Decomposition of acoustic and elastic R_P into contributions from one-parameter reflection coefficients

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ABSTRACT

In AVO/AVA inversion, a linearized form of the Zoeppritz equations known as the Aki-Richards approximation and variants are used to model R_P. This approximation can be viewed as a linear decomposition of the full reflection coefficient into contributions from the reflectivities of individual medium parameters. A forward/inverse series framework leads to an alternative approach to this type of decomposition. The first order terms in the decomposition are qualitatively similar to the Aki-Richards approximation, with second- and third-order terms correcting the approximation at large angle and large contrast. We test the approach both for acoustic and elastic reflection coefficients. In the elastic case, where forward/inverse methods of the kind we use require the incorporation of both R_P and R_S, we proceed in an approximate fashion using R_P only. The elastic nonlinear corrections, in spite of the approximation, provide a significant increase in accuracy over the linear/Aki-Richards approximation in several large contrast/large angle model regimes. Separately determining individual reflectivities could provide useful input to bandlimited impedance inversion algorithms, or the ability to extrapolate data from small to large angle.

INTRODUCTION

Practical inversion of amplitude information in reflection seismic data (Castagna and Backus, 1993) is based on linear-approximate solutions of the Zoeppritz equations, in particular that of Aki and Richards (2002) (hereafter AR). Although the Zoeppritz equations can be solved numerically (and even analytically, if you don't mind a mess), the linearized solutions have historically won out over the more complex exact forms as practical tools.

One of the reasons for this is that the linear approximations represent direct decompositions of the full R_P coefficient into weighted contributions from reflectivities due to individual parameter variations (e.g., Goodway et al., 2006). For instance, the AR approximation

$$R_P(\theta) = \frac{1}{2}\left(\frac{\Delta V_P}{V_P} + \frac{\Delta\rho}{\rho}\right) - 2\frac{V_S^2}{V_P^2}\sin^2\theta\left(2\frac{\Delta V_S}{V_S} + \frac{\Delta\rho}{\rho}\right) + \frac{1}{2}\tan^2\theta\frac{\Delta V_P}{V_P}, \qquad (1)$$

in which V_P, V_S and ρ represent the mean value of P-wave velocity, S-wave velocity and density respectively across the boundary, can be seen to explicitly express R_P in terms of

$$\frac{\Delta V_P}{V_P} \approx \frac{V_P^L - V_P^U}{V_P^L + V_P^U}, \quad \frac{\Delta V_S}{V_S} \approx \frac{V_S^L - V_S^U}{V_S^L + V_S^U}, \quad \frac{\Delta\rho}{\rho} \approx \frac{\rho^L - \rho^U}{\rho^L + \rho^U}, \qquad (2)$$

where superscript L signifies the lower medium and superscript U signifies the upper medium. These fractions are evidently equivalent to the reflection coefficients at normal incidence that would have been measured had only those individual parameters varied. The power of such a decomposition, beyond the analytical clarity it brings, is that with these reflectivities in hand, well-developed methods for normal-incidence, single-parameter bandlimited impedance inversion (e.g., Walker and Ulrych, 1983; Oldenburg et al., 1983) may be straightforwardly employed to complete the inversion.

Still, there is the matter of the inexact nature of AR and the many approximations deriving from it (e.g., Shuey, 1985), and the error that accrues at large contrasts and large angles. There have been several notable attempts to enhance accuracy by for instance providing higher-order corrections to AR. Such corrections have been constructed based on Taylor's series expansions with respect to both the model parameters within the Zoeppritz equations (Ursin and Dahl, 1992), and with respect to the ray parameter (Wang, 1999).

In this paper we will take another approach, using the tools of direct inversion, which have been developed of late for the determination of parameter contrasts from reflection coefficient information (e.g., Zhang and Weglein, 2009), to decompose acoustic and elastic reflection coefficients into their component reflectivities.

We begin by considering various acoustic configurations, i.e., reflections from contrasts in sets of parameters with acoustic analogues (e.g., including P-wave velocity, density, Q, etc., but not S-wave velocity). We develop a formula for the linear and nonlinear reconstitution of the full acoustic multiparameter reflection coefficient in terms of the relevant individual reflectivities. Remarkably, within this multiparameter acoustic configuration, the *same formula* is found to approximate R, regardless of which parameters vary, how many of them vary, and regardless of which experimental variable(s) R varies over. We then proceed to the elastic problem. The resulting formulas are only approximate, since the full problem must be posed using contributions from both R_P and R_S reflectivities, but in many regimes of large contrast/angle the accuracies of the formula are significantly higher than AR and other linearizations. We end by discussing some of the consequences of this approach to AVO modeling and to inversion, and some potentially fruitful directions in which to push this research in the near future.

MULTIPARAMETER ACOUSTIC REFLECTION COEFFICIENTS

Let R_P be the reflection coefficient associated with an interface across which N acoustic parameters, $\mu = (\mu_1, \mu_2, ...\mu_N)$, have varied, from μ^0 in the incidence medium, to μ^1 in the target medium*. We introduce N additional reflection coefficients $R_\mu = (R_{\mu_1}, R_{\mu_2}, ...R_{\mu_N})$, where R_{μ_i} is the reflection coefficient associated with an interface across which only μ_i has changed†. The approximate solution for R_P is, explicitly to third order,

$$R_P = \sum_{i=1}^{N} R_{\mu_i} - \frac{1}{3}\left[\left(\sum_{i=1}^{N} R_{\mu_i}\right)^3 - \left(\sum_{i=1}^{N} R_{\mu_i}^3\right)\right] + ..., \tag{3}$$

*For instance, these μ might represent P-wave velocity and density, in which case $\mu = (c, \rho)$ varies from $\mu^0 = (c^0, \rho^0)$ to $\mu^1 = (c^1, \rho^1)$.

†For instance, R_ρ is the P-P reflection coefficient associated with an interface across which density varied from ρ^0 to ρ^1, and all other parameters remained constant.

with fifth order and higher corrections available if desired. The derivation of this formula for a three-parameter case is included in Appendix A. The general formula above was obtained by carrying out the same derivation on a range of different types of parameter contrast, and noticing that they followed the same pattern. Next, let us form various realizations of equation (3), and compare them with their exact counterparts.

Example I: velocity and density contrast

If density ρ_0 and P-wave velocity c_0 change to ρ_1 and c_1 respectively across an interface, the angle dependent reflection coefficient is (e.g. Keys, 1989)

$$R(\theta) = \frac{(\rho_1/\rho_0)\,(c_1/c_0)\cos\theta - \sqrt{1-(c_1/c_0)^2\sin^2\theta}}{(\rho_1/\rho_0)\,(c_1/c_0)\cos\theta + \sqrt{1-(c_1/c_0)^2\sin^2\theta}}. \tag{4}$$

The individual reflectivities are

$$R_\rho = \frac{(\rho_1/\rho_0)-1}{(\rho_1/\rho_0)+1} \tag{5}$$

across a contrast in density only, and

$$R_c(\theta) = \frac{(c_1/c_0)\cos\theta - \sqrt{1-(c_1/c_0)^2\sin^2\theta}}{(c_1/c_0)\cos\theta + \sqrt{1-(c_1/c_0)^2\sin^2\theta}} \tag{6}$$

across a contrast in P-wave velocity only. By the formula in equation (3), we approximate the full reflection coefficient to third order in these reflectivities by

$$R(\theta) \approx R_c(\theta) + R_\rho - \left[R_c^2(\theta)R_\rho + R_\rho^2 R_c(\theta)\right]. \tag{7}$$

We investigate this formula for large contrast models in Figure 1. Approximations are plotted for three different configurations of incidence and target medium properties. In each plot, first order (blue), third order (red), and fifth order (green) approximations are compared to the exact reflection coefficient (black). The derivation of the fifth order formula is presented in a coming section. The linearization is often close to the result achieved by the AR approximation, and though there are parameter configurations that cause the two to deviate from each other dramatically, the blue line is a reasonably faithful guide to the accuracy to be expected from AR in each circumstance.

Example II: velocity and Q contrast

The flexibility of equation (3) is best illustrated by applying it to a problem that differs in *which* parameters vary and *over what experimental variable* they are examined. If an acoustic medium with P-wave velocity c_0 is perturbed at an interface such that c_0 changes to c_1, and the target medium additionally takes on an anacoustic character through introduction of a finite quality factor Q_1, at normal incidence we generate a complex, frequency

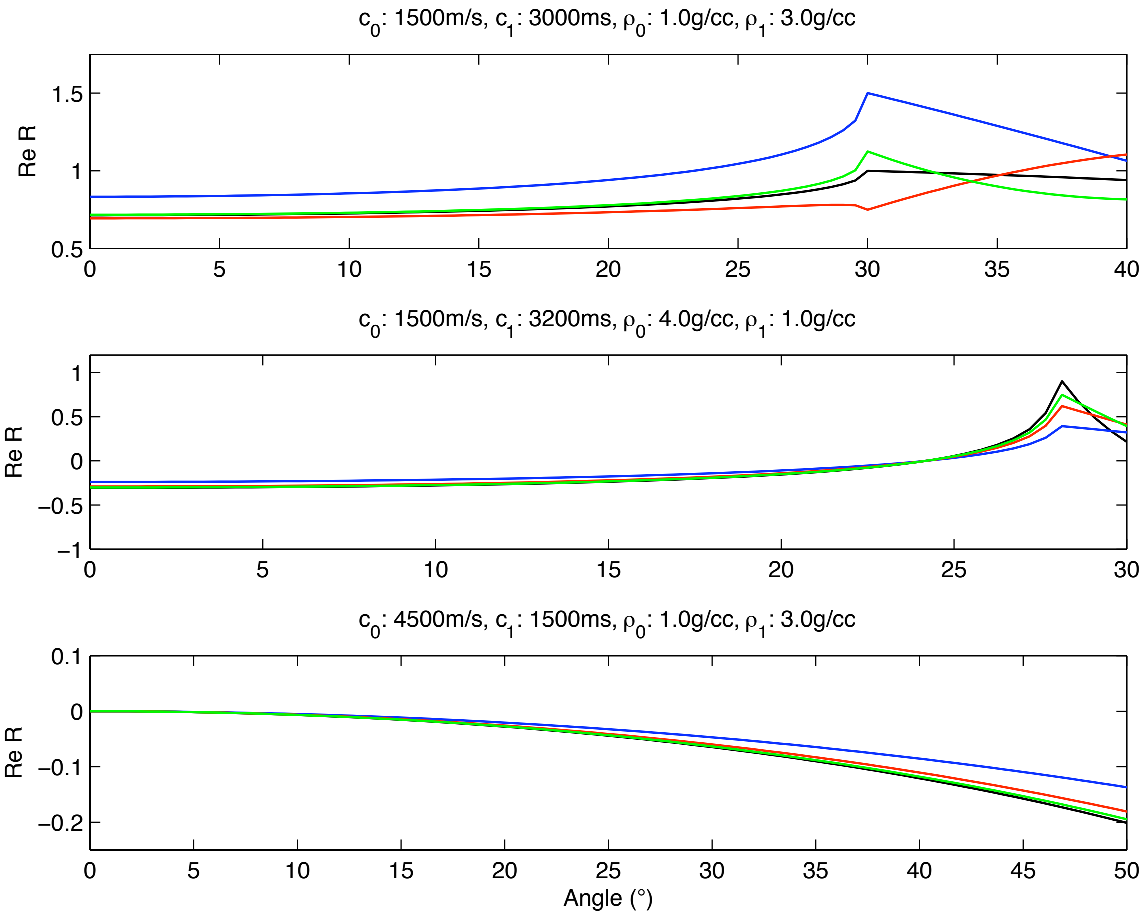

FIG. 1. First, third, and fifth order approximations to the Zoeppritz equations in terms of density-only, and P-wave velocity only reflectivities. Three configurations of incidence and target medium parameters, corresponding to large contrast and (relative to the contrasts) large angle are shown. Black: exact R; blue: first order; red: third order; green: fifth order.

dependent reflection coefficient:

$$R(\omega) = \frac{1 - \left(\frac{c_0}{c_1}\right)\left[1 + \frac{F(\omega)}{Q_1}\right]}{1 + \left(\frac{c_0}{c_1}\right)\left[1 + \frac{F(\omega)}{Q_1}\right]} \tag{8}$$

where

$$F(\omega) = \frac{i}{2} - \frac{1}{\pi}\log\left(\frac{\omega}{\omega_0}\right) \tag{9}$$

and ω_0 is a reference frequency. Equation (8) is consistent with the nearly constant Q model discussed by Aki and Richards (2002). The individual reflectivities are

$$R_c = \frac{1 - \left(\frac{c_0}{c_1}\right)}{1 + \left(\frac{c_0}{c_1}\right)}, \tag{10}$$

and

$$R_Q(\omega) = -\frac{\frac{F(\omega)}{Q_1}}{2 + \frac{F(\omega)}{Q_1}}, \tag{11}$$

and hence the approximation to third order is

$$R(\omega) \approx R_c + R_Q(\omega) - \left[R_c^2 R_Q(\omega) + R_Q^2(\omega) R_c\right]. \tag{12}$$

In Figure 2 we again examine the accuracy of the formula in the presence of some large contrast configurations over a wide range of frequencies. Similar results are noted.

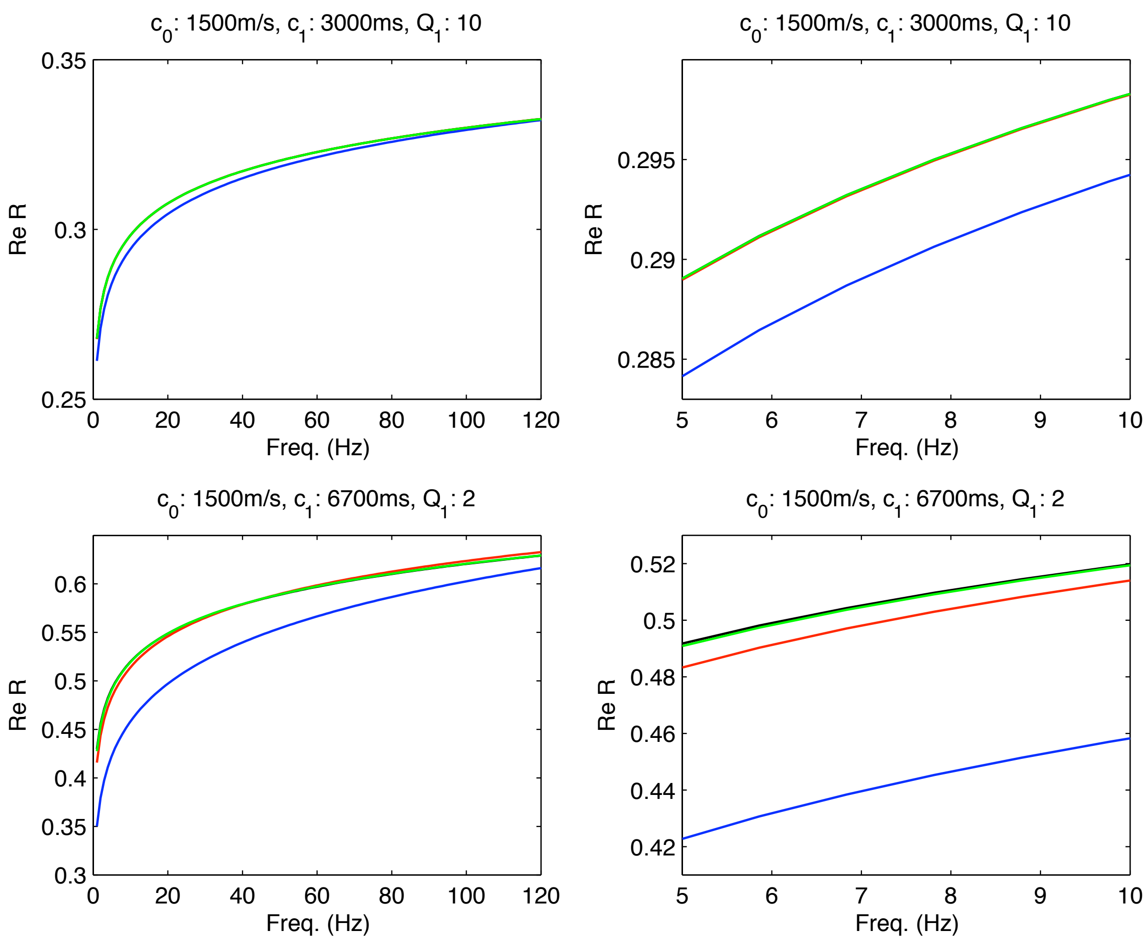

FIG. 2. Anacoustic reflection coefficient approximations. Black: exact; blue: linear; red: third order; green: fifth order. Top left: a moderate contrast example; top right: detail of same. Bottom left: large contrast example; bottom right: detail of same.

Example III: velocity, density and Q contrast

To illustrate the form of the approximation with more than two parameters varying, let us consider the case in which P-wave velocity, Q, and density all vary, in which case we have

$$R(\theta)=\frac{(\rho_1/\rho_0)\,(c_1/c_0)\left[1+\frac{F(\omega)}{Q_1}\right]^{-1}\cos\theta-\sqrt{1-(c_1/c_0)^2\left[1+\frac{F(\omega)}{Q_1}\right]^2\sin^2\theta}}{(\rho_1/\rho_0)\,(c_1/c_0)\left[1+\frac{F(\omega)}{Q_1}\right]^{-1}\cos\theta+\sqrt{1-(c_1/c_0)^2\left[1+\frac{F(\omega)}{Q_1}\right]^2\sin^2\theta}} \tag{13}$$

Using the primitive reflection coefficients associated with density, velocity, and Q in equations (5), (10) and (12) respectively, we obtain the expansion

$$\begin{aligned} R =& R_c + R_\rho + R_Q \\ &- R_c^2(R_\rho + R_Q) - R_\rho^2(R_c + R_Q) - R_Q^2(R_c + R_\rho) - 2R_cR_\rho R_Q + ... \quad . \end{aligned} \tag{14}$$

In Appendix A, we use this three-parameter framework to derive the formula in equation (3).

MULTIPARAMETER ELASTIC REFLECTION COEFFICIENTS

Let us consider the extension of the previous methods to the three parameter elastic case. There is a significant caveat that goes along with this extension. It has been established (Zhang and Weglein, 2009) that for nonlinear elastic inversion all four of PP, PS, SP, and SS components of the data are necessary. In the acoustic case, in contrast, the nonlinear inverse problem is well posed given the one type of reflection coefficient. For an incident P-wave, there are two reflected modes, PP and PS. It follows that in order to correctly decompose *either* R_P or R_S in individual reflectivity values using our forward/inverse series approach, *both* data types will need to be invoked. We leave that more general problem for future research. What we will find when posing the inconsistent version of the problem, involving R_P only, is that under many important large contrast, large angle circumstances highly accurate approximations are produced.

Using the same approach as in the previous section, the full reflection R_P is decomposed directly in terms of individual V_P (here denoted α), V_S (here denoted β) and ρ reflectivities through the formula

$$R_P(\theta) = R_1(\theta) + R_2(\theta) + R_3(\theta) + ... \tag{15}$$

where, the first order term is given by

$$R_1(\theta) = R_\alpha(\theta) + R_\beta(\theta) + R_\rho(\theta), \tag{16}$$

the second order term by

$$R_2(\theta) = W_1\ R_\beta^2(\theta) + W_2\ R_\beta(\theta)R_\rho(\theta), \tag{17}$$

and the third order term by

$$\begin{aligned} R_3(\theta) = & W_3\ R_\alpha^2(\theta)R_\beta(\theta) + W_4\ R_\alpha^2(\theta)R_\rho(\theta) \\ &+ W_5\ R_\alpha(\theta)R_\beta^2(\theta) + W_6\ R_\beta^2(\theta)R_\rho(\theta) \\ &+ W_7\ R_\beta(\theta)R_\rho^2 + W_8\ R_\alpha(\theta)R_\rho^2(\theta) \\ &+ W_9\ R_\alpha(\theta)R_\beta(\theta)R_\rho(\theta), \end{aligned} \tag{18}$$

where

$$W_1 = \frac{1}{BX^2}\left(\frac{1}{2} - \frac{1}{B}\right) - \frac{1}{8}, \quad W_2 = 1 - 2B, \quad W_3 = W_4 = W_8 = -1, \quad W_5 = -\frac{1}{2}\frac{1}{BX^2},$$
$$W_6 = \frac{1}{2X^2}\left(1 - \frac{9}{2B} + \frac{7}{4B^2}\right), \quad W_7 = \frac{3}{2} - \frac{B}{2}, \quad \text{and} \quad W_9 = 4B - 2 \tag{19}$$

and $X = \sin\theta$. The derivation of this formula is provided in Appendix B.

In Figure 3 we illustrate three cases in which the decomposition, expressed to third order, produces significantly increased accuracy as compared to the linear and AR approximations. Empirically, we find that the approximation works best in extremely large contrast cases, in which either (1) all three parameters vary across the interface by roughly 100%, or (2) either V_P or ρ vary by that amount. The inconsistent/approximate nature of the formula seems to be felt at lower contrast, and in particular where V_P and ρ undergo small contrasts and V_S undergoes a large contrast. In its current form, the approximation would therefore likely be best used in concert with the linear or AR approximation, with one or the other being utilized depending on known or assumed contrast magnitudes.

CONCLUSIONS: IMPLICATIONS FOR MODELING AND INVERSION

The approximations we have produced are relationships between the actual (i.e., measured) reflection coefficient and the individual parameter reflectivities underlying it. There are several ways these relationships could be used in a practical way. First, through a nonlinear regression the best-fit reflectivities could be estimated and evaluated at normal incidence. Then, standard bandlimited impedance inversion procedures could be used on each reflectivity function to determine the parameter profiles. Second, the equations could be used to extrapolate data with limited offset to higher angles, again using a nonlinear regression.

It is worth emphasizing that there is no reason to limit the reflectivity decompositions to those in V_P, V_S, and ρ. As discussed by Goodway et al. (2006), often Lamé parameters λ, μ and ρ, or Lamé impedances $\lambda\rho$ and $\mu\rho$ are more useful products; the approach we have described in this note would extend readily to include one-parameter reflectivities with any of these parametrizations.

What we have developed exists in the plane-wave domain, i.e., it is an AVA theory; none of the issues surrounding transformation to an AVO theory, which describes space- and time-domain amplitudes have been broached as of yet. Also, we emphasize that posing the elastic problem consistently, using both S- and P-reflectivities, thus going beyond the current approximation, is a conceptually straightforward but important aspect of the problem.

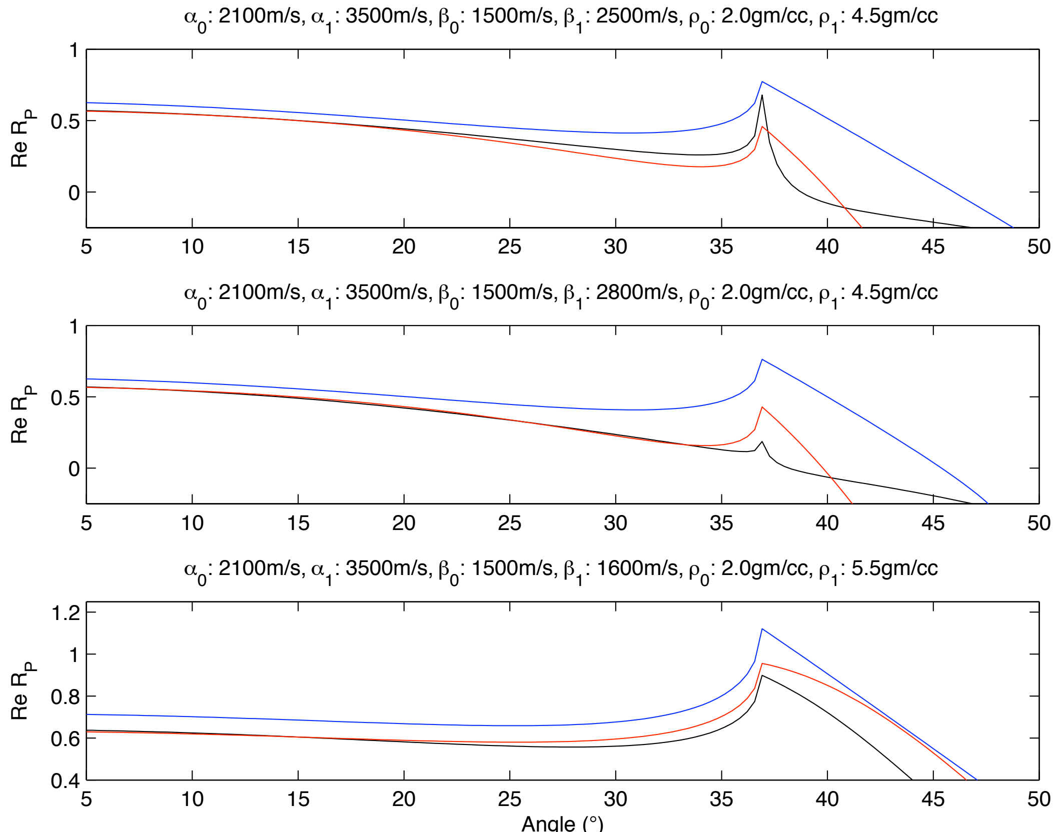

FIG. 3. Decomposition of elastic R_P into one-parameter reflectivities. Black: exact R_P; blue: linear decomposition; red: third order. This approximation (a full expression is expected to involve R_S as well as R_P reflectivity contributions) seems to perform particularly well in comparison to the AR approximation when either all three parameters undergo large contrasts, or V_P and ρ undergo large contrasts.

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APPENDIX A: DERIVATION OF ACOUSTIC DECOMPOSITION FORMULA

We shall derive formula (3) for the case of the three-parameter (velocity, density and Q contrast) example. All other examples are derived in the same way, and all lead to the same formula. We begin with the full expression for the three-parameter reflection coefficient in equation (13), re-written with the contrasts expressed as perturbations on the incident medium parameter. Defining

$$a_c = 1 - c_0^2/c_1^2, \quad a_\rho = 1 - \rho_0/\rho_1, \quad a_Q = 1/Q_1, \tag{20}$$

expanding about both these three perturbations and $\sin^2\theta$, and retaining only terms at zero'th order in the latter, we have

$$R(\theta = 0, \omega) = R_1 + R_2 + R_3 + ..., \tag{21}$$

where

$$R_1(\omega) = \frac{1}{4}a_c + \frac{1}{2}a_\rho - \frac{F}{2}a_Q, \tag{22}$$

$$R_2(\omega) = \frac{1}{8}a_c^2 + \frac{1}{4}a_\rho^2 + \frac{F^2}{4}a_Q^2, \tag{23}$$

$$\begin{aligned} R_3(\omega) =& \frac{5}{64}a_c^3 + \frac{1}{8}a_\rho^3 - \frac{F^3}{8}a_Q^3 + \frac{F}{32}a_c^2 a_Q - \frac{1}{32}a_c^2 a_\rho - \frac{F^2}{16}a_c a_Q^2 \\ &- \frac{F^2}{8}a_\rho a_Q^2 - \frac{1}{16}a_\rho^2 a_c + \frac{F}{8}a_\rho^2 a_Q + \frac{F}{8}a_c a_\rho a_Q, \end{aligned} \tag{24}$$

etc. We next perform the same expansions on the primitives in equations (5), (10) and (12), obtaining either by similar analysis or by simply picking the coefficients directly from equation (24)

$$\begin{aligned} R_c &= \frac{1}{4}a_c + \frac{1}{8}a_c^2 + \frac{5}{64}a_c^3 + ..., \\ R_\rho &= \frac{1}{2}a_\rho + \frac{1}{4}a_\rho^2 + \frac{1}{8}a_\rho^3 + ..., \\ R_Q &= -\frac{F}{2}a_Q + \frac{F^2}{4}a_Q^2 - \frac{F^3}{8}a_Q^3 + ... \end{aligned} \tag{25}$$

Next we solve for the perturbations a_c, a_ρ and a_Q in terms of R_c, R_ρ, and R_Q respectively. In simple cases like this one, exact, closed-form solutions are available. However, such that we provide a derivation applicable to cases in which such solutions are not available, let us instead form inverse series for the perturbations. Letting $a_c = a_{c_1} + a_{c_2} + a_{c_3} + ...$, $a_\rho = a_{\rho_1} + a_{\rho_2} + a_{\rho_3} + ...$, and $a_Q = a_{Q_1} + a_{Q_2} + a_{Q_3} + ...$, in which subscript i indicates that the term is ith order in the respective one-parameter reflection coefficient, substituting these into equation (25), and equating like orders, we reconstruct the three perturbations:

$$\begin{aligned} a_c &= 4(R_c - 2R_c^2 + 3R_c^3 + ...), \\ a_\rho &= 2(R_\rho - R_\rho^2 + R_\rho^3 + ...), \\ a_Q &= -(2/F)(R_Q - R_Q^2 + R_Q^3 + ...). \end{aligned} \tag{26}$$

Having thus expressed the individual perturbations involved in the construction of the full reflection coefficient R in terms of the three reflectivities R_c, R_ρ, and R_Q, we eliminate a_c, a_ρ and a_Q in equation (21) in favour of R_c, R_ρ, and R_Q via equation (26), finally obtaining the result quoted in the previous section:

$$\begin{aligned} R =& R_c + R_\rho + R_Q \\ & - R_c^2(R_\rho + R_Q) - R_\rho^2(R_c + R_Q) - R_Q^2(R_c + R_\rho) - 2R_cR_\rho R_Q + ... \ . \end{aligned} \tag{27}$$

Notice that the derivation has taken place entirely within the small offset/normal incidence regime. Yet, as we see in Figure 1, the resulting approximation is very accurate at large angles (indeed this is the whole point of the result). The variability of R with θ is provided by reinstating the angle dependence of the input reflectivities, *after* the elimination of the perturbations. The same procedure can evidently be carried forward to higher orders, whence derive the fifth-order examples of the previous sections.

APPENDIX B: DERIVATION OF ELASTIC DECOMPOSITION FORMULA

We begin by examining exact solutions to the Zoeppritz equations associated with an incident plane P-wave, generated using Cramer's rule (e.g., Keys, 1989). Let us define $X = \sin\theta$, $S_Y = \sqrt{1 - Y^2X^2}$ and $S^Y = (1 - 2Y^2X^2)$ where Y can take on the values 1, B, C or D. For an incident P wave, the Zoeppritz equations are embodied in the system

$$\mathcal{A} \begin{bmatrix} R_P \\ R_S \\ T_P \\ T_S \end{bmatrix} = \mathbf{b}, \tag{28}$$

where

$$\mathcal{A}(A, C, D) = \begin{bmatrix} -X & -S_B & CX & -S_D \\ S_1 & -BX & S_C & DX \\ 2B^2XS_1 & BS^B & 2AD^2XS_C & -ADS^D \\ -S^B & 2B^2XS_B & ACS^D & 2AD^2XS_D \end{bmatrix}, \tag{29}$$

and $\mathbf{b} = (X, S_1, 2BS_1X, S^B)^T$, and the constants A, B, C and D contain elastic medium parameters:

$$A = \frac{\rho}{\rho_0}, \ B = \frac{\beta_0}{\alpha_0}, \ C = \frac{\alpha}{\alpha_0}, \ D = B\frac{\beta}{\beta_0}. \tag{30}$$

We will also make use of an auxiliary matrix $\mathcal{A}_P(A,C,D)$ formed by replacing the first column of $\mathcal{A}$ with $\mathbf{b}$. The solution of interest, R_P, is formed using Cramer's rule:

$$R_P(\theta) = \frac{\det\mathcal{A}_P(A,C,D)}{\det\mathcal{A}(A,C,D)}. \tag{31}$$

In the coming sections we will also make use of special instances of equation (31) when we discuss individual reflectivities associated with variations in each of α, β, and ρ:

$$R_\alpha(\theta) = \frac{\det\mathcal{A}_P(1,C,B)}{\det\mathcal{A}(1,C,B)},\ R_\beta(\theta) = \frac{\det\mathcal{A}_P(1,1,D)}{\det\mathcal{A}(1,1,D)},\ R_\rho(\theta) = \frac{\det\mathcal{A}_P(A,1,B)}{\det\mathcal{A}(A,1,B)}. \tag{32}$$

To begin to derive the approximation in equation (15), we must form an expansion of R_P in orders of perturbations of the three parameters α, β, and ρ, as well as $\sin^2\theta$. We introduce $a_\alpha = 1-\alpha_0^2/\alpha^2$, $a_\beta = 1-\beta_0^2/\beta^2$, and $a_\rho = 1-\rho_0/\rho$, substitute these forms into A, C, and D in equation (30), and express the three results as binomial series expansions. When these are in turn substituted into $\mathcal{A}$ and $\mathcal{A}_P$, with all S_Y quantities also expressed in series form, their determinants may be organized in increasing order in the five perturbations:

$$\begin{aligned} \det\mathcal{A} &= \det\mathcal{A}^{(0)} + \det\mathcal{A}^{(1)} + \det\mathcal{A}^{(2)} + ..., \\ \det\mathcal{A}_P &= \det\mathcal{A}_P^{(1)} + \det\mathcal{A}_P^{(2)} + ..., \\ \det\mathcal{A}_S &= \det\mathcal{A}_S^{(1)} + \det\mathcal{A}_S^{(2)} + ..., \end{aligned} \tag{33}$$

where superscript (i) indicates i'th order in any combination of the three perturbations (e.g., a term containing $a_\rho^2 a_\beta$ is considered "third order"). This leads to a series expression for R_P as follows:

$$R_P = \frac{\hat{\det}\mathcal{A}_P}{\hat{\det}\mathcal{A}} = \hat{\det}\mathcal{A}_P^{(1)} + \left(\hat{\det}\mathcal{A}_P^{(2)} - \hat{\det}\mathcal{A}_P^{(1)}\hat{\det}\mathcal{A}^{(1)}\right) + ...\ , \tag{34}$$

where for any $\mathcal{Y}$, $\hat{\det}\mathcal{Y} \equiv \det\mathcal{Y}/\det\mathcal{A}^{(0)}$. To third order equation (34) has the explicit form

$$R_P(\theta) = R_{P_1} + R_{P_2} + R_{P_3} + ..., \tag{35}$$

where

$$R_{P_1}(\theta) = \Gamma_{11}a_\alpha + \Gamma_{12}a_\beta + \Gamma_{13}a_\rho \tag{36}$$

$$R_{P_2}(\theta) = \Gamma_{21}a_\alpha^2 + \Gamma_{22}a_\beta^2 + \Gamma_{23}a_\rho^2 + \Gamma_{24}a_\alpha a_\beta + \Gamma_{25}a_\alpha a_\rho + \Gamma_{26}a_\beta a_\rho \tag{37}$$

and

$$\begin{aligned} R_{P_3}(\theta) = {}& \Gamma_{31}a_\alpha^3 + \Gamma_{32}a_\beta^3 + \Gamma_{33}a_\rho^3 + \Gamma_{34}a_\alpha^2 a_\beta + \Gamma_{35}a_\alpha^2 a_\rho \\ & + \Gamma_{36}a_\beta^2 a_\alpha + \Gamma_{37}a_\beta^2 a_\rho + \Gamma_{38}a_\rho^2 a_\alpha + \Gamma_{39}a_\rho^2 a_\beta + \Gamma_{310}a_\alpha a_\beta a_\rho. \end{aligned} \tag{38}$$

The prefactors Γ_{ij}, which are in general functions of X (that is, θ) and B, are provided in Appendix C. Next, we form similar expansions of the individual reflectivities in equation

(32):

$$
\begin{aligned}
R_\alpha &= \frac{1}{4}(1+X^2)a_\alpha + \left(\frac{1}{8}+\frac{1}{4}X^2\right)a_\alpha^2 + \frac{5}{64}(1+3X^2)\alpha^3 + ... \\
R_\beta &= -2B^2X^2a_\beta + B^2X^2(B-2)a_\beta^2 + B^2X^2\left(\frac{7}{4}B-2\right)a_\beta^3 + ... \\
R_\rho &= \frac{1}{2}(1-4B^2X^2)a_\rho + \left[\frac{1}{4}+BX^2\left(B^2-B-\frac{1}{4}\right)\right]a_\rho^2 \\
&\quad + \left[\frac{1}{8}+BX^2\left(\frac{1}{2}B^2-\frac{3}{8}\right)\right]a_\rho^3 + ... \;.
\end{aligned}
\tag{39}
$$

The P-wave velocity reflection coefficient R_α and the density coefficient R_ρ may be inverted as before. Setting $\theta = 0$, forming inverse series $a_\alpha = a_{\alpha_1} + a_{\alpha_2} + ...$ and $a_\rho = a_{\rho_1} + a_{\rho_2} + ...$, substituting them into the first two equations in (39), and equating like orders, we construct the following expressions for the two perturbations:

$$
\begin{aligned}
a_\alpha &= 4R_\alpha - 8R_\alpha^2 + 12R_\alpha^3 - ..., \\
a_\rho &= 2R_\rho - 2R_\rho^2 + 2R_\rho^3 + ... \;.
\end{aligned}
\tag{40}
$$

For the S-wave velocity reflection coefficient R_β we truncate all terms beyond X^{-2}, and consider the series

$$
\frac{R_\beta}{B^2X^2} = \begin{cases} -2a_\beta - (B-2)a_\beta^2 + (7B/4-2)a_\beta^3, & \text{if} \quad \theta \neq 0 \\ 0, & \text{if} \quad \theta = 0 \end{cases}, \tag{41}
$$

which, upon similar inversion returns

$$
a_\beta = -\frac{1}{2B^2X^2}\left[R_\beta + \frac{1}{2BX^2}\left(\frac{1}{2}-\frac{1}{B}\right)R_\beta^2 + \frac{1}{8B^2X^4}\left(1-\frac{1}{2B}\right)R_\beta^3 + ...\right]. \tag{42}
$$

Here some of the inconsistency of the approach becomes apparent, as the series grows in reciprocal orders of $\sin\theta$. Nevertheless, with the knowledge that we may face a certain degree of inaccuracy/instability, we may now use equations (40) and (42) to eliminate a_α, a_β, and a_ρ in favour of R_α, R_β, and R_ρ in equation (35), which recovers the series in equation (15).

APPENDIX C: COEFFICIENTS OF EXPANSION OF ELASTIC R_P

Equations (36)–(38) contain terms in the expansion of R_P in terms of contrasts in α, β, ρ, and $X = \sin\theta$. The coefficients Γ_{ij} are, explicitly,

$$
\begin{aligned}
\Gamma_{11} &= \frac{1}{4}(1+X^2), \\
\Gamma_{12} &= -2B^2X^2, \\
\Gamma_{13} &= \frac{1}{2}(1-4B^2X^2),
\end{aligned}
\tag{43}
$$

$$
\begin{aligned}
&\Gamma_{21} = \frac{1}{8} + \frac{1}{4}X^2,\ \Gamma_{22} = B^2X^2(B-2),\\
&\Gamma_{23} = \frac{1}{4} + BX^2\left(B^2 - B - \frac{1}{4}\right),\ \Gamma_{24} = \Gamma_{25} = 0,\\
&\Gamma_{26} = B^2X^2(2B-1),
\end{aligned}
\tag{44}
$$

and

$$
\begin{aligned}
&\Gamma_{31} = \frac{5}{64}(1+3X^2),\ \Gamma_{32} = B^2X^2\left(\frac{7}{4}B - 2\right),\\
&\Gamma_{33} = \frac{1}{8} + BX^2\left(\frac{1}{2}B^2 - \frac{3}{8}\right),\ \Gamma_{34} = \frac{1}{8}B^2X^2,\\
&\Gamma_{35} = -\frac{1}{32} + \left(\frac{1}{8}B^2 - \frac{1}{16}\right)X^2,\ \Gamma_{36} = -\frac{1}{2}B^3X^2,\\
&\Gamma_{37} = \left(2B - \frac{3}{4}\right)B^2X^2,\ \Gamma_{38} = -\frac{1}{16} + \left(\frac{1}{2}B^2(1-B) - \frac{1}{8}B - \frac{1}{16}\right)X^2,\\
&\Gamma_{39} = \left(\frac{3}{4}B^2 + \frac{1}{4}B - \frac{1}{16}\right)BX^2,\ \Gamma_{310} = B^2X^2\left(\frac{1}{2} - B\right).
\end{aligned}
\tag{45}
$$