - \frac{k_i^2 \delta^2 (\bar{c}_{55})_n}{\rho_n} \right] R_n^m +$$
$$\frac{\delta^2}{\rho_n h^2}\left[(\bar{c}_{33})_{n+1/2} R_{n+1}^m + (\bar{c}_{33})_{n-1/2} R_{n-1}^m \right] +$$
$$\frac{k_i \delta^2}{2h\rho_n}\left[(\bar{c}_{13})_{n+1/2} + (\bar{c}_{55})_{n+1/2} \right] S_{n+1}^m - \tag{28}$$
$$\frac{k_i \delta^2}{2h\rho_n}\left[(\bar{c}_{13})_{n-1/2} + (\bar{c}_{55})_{n+1/2} \right] S_{n-1}^m -$$
$$\left. \left(1 - \frac{\gamma_n (\Delta t)}{2\rho_n} \right) R_n^{m-1} \right\} \Big/ \left(1 + \frac{\gamma_n (\Delta t)}{2\rho_n} \right) + \delta^2 \hat{F}_z$$

Alternate methods of introducing attenuating boundary conditions at fictitious boundaries which produce spurious reflections on the synthetic traces may be found in the well known works of Cerjan et al. (1985), Clayton and Enquist (1977) and Reynolds (1979).

An explosive point $qP-$wave source has

$$\hat{F}_r = \frac{1}{2\pi} \delta(z-d) f(t) \tag{29}$$

$$\hat{F}_z = \frac{1}{2\pi} \frac{d}{dz}\left[\delta(z-d) \right] f(t) \tag{30}$$

This matter will be dealt with later in this report in Appendix D.

Stability Condition

This formula is given without derivation as

$$\frac{(\Delta t)^2}{(\Delta z)^2}\left(V_{qP}^2+V_{qS_V}^2\right)+\frac{(\Delta t)^2 k_i^2}{4}\left(V_{qP}^2+V_{qS_V}^2\right)\le 1 \quad \approx \quad \frac{(\Delta t)^2}{(\Delta z)^2}\left(V_{qP}^2+V_{qS_V}^2\right)\le 1 \tag{31}$$

where $V_{qP}^2=\max\left(a_{11},a_{33}\right)$, that is, the maximum value that either a_{11} or a_{33} $\left(a_{ii}=c_{ii}/\rho\right)$ attains on the finite difference grid and $V_{qS_V}^2=\max\left(a_{55}\right)$, where the previous definitions hold. The value of k_i is the minimum non zero value in the series $\left(k_i, i=1,N\right)$, with N being the maximum number of roots used in the infinite series summation approximation. A more comprehensive derivation of this may be found in almost any text on finite difference methods. As there is a possibility that $\max\left(a_{55}\right)$ may not give the maximum value of the square of the shear wave velocity on the grid, and if problems occur, a safer estimate of Δt may be obtained by substituting $V_{qS_V}^2=\max\left(a_{11},a_{33}\right)$ in equation (31). If this produces acceptable results, Δt may be made larger by using some value for $V_{qS_V}^2$, obtained by numerical experimentation, between $\max\left(a_{55}\right)$ and $\max\left(a_{11},a_{33}\right)$.

SUMMARY AND CONCLUSIONS

The theory and development of finite difference analogues for coupled $qP-qS_V$ wave propagation in a plane parallel layered transversely isotropic model has been presented. The radial coordinate was removed using a finite Hankel transform prior to implementation of finite difference process. What results are a coupled system of finite difference equations in only depth and time. The infinite inverse series summation may be truncated if a band limited source wavelet is used.

The finite difference analogues given are accurate to second order in both time and space (depth). The analogues for a surface point as well as general points within the medium are given. Provisions for either a vertical or explosive point source of P – waves are included in the derivations. A number of points regarding this seismic modeling process, especially where some mathematical rigor is required are dealt with in a series of Appendices.

This is the second in a sequence of six (currently) related to modeling using hybrid methods in anisotropic medium. At present, a plane layered orthorhombic structure with no anisotropic parameter variations in the two lateral spatial directions, is being tested. Two finite Fourier transforms are employed to remove dependence on the lateral spatial Cartesian coordinates. The theory presently being developed employs these two finite Fourier transforms to reduce the finite difference problem from 3 spatial dimensions and time to one spatial dimension (depth) and time. In the most recent development, the anisotropic parameters are allowed to “slowly” vary with the two lateral coordinates in an orthorhombic medium.

Using the formulae presented here it should be possible write a hybrid finite difference – finite integral transform programs for a transversely isotropic medium for a variety of source – receiver configurations including AVO and VSP.

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APPENDIX A: FINITE HANKEL TRANSFORM

Although the two following finite Hankel transform methods may be found in the literature (Sneddon, 1972, for example), it was felt that for completeness they should be included here, at least in an abbreviated theorem formulation. The finite Hankel transform of the first kind is a direct application of the following theorem.

Theorem I: If $f(x)$ satisfies Dirichlet's conditions in the interval $(0,a)$ and if its Hankel transform in that range is defined to be

$$H_{\mu}^{(1)}\left[f(x)\right] \equiv f_J\left(\xi_j\right) = \int_0^a x\, f(x)\, J_{\mu}\left(\xi_j x\right) dx \qquad \text{(A.1)}$$

where ξ_j is a root of the transcendental equation

$$J_{\mu}\left(\xi_j a\right) = 0 \qquad \text{(A.2)}$$

then, at any point in the interval $(0,a)$ at which the function $f(x)$ is continuous ,

$$f(x) = \frac{2}{a^2}\sum_{j=1}^{\infty} f_J\left(\xi_j\right) \frac{J_{\mu}\left(\xi_j x\right)}{\left[J_{\mu+1}\left(\xi_j x\right)\right]^2} \qquad \text{(A.3)}$$

where the sum is taken over all the positive roots of equation (A.2).

The finite Hankel transform and inverse of the second kind used in the text are given as follows:

Theorem II: If $f(x)$ satisfies Dirichlet's conditions in the interval $(0,a)$ and if its Hankel transform in that range is defined to be

$$H_{\mu}^{(1)}\left[f(x)\right] \equiv f_J\left(\xi_j\right) = \int_0^a x\, f(x)\, J_{\mu}\left(\xi_j x\right) dx \qquad \text{(A.4)}$$

in which ξ_j is a root of the transcendental equation

$$\xi_j J'_{\mu}\left(\xi_j a\right) + h\, J_{\mu}\left(\xi_j a\right) = 0 \qquad \text{(A.5)}$$

then, at each point in the interval $(0,a)$ at which the function $f(x)$ is continuous,

$$f(x) = \frac{2}{a^2}\sum_{j=1}^{\infty} \frac{\xi_j^2 f_J\left(\xi_j\right)}{h^2 + \left(\xi_j^2 - \mu^2/a^2\right)} \frac{J_{\mu}\left(\xi_j x\right)}{\left[J_{\mu}\left(\xi_j x\right)\right]^2} \qquad \text{(A.6)}$$

where the sum is taken over all the positive roots of (A.5) and h is determined from a boundary operator **N** at $x = a$ defined as

$$\mathbf{N}[f] = \frac{df(a)}{dx} + hf(a) = 0 \qquad \text{(A.7)}$$

APPENDIX B: FINITE DIFFERENCE ANALOGUE

For determining the finite difference analogue in the case of an operation of the type

$$\frac{\partial}{\partial z}\left[\zeta(z)\frac{dB(z)}{\partial z}\right] \tag{B.1}$$

let

$$w(z) = \zeta(z)\frac{\partial B(z)}{\partial z} \tag{B.2}$$

$$w(z)\frac{\partial z}{\zeta(z)} = \partial B(z) \tag{B.3}$$

$$w_{k-1/2}\int_{z_{k-1}}^{z_k}\frac{dz}{\zeta(z)} = B_k - B_{k-1} \tag{B.4}$$

or

$$w_{k-1/2} = \left(B_k - B_{k-1}\right)\left[\int_{z_{k-1}}^{z_k}\frac{dz}{\zeta(z)}\right]^{-1} \tag{B.6}$$

$$w_{k+1/2}\int_{z_k}^{z_{k+1}}\frac{dz}{\zeta(z)} = B_{k+1} - B_k \tag{B.7}$$

or

$$w_{k+1/2} = \left(B_{k+1} - B_k\right)\left[\int_{z_k}^{z_{k+1}}\frac{dz}{\zeta(z)}\right]^{-1} \tag{B.8}$$

so that

$$\frac{\partial w(z)}{\partial z} = \frac{\partial}{\partial z}\left[\zeta(z)\frac{\partial B(z)}{\partial z}\right] \tag{B.9}$$

whose finite difference analogue is of the form

$$\frac{\partial w(z)}{\partial z} \approx \frac{\left(w_{k+1/2} - w_{k-1/2}\right)}{\Delta z} \tag{B.10}$$

which in terms of $\chi(z)$ and B_j

$$\frac{\partial}{\partial z}\left[\zeta(z)\frac{\partial B(z)}{\partial z}\right]\approx \frac{(B_{k+1}-B_k)}{(\Delta z)^2}\left[\frac{1}{\Delta z}\int_{z_k}^{z_{k+1}}\frac{dz}{\zeta(z)}\right]^{-1}-\frac{(B_k-B_{k-1})}{(\Delta z)^2}\left[\frac{1}{\Delta z}\int_{z_{k-1}}^{z_k}\frac{dz}{\zeta(z)}\right]^{-1} \tag{B.11}$$

from which it follows that

$$\frac{\partial}{\partial z}\left[\zeta(z)\frac{\partial B(z)}{\partial z}\right]\approx\frac{\chi_{k+1}B_{k+1}-(\chi_{k+1}+\chi_k)B_k+\chi_k B_{k-1}}{(\Delta z)^2} \tag{B.12}$$

with the staggered grid harmonic average of $\zeta\left(z_{k+1/2}\right)$ is given by

$$\chi_k=\left[\frac{1}{\Delta z}\int_{z_{k-1}}^{z_k}\frac{dz}{\zeta(z)}\right]^{-1}=\frac{2\zeta_k\zeta_{k-1}}{\zeta_k+\zeta_{k-1}} \tag{B.13}$$

The second finite difference analogue required in this report is

$$\begin{aligned} s(z)&=\zeta(z)B(z)\\ \frac{\partial s(z)}{dz}&=\frac{\partial}{dz}\left[\zeta(z)B(z)\right]\end{aligned} \tag{B.14}$$

which has the analogue

$$\begin{aligned} \frac{\partial s(z)}{dz}&=\frac{B_{k+1}\chi_{k+1}-B_{k-1}\chi_k}{2\Delta z}\\ \frac{\partial s(z)}{dz}&=\frac{B_{k+1}\chi_{k+1}-(\chi_{k+1}-\chi_k)B_k-B_{k-1}\chi_k}{2\Delta z}\end{aligned} \tag{B.15}$$

where the staggered grid harmonic averages χ_{k+1} and χ_k have been previously defined above.

APPENDIX C: TERMS IN INVERSE SUMMATION SERIES

The analytic Fourier transform of the Gabor wavelet is, apart from some constant multiplicative terms,

$$F(\omega)=\frac{\pi^{1/2}}{\omega_0}\exp\left[-\frac{\gamma^2}{4}(1+\omega/\omega_0)^2\right]\cosh\left(\frac{\omega\gamma^2}{\omega_0}\right) \tag{C.1}$$

The horizontal wave number, k, in a coordinate system with cylindrical symmetry is related to the angular frequency as

$$k = \frac{\omega}{v} \tag{C.2}$$

where v is velocity $\left(v = \sqrt{a_{ij}}\right)$ and ω is the circular frequency. It will be assumed that some upper bound, ω_u, on the band limited spectrum of the source wavelet has been determined, often through numerical integration of the spectrum and then reintegration to the value ω_u up to which about 99.99% of the initial integration. Once ω_u has been determined, the value of k_u may be obtained as

$$k_u = \frac{\omega_u}{v_{\min}} \tag{C.3}$$

with $v_{\min}$ being the minimum velocity qP $\left(\sqrt{A_{33}}\right)_{\min}$ or qS_V $\left(\sqrt{A_{55}}\right)_{\min}$ encountered on the spatial grid which is one dimensional. It is known from numerical experiments that a good approximation for the duration of the Gabor wavelet in the time domain is γ/f_0. For some arbitrary k_i in the inverse series,

$$k_i = \frac{\zeta_i}{a} \tag{C.4}$$

where the values of ζ_i are the roots of the transcendental equation

$$J\left(\zeta_i\right) = 0 \tag{C.5}$$

so that

$$k_u = \frac{\omega_u}{v_{\min}} = \frac{\zeta_u}{a} \tag{C.6}$$

or equivalently

$$\zeta_u = \frac{a\omega_u}{v_{\min}} \tag{C.7}$$

indicating that the number of terms which must be considered to adequately approximate the infinite series summation increases linearly with a. It may be seen upon examination of equation (C.1) that the spectral width of the Gabor wavelet decreases with increasing values of γ. With the value of $\gamma = 4$ used here, $\omega_u \approx 2\omega_0$, so that with the predominant wavelength defined in terms of the predominant circular frequency and the minimum velocity encountered, $\lambda_0 = f_0/v_{\min}$ equation (C.7) becomes

$$\zeta_u = 4\pi a \tag{C.8}$$

In the above equation, $\alpha = a/\lambda_0$, a dimensionless quantity relating the predominant wavelength with the pseudo – boundary introduced at $r = a$. For large values of *i*, the relation approximate relation $\zeta_i \approx \pi i$ holds (Abramowitz and Stegun, 1980). Thus the number of terms *N*, required to approximate the infinite series is, with $\zeta_u = \pi N$, given as

$$N = 4a \tag{C.9}$$

For comparison purposes, going through the derivation with $\gamma = 5$ results in the value of *N* being given as

$$N = 8a/5 \tag{C.10}$$

which is less than that estimated for $\gamma = 4$, as would be expected.

APPENDIX D: SOURCE TERM

The finite Hankel transforms of the two components, radial and vertical, of an explosive *qP* point source term located at the point (r_s, z_s) within the *TI* medium are given as:

Radial component:

$$\begin{aligned}\hat{F}_r &= \delta(z - z_s) f(t) \int_0^a \frac{\partial}{\partial r}\left[\frac{\delta(r - r_s)}{2\pi r}\right] J_1(k_i a)\, r\, dr \\ &= -\frac{\delta(z - z_s) f(t) k_i}{2\pi} \int_0^a \delta(r - r_s) J_0(k_i a)\, dr\end{aligned} \tag{D.1}$$

where r_s is the radial source position, so that the transformed radial component has the form

$$\hat{F}_r = -\frac{k_i J_0(k_i a)}{2\pi} \delta(z - z_s) f(t). \tag{D.2}$$

Vertical component:

$$F_z = \frac{1}{2\pi}\frac{d}{dz}\left[\delta(z - z_s)\right]\delta(r - r_s) f(t) \tag{D.3}$$

$$\begin{aligned}F_z &= \frac{1}{2\pi}\frac{d}{dz}\left[\delta(z - z_s)\right] f(t) \int_0^a \frac{\delta(r - r_s)}{r} J_0(k_i r)\, r\, dr \\ &= \frac{1}{2\pi}\frac{d}{dz}\left[\delta(z - z_s)\right] J_0(k_i a) f(t)\end{aligned} \tag{D.4}$$

The partial derivative of the delta function with respect to ς can be written using a standard finite difference analogue as

$$\frac{\partial}{\partial \zeta}\left[\delta\left(\zeta-\zeta_s\right)\right] \approx \frac{\delta\left(\zeta-\zeta_{s+1}\right)-\delta\left(\zeta-\zeta_{s-1}\right)}{2(\Delta\zeta)} = \frac{\delta\left(\zeta-\zeta_{s+1}\right)}{2(\Delta\zeta)} - \frac{\delta\left(\zeta-\zeta_{s-1}\right)}{2(\Delta\zeta)}. \quad \text{(D.5)}$$

so that the partial derivative of this function at some grid point $\varsigma_S = n_S \Delta\varsigma$ is specified at the grid points at $\varsigma_{S+1} = n_{S+1}\Delta\varsigma$ and $\varsigma_{S-1} = n_{S-1}\Delta\varsigma$. From past experience with programs of the Mikhailenko type and standard finite difference formulations, the above produces accurate results when compared to other exact computational methods such as numerical integration.