Higher order terms of the asymptotic ray theory series solution for the acoustic wave equation in *2, 3* and higher dimensions

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ABSTRACT

Often referred to, but rarely derived in practice, are the transport equations for higher order terms in an Asymptotic Ray Theory (ART) solution method for hyperbolic (wave) equations. In most instances in the literature only the first term (zero order) term in the asymptotic series is used in the computation of dynamic (amplitude) quantities. Higher order terms in the series will be derived here in any number of dimensions, with the emphasis on the two and three dimensional cases, and compared with the exact solution. The type of medium propagation will be assumed to be an infinite space and the hyperbolic equation used will be the simple acoustic wave equation with a constant velocity – homogeneous medium. In addition, the summation of the series $\sum_{n=0}^{\infty}(i\omega)^{-n}$ will be presented for use in a solution which has been assumed to be high frequency.

INTRODUCTION

To keep matters as simple as possible, an acoustic wave type propagating in an infinite isotropic homogeneous acoustic medium is assumed. Physically, this media type is usually taken to be a fluid. An asymptotic ray series or geometrical optics solution is sought that describes the wave propagation in an N-spatial dimension medium, $(N \geq 1)$. No source, initial or boundary conditions required. When compared to the formulation required for the exact solution of the scalar wave equation (Červený and Ravindra, 1971), the conditions that need to be satisfied to employ an ART solution are minimal in comparison. What is required is that appropriate radiation conditions are assumed to provide a physically realizable solution.

A definition of an asymptotic series might be of use here as these series types need not be convergent. An asymptotic series or expansion is a formal series of functions having the property that truncation of the series after a finite number of terms provides an approximation to given function or solution method, as the argument of the function tends towards a particular point, usually infinity. A Taylor series fits this definition. However, a Taylor series is always assumed to be convergent.

Exact solutions are not derived here as they may be found in almost any text on the subject of wave propagation. They will be written down as required for comparison purposes. So as not to introduce unnecessary references into this discussion a minimum of these will be cited, which together provide enough information for the pursuit of this tutorial problem. They are Červený, (2001), Hron and Kanasewich, (1971) and Hildebrand (1962).

BASIC THEORY

The scalar type wave equation, with a constant velocity, α, may be written in N dimensions, $N \ge 1$ as

$$\nabla^2 \phi(\mathbf{x},t) - \frac{1}{\alpha^2}\frac{\partial^2 \phi(\mathbf{x},t)}{\partial t^2} = 0 \tag{1}$$

with $\nabla^2 = \frac{\partial^2}{\partial x_i^2}$ $(i = 1, N)$ being the Laplacian operator in an N dimensional Cartesian space. The geometrical optics, high frequency or Asymptotic Ray Theory (ART) solution is defined in the following manner, where it has been implicitly assumed that $A_n(\mathbf{x})$ is complex valued for some N – spatial dimension scalar amplitude series as

$$\phi(\mathbf{x},t) = \sum_{n=0}^{\infty} A_n(\mathbf{x}) \frac{\exp\left[i\omega\left(t - \tau(\mathbf{x})\right)\right]}{(i\omega)^n} \tag{2}$$

The more general form of the above equation is

$$\phi(\mathbf{x},t) = \sum_{n=0}^{\infty} A_n(\mathbf{x}) f_n\left(t - \tau(\mathbf{x})\right) \tag{3}$$

for some generalized function $f_n(\zeta)$ which has the properties that $df_n(\zeta)/d\zeta = f_{n-1}(\zeta)$ and $f_n(\zeta) \equiv 0, \forall\, n : n < 0$. The expression used in equation (2) for this function satisfies the first property and the second property is imposed, where $\left(A_n(\mathbf{x}) \equiv 0, \forall\, n : n < 0\right)$ and $\tau(\mathbf{x})$ is some phase function related to the travel time of the wavefront through what has been assumed to be an isotropic homogeneous acoustic medium. The quantities t, ω and $\mathbf{x}$ are time, circular frequency and an $N-$dimensional position vector.

Substituting equation (2) into (1) yields, apart from the related eikonal equation problem, the following, which are used in the determination of the amplitude terms, $A_n(\mathbf{x})$

$$\begin{aligned} e^{i\omega\left(t-\tau(\mathbf{x})\right)} \Bigg\{ & \sum_{n=0}^{\infty} \frac{\nabla^2 A_n}{(i\omega)^n} - 2\sum_{n=0}^{\infty} \frac{\left(\nabla A_n \cdot \nabla \tau\right)}{(i\omega)^{n-1}} - \sum_{n=0}^{\infty} \frac{A_n \nabla^2 \tau}{(i\omega)^{n-1}} + \\ & \sum_{n=0}^{\infty} \frac{A_n \left(\nabla \tau\right)^2}{(i\omega)^{n-2}} - \frac{1}{\alpha^2} \sum_{n=0}^{\infty} \frac{A_n}{(i\omega)^{n-2}} \Bigg\} = 0 \end{aligned} . \tag{4}$$

A rearrangement of the terms in the individual series results in

$$e^{i\omega(t-\tau(\mathbf{x}))}\sum_{n=0}^{\infty}(i\omega)^{-n}\left\{(i\omega)^2\nabla^2 A_{n-2}+(i\omega)\left[2\nabla A_{n-1}\cdot\nabla\tau+A_{n-1}\nabla^2\tau\right]+\right.$$
$$\left.A_n(\nabla\tau)^2-\alpha^{-2}A_n\right\}=0 \tag{5}$$

As equation (5) must hold for any value of the frequency, ω, the coefficients of each power of ω must vanish. Therefore

$$\left[(\nabla\tau)^2-\alpha^{-2}\right]A_n=0 \quad \forall\, n:0\le n<\infty \tag{6}$$

Under the assumption that the zero order term in the asymptotic expansion is not equal to zero, then a solution of this problem exists if and only if

$$(\nabla\tau)^2-\alpha^{-2}\equiv 0\,. \tag{7}$$

This equation is the known as the eikonal equation which is related to the Hamiltonian of the system (Courant and Hilbert, 1962) from which the solution for the propagation of rays or characteristics may be determined. It is along the rays that the energy is carried from one point to another related point in the medium. The second conditional equation, which may be obtained from (5) and the subsequent statement, is

$$\left[2\nabla A_0\cdot\nabla\tau+A_0\nabla^2\tau\right]+\left[\left[(\nabla\tau)^2-\alpha^{-2}\right]A_1\right]=0\,. \tag{8}$$

As equations (5) and (7) must be valid for all values of *n*, equation (8) reduces to the transport equation most often encountered in the literature

$$\left[2\nabla A_0\cdot\nabla\tau+A_0\nabla^2\tau\right]=0 \tag{9}$$

The final term from equation (5) is

$$\nabla^2 A_{n-2}+\left[2\nabla A_{n-1}\cdot\nabla\tau+A_{n-1}\nabla^2\tau\right]+\left[\left[(\nabla\tau)^2-\alpha^{-2}\right]A_n\right]=0,\quad n\ge 2 \tag{10}$$

This is a recursive transport equation with each term in the series dependent on the previous term in the infinite series. However, as the eikonal equation, equation (7), is valid for all *n*, equation (10) becomes

$$\nabla^2 A_{n-2}+\left[2\nabla A_{n-1}\cdot\nabla\tau+A_{n-1}\nabla^2\tau\right]=0,\quad n\ge 2 \tag{11}$$

It follows from (11) that the transport equation for the second (first order) term, A_1, is given by

$$2\left(\nabla A_1\cdot\nabla\tau\right)+A_1\nabla^2\tau=\nabla^2 A_0 \tag{12}$$

and the A_2 term by

$$2\left(\nabla A_2 \cdot \nabla \tau\right) + A_2 \nabla^2 \tau = \nabla^2 A_1 \tag{13}$$

The equation for the solution of the second (first order) term in the asymptotic series is the most general for the higher order terms in the asymptotic series and it is used to obtain the terms in the asymptotic series, A_n, $n \geq 1$.

WAVE EQUATION IN TWO DIMENSIONS

For $N = 2$, the two-dimensional scalar wave equation in the (x,z) Cartesian plane, using angular invariant polar (radial) coordinates allows the resultant amplitude determination problem to be formulated in terms of the single independent variable $R = \left(x^2 + z^2\right)^{1/2}$ or more generally, assuming a source location at (x_0, z_0) rather than at $(0,0)$, $R = \left(\left(x - x_0\right)^2 + \left(z - z_0\right)^2\right)^{1/2}$. Using this dependent variable the following solution for the zero order transport equation, equation (9), results[1]

$$A_0(x,z) = A_0(R) = \frac{1}{\sqrt{R}} \tag{14}$$

The second term in the asymptotic equation for $A_1(R)$ is the solution of the following equation, which is dependent on $A_0(R)$, and is obtained as

$$A_1 \nabla^2 \tau + 2\nabla A_1 \cdot \nabla \tau = \nabla^2 A_0 \tag{15}$$

with the quantity on the RHS of (15) being given as

$$\nabla^2 A_0 = \frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial}{\partial R}\left(\frac{1}{\sqrt{R}}\right)\right) = \frac{1}{4R^{5/2}} \tag{16}$$

and A_1 becoming the solution of

$$\frac{dA_1}{dR} + \frac{A_1}{2R} = \frac{\alpha}{8R^{5/2}} \tag{17}$$

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[1] $\tau(x,z) = \dfrac{\left(\left(x - x_0\right)^2 + \left(z - z_0\right)^2\right)^{1/2}}{\alpha} = \dfrac{R}{\alpha}$

$$\nabla \tau = \frac{\partial \tau}{\partial R}\mathbf{u_R} = \frac{\mathbf{u_R}}{\alpha}$$

$$\nabla^2 \tau = \frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial \tau}{\partial R}\right) = \frac{1}{\alpha R}$$ (2 Dimensional *Polar* Coordinates).

This may be rewritten after introducing an integration factor[2] as

$$\frac{d}{dR}\left(R^{1/2}A_1\right)=\frac{\alpha}{8R^2} \tag{18}$$

which has the solution

$$A_1=-\left(\frac{1}{\sqrt{R}}\right)\frac{\alpha}{R}. \tag{19}$$

The third or second order term in the asymptotic series may then be determined in the following manner

$$2\left(\nabla A_2\cdot\nabla\tau\right)+A_2\nabla^2\tau=\nabla^2 A_1. \tag{20}$$

With the relations

$$\tau=\frac{R}{\alpha}\qquad \nabla\tau=\frac{\partial\tau}{\partial R}\mathbf{u_R}=\frac{\mathbf{u_R}}{\alpha}\qquad \nabla^2\tau=\frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial\tau}{\partial R}\right)=\frac{1}{\alpha R}., \tag{21}$$

it follows that

$$A_1=\frac{-\alpha}{8R^{3/2}}\qquad \nabla A_1=\frac{\partial A_1}{\partial R}\mathbf{u_R}\qquad \frac{\partial A_1}{\partial R}=\frac{3\alpha}{16R^{5/2}} \tag{22}$$

resulting in

$$\nabla^2 A_1=\frac{9\alpha}{32R^{7/2}}. \tag{23}$$

Using equation (11) results in the following, after determining an integrating factor,

$$\frac{d}{dR}\left(R^{1/2}A_2\right)=\frac{9\alpha^2}{64R^3}\quad . \tag{24}$$

Continuing with the solution method yields

$$A_2=-\frac{9\alpha^2}{2!8^2R^{5/2}}. \tag{25}$$

Thus, the first three terms in the ART solution for the amplitude term, which is a series in terms of $(i\omega)^{-n}$, $(n=0,1,\ldots)$, for the two-dimensional scalar wave equation is of the form

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[2] Integration factor: $I_F=\exp\left(\int\frac{dR}{2R}\right)=\sqrt{R}$

$$\phi_{ART}(R,t) \approx C\left[\frac{e^{i\omega(t-\tau)}}{\sqrt{R}}\left\{1-\frac{1}{8ikR}+\frac{9}{2!8^2(ikR)^2}\cdots\right\}\right] \quad \text{with} \quad \left(k=\frac{\omega}{\alpha}\right) \tag{26}$$

The quantity, C, is some integration constant to be determined or specified. If compared with the asymptotic expansion of the exact solution for this problem, which is $\phi(x,z,t)=e^{i\omega t}H_0^{(2)}(\xi)$ for large $\xi = kR$ [or equivalently $\phi(x,z,t)=e^{-i\omega t}H_0^{(1)}(\xi)$] depending on which sign of the Fourier time transform is taken in the exact solution. The asymptotic expansion for large argument values is given by

$$\phi_{ex}(R,t)=e^{i\omega t}H_0^{(2)}(kR)\approx e^{i(\omega t-i\omega\tau)}e^{i\pi/4}\sqrt{\frac{2}{\pi kR}}\left\{1-\frac{1}{8ikR}+\frac{9}{2!8^2(ikR)^2}-\cdots\right\}. \tag{27}$$

It becomes clear that the time harmonic asymptotic solution series in equation (26) is, apart a the factor, $\sqrt{2/k\pi}\; e^{-i\pi/4}$, equivalent to the Hankel function of type (2) and order zero.

The second term in equation (27) is not the exact expression for the near field term but is as reasonable approximation. It is of equivalent merit in this capacity as the first term is for the far field term, and it follows from a derivation of reasonable rigor. It may also be seen that this solution displays an oscillatory but damped behaviour.

THREE DIMENSIONAL CASE

The transport equation (zero order) in the three dimensional case is as in the $2D$ case

$$A_0\nabla^2\tau + 2\nabla A_0\cdot\nabla\tau = 0 \tag{28}$$

For $N=3$, the three-dimensional scalar wave equation in (x,y,z) space, (in spherical coordinates) in terms of the independent variable $R=\left(x^2+y^2+z^2\right)^{1/2}$ or more generally, $R=\left((x-x_0)^2+(y-y_0)^2+(z-z_0)^2\right)^{1/2}$), has the following solution for the zero order transport equation (equation (14))[3]

[3] $$\tau(x,y,z)=\frac{\left((x-x_0)^2+(y-y_0)^2(z-z_0)^2\right)^{1/2}}{\alpha}=\frac{R}{\alpha}$$

$$\nabla\tau=\frac{\partial\tau}{\partial R}\mathbf{u_R}=\frac{\mathbf{u_R}}{\alpha}$$

$$\nabla^2\tau=\frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial\tau}{\partial R}\right)=\frac{2}{\alpha R} \quad\rightarrow\quad \nabla^2 * =\frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial *}{\partial R}\right) \quad \text{(Polar Coordinates)}$$

$$A_0\left(x,y,z\right)=A_0\left(R\right)=\frac{1}{R} \tag{29}$$

The second term in the asymptotic equation for $A_1\left(R\right)$ is the solution of the following equation, which is dependent on $A_0\left(R\right)$, and is obtained from

$$A_1\nabla^2\tau+2\nabla A_1\cdot\nabla\tau=\nabla^2 A_0 \tag{30}$$

where $A_0\left(\mathbf{x}\right)$ is known. In the $3D$ case, the quantity on the RHS of (30) is given by

$$\nabla^2 A_0=\frac{1}{R}\frac{\partial}{\partial R}\left(R^2\frac{\partial}{\partial R}\left(\frac{1}{R}\right)\right)=0\,. \tag{31}$$

The following sequence of operations

$$\begin{aligned}
&A_1\frac{2}{\alpha R}+\frac{2}{\alpha}\frac{\partial A_1}{\partial R}=0\\
&\frac{A_1}{R}+\frac{\partial A_1}{\partial R}=0\\
&\frac{1}{R}\frac{\partial\left(RA_1\right)}{\partial R}=0
\end{aligned} \tag{32}$$

results in

$$RA_1=\text{constant}\quad\rightarrow\quad A_1=\frac{\text{constant}}{R} \tag{33}$$

The constant is arbitrary, so as previously it is chosen equal to 1. In a similar manner, $A_2=1/R$.

Thus as a result of the above equations all terms in the asymptotic series, $n\geq 0$, are of the form

$$A_n=\frac{1}{R\left(i\omega\right)^n} \tag{34}$$

so that the asymptotic series defined in equation (2), for this $3D$ case, apart from some multiplicative integration constant, is

$$\phi\left(R,t\right)=e^{i\omega t}\left[\frac{e^{-ikR}}{R}\right]\sum_{n=0}^{\infty}\frac{1}{\left(i\omega\right)^n} \tag{35}$$

The exact (time harmonic) solution for the *3D* wave equation for the geometry used here is given by Cerveny and Ravindra (1971) as

$$\phi(R,t)=\frac{\exp\left[i\omega t-ikR\right]}{ikR}\quad\left[kR=\omega\left(R/\alpha\right)=\omega\tau\left(R\right)\right] \tag{36}$$

and from this, the ART solution is

$$\phi(R,t)=\frac{\exp\left[i\omega\left(t-\tau\left(R\right)\right)\right]}{R}\sum_{n=0}^{\infty}\frac{1}{\left(i\omega\right)^{n}} \tag{37}$$

For these two solution method to have the same basic form, it is required that the series, $\left[\sum_{n=0}^{\infty}\frac{1}{\left(i\omega\right)^{n}}\right]\to 1$, in the high frequency approximation. This series summation is discussed in Appendix A. It may be observed that the $3D$ solution displays a much different behaviour than the $2D$ case, as there is no oscillatory motion.

N DIMENSIONAL CASE

The transport equation (zero order) in the $N-$dimensional case is as in the $2D$ and $3D$ cases

$$A_0\,\partial_{x_i x_i}\tau+2\,\partial_{x_i}A_0\,\partial_{x_i}\tau=0,\quad\left(i=1,N\right) \tag{38}$$

The solution for the N-dimensional scalar wave equation in the x_i hyperspace is in terms of the independent variable $R=\left(\sum_{i=1}^{N}x_i^2\right)^{1/2}$, or more generally, $R=\left(\sum_{i=1}^{N}\left(x_i-x_i^0\right)^2\right)^{1/2}$. Defining $\tau\left(x_i\right)$ in terms of R and the constant velocity α has $\tau\left(x_i\right)=\frac{R}{\alpha}$

In $N-$dimensions, the operator $\partial_{x_i x_i}\left[\nabla^2\right]$ is the equivalent of that in 3-dimensional polar coordinates having the form, $\frac{1}{R^{N-1}}\frac{\partial}{\partial R}\left(R^{N-1}\frac{\partial}{\partial R}\right)$. Starting with the expression for $\tau\left(x_i\right)$ in $N-$dimensions and applying the $N-$dimensional Laplacian the following results

$$\begin{aligned}\partial_{x_i x_i}\tau&=\frac{1}{R^{N-1}}\frac{\partial}{\partial R}\left(R^{N-1}\frac{\partial\tau}{\partial R}\right)=\frac{1}{\alpha R^{N-1}}\frac{\partial}{\partial R}\left(R^{N-1}\frac{\partial R}{\partial R}\right)\\&=\frac{1}{\alpha R^{N-1}}\frac{\partial}{\partial R}\left(R^{N-1}\right)=\frac{\left(N-1\right)R^{N-2}}{\alpha R^{N-1}}=\frac{\left(N-1\right)}{\alpha R}\end{aligned} \tag{39}$$

Using this relationship, it may be seen through a sequence of steps that

$$\begin{aligned}
&\frac{A_0\left(N-1\right)}{\alpha R}+\frac{2\partial A_0}{\alpha\partial R}=0\\
&\frac{\partial A_0}{A_0}=-\frac{\partial R}{\left[\left(N-1\right)/2\right]R}\\
&\ln A_0=-\frac{1}{\left[\left(N-1\right)/2\right]}\ln R\\
&A_0=\frac{1}{R^{(N-1)/2}}
\end{aligned}\tag{40}$$

which is consistent with what was derived for N=2 and 3. To obtain the first order term, $\nabla^2 A_0$ is required and given by

$$\nabla^2 A_0=\frac{1}{R^{N-1}}\frac{\partial}{\partial R}\left(R^{N-1}\frac{\partial}{\partial R}\left(\frac{1}{R^{(N-1)/2}}\right)\right)=\frac{-\left(N-1\right)\left(N-3\right)}{2^2 R^{(N+3)/2}}\tag{41}$$

For $N=4$ and $N=5$ equation (41) becomes

$$\nabla^2 A_0=\frac{-\left(4-1\right)\left(4-3\right)}{2^2 R^{(4+3)/2}}=\frac{-3}{4R^{7/2}}\quad\left(N=4\right)\tag{42}$$

and

$$\nabla^2 A_0=\frac{-\left(5-1\right)\left(5-3\right)}{4R^{(5+3)/2}}=\frac{-2}{R^4}\quad\left(N=5\right)\tag{43}$$

Continuing with the solution method used for the $N=2$ and 3 cases has

$$A_1\nabla^2\tau+2\nabla A_1\cdot\nabla\tau=\nabla^2 A_0\tag{44}$$

or in the equivalent of an N dimensional polar coordinate system

$$\frac{\partial A_1}{\partial R}+A_1\frac{\left(N-1\right)}{2R}=\frac{-\alpha\left(N-1\right)\left(N-3\right)}{2^3 R^{(N+3)/2}}\tag{45}$$

Introducing an integrating factor into (45) results in

$$\frac{\partial}{\partial R}\left[R^{(N-1)/2}A_1\right]=\frac{-\alpha\left(N-1\right)\left(N-3\right)\partial R}{2^3 R^2}\tag{46}$$

the solution of which is

$$A_1=\frac{\alpha\left(N-1\right)\left(N-3\right)}{2^3 R^{(N+1)/2}}\tag{47}$$

The second order term is determined in a manner similar to that for the first order term. First evaluate $\nabla^2 A_1$

$$\nabla^2 A_1 = \frac{\alpha(N-1)(N-3)}{2^3 R^{N-1}} \frac{\partial}{\partial R}\left(R^{N-1}\frac{\partial}{\partial R}\left(\frac{1}{R^{(N+1)/2}}\right)\right) \tag{48}$$

$$\nabla^2 A_1 = \frac{-\alpha(N+1)(N-1)(N-3)(N-5)}{2^5 R^{(N+5)/2}} \tag{49}$$

Proceeding requires the solution of

$$A_2 \nabla^2 \tau + 2\nabla A_2 \cdot \nabla \tau = \nabla^2 A_1 , \tag{50}$$

which again after the integrating factor implementation becomes

$$\frac{\partial\left[R^{(N-1)/2} A_2\right]}{\partial R} = \frac{-\alpha^2 (N+1)(N-1)(N-3)(N-5) R^{(N-1)/2}}{2^6 R^{(N+5)/2}} . \tag{51}$$

Solving the above equation yields

$$A_2 = \frac{\alpha^2 (N+1)(N-1)(N-3)(N-5)}{2^7 R^{(N+3)/2}} . \tag{52}$$

Combining the terms in the asymptotic series has, apart from some integration constant

$$\sum_{j=1}^{\infty} \frac{A_j(R)}{(i\omega)^j} e^{i\omega(t-\tau)} = \frac{e^{i\omega(t-\tau)}}{R^{(N-1)/2}}\left[1 + \frac{(N-1)(N-3)}{(ik)2^3 R} + \frac{(N+1)(N-1)(N-3)(N-5)}{(ik)^2 2^7 R^2} + \right. \tag{53}$$

Inspection can indicate the next terms in the series. For $N = 3$ one term is present while for $N = 5$, two terms. For $N = 3$, this is consistent with that derived earlier and given in equation (35).

$$\phi(R,t) = \frac{e^{i\omega(t-\tau(R))}}{ikR} \tag{54}$$

Comparison of (35) and (54) results in a difference of importance. The expression given in equation (54) does not require that the series $\sum_{n=0}^{\infty} \frac{1}{(i\omega)^n}$ be summed. For a higher dimensional solution $(N = 5)$

$$\phi(R,t) = \frac{e^{i\omega(t-\tau(R))}}{R^2}\left[1 + \frac{1}{ikR}\right] \tag{55}$$

If N is even the solution is as in the case of the $N = 2$ problem an infinite series displaying a damped oscillation for all values of N.

Compressional wave displacement, $\mathbf{u}\left(R,t\right)$, is often defined in terms of this type of potential as

$$\mathbf{u}\left(R,t\right)=\nabla\phi\left(R,t\right). \tag{56}$$

This relation may be applied to (54) and (55) to obtain displacements.

CONCLUSIONS

It has been shown that the asymptotic solutions for the 2 and 3 dimensional scalar wave equations in an infinite isotropic homogeneous medium are what one would assume to be when compared with the exact solutions for the same problem. The asymptotic solutions are obtained using the inhomogeneous wave equation with no source conditions required to be specified which is not the case when the exact solution is sought. This fairly loose manner of setting out the problem allows for many options in the solution method when more complex medium types are considered. This includes the introduction of interfaces across which the velocity may be discontinuous and the possibility of an arbitrary inhomogeneous specification of the velocity field. These added complexities cannot be solved using a Sommerfeld type integral method employed in obtaining the exact solution in an infinite halfspace of a similar type.

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APPENDIX: SERIES SUMMATION

Assume a series expression of the form, with $i=\sqrt{-1}$, where n is such that $n\to\infty$,

$$S_n=\sum_{n=0}^{n}\frac{1}{\left(i\omega\right)^n}. \tag{A.1}$$

Expanding the above into individual terms to obtain

$$S_n=\sum_{n=0}^{n}\frac{1}{\left(i\omega\right)^n}=1-\frac{i}{\left(\omega\right)^1}-\frac{1}{\left(\omega\right)^2}+\frac{i}{\left(\omega\right)^3}+\frac{1}{\left(\omega\right)^4}-\frac{i}{\left(\omega\right)^5}\;\cdots\;+\frac{1}{\left(i\omega\right)^n} \tag{A.2}$$

Separate the expanded series into real and imaginary parts

$$S_n = S_n^{(R)} + iS_n^{(I)} \tag{A.3}$$

yielding

$$S_n^{(R)} = 1 - \frac{1}{(\omega)^2} + \frac{1}{(\omega)^4} \ldots + \frac{(-1)^{n/2}}{(\omega)^{2n}} \quad (n \in \mathbb{N}) \tag{A.4}$$

and

$$S_n^{(I)} = -\frac{1}{(\omega)^1} + \frac{1}{(\omega)^3} - \frac{1}{(\omega)^5} \ldots + \frac{(-1)^{(n+1)/2}}{(\omega)^{2n+1}} \quad (n \in \mathbb{N}) \tag{A.5}$$

Multiple both sides of (A.4) by $\frac{1}{\omega^2}$

$$\frac{S_n^{(R)}}{\omega^2} = \frac{1}{\omega^2} - \frac{1}{\omega^4} + \frac{1}{\omega^6} - \cdots \frac{1}{\omega^n} \tag{A.6}$$

which after rearrangement becomes

$$S_n^{(R)} = 1 - \frac{1}{\omega^2} + \frac{1}{\omega^4} - \frac{1}{\omega^6} + \cdots \frac{1}{\omega^n}. \tag{A.7}$$

Adding this formulae to (A.4) results in

$$\left(1 + \frac{1}{\omega^2}\right) S_n^{(R)} = 1 \quad \rightarrow \quad S_n^{(R)} = \frac{\omega^2}{\left(1+\omega^2\right)} \approx \frac{\omega^2}{\omega^2}. \tag{A.8}$$

so that in the high frequency limit, $\omega \gg 1$, it may be seen that (A.8) has the limit

$$S_n^{(R)} = \frac{\omega^2}{\left(1+\omega^2\right)} \quad \rightarrow \quad 1 \quad \text{for } \omega \gg 1. \tag{A.9}$$

Now consider the imaginary part of the series (A.2), which may be written as

$$S_n^{(I)} = -\frac{1}{\omega} + \frac{1}{\omega^3} - \frac{1}{\omega^5} + \frac{1}{\omega^7} - \cdots \frac{1}{\omega^n}. \tag{A10}$$

Multiply both sides of equation (A.10) by $1/\omega^2$ to obtain

$$\frac{1}{\omega^2} S_n^{(I)} = -\frac{1}{\omega^3} + \frac{1}{\omega^5} - \frac{1}{\omega^7} + \frac{1}{\omega^9} \cdots \frac{1}{\omega^{n+2}} \cdots. \tag{A.11}$$

Adding this result and equation (A.5) produces

$$\left(1+\frac{1}{\omega^2}\right)S_n^{(I)} = -\frac{1}{\omega} \qquad \text{(A.12)}$$

or after some rearrangement

$$S_n^{(I)} = -\frac{\omega}{\left(\omega^2+1\right)} \approx -\frac{1}{\omega}. \qquad \text{(A.13)}$$

Again assuming that $\omega \gg 1$, equation (A.13) may be written as

$$S_n^{(I)} \to 0 \quad \left[\omega \to \infty\right]. \qquad \text{(A.15)}$$

Thus the high frequency limit of the series (A.1) as $\omega \to \infty$ is simply

$$S_n = 1. \qquad \text{(A.16)}$$

It should be mentioned that if the condition $\omega \gg 1$ is imposed, the series summation is independent of the number of terms in the series, i.e., independent of n, but is valid for any value n, $n \ge 1$.