A hybrid method applied to a scalar (almost) 3D S_H wave equation

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ABSTRACT

Seismic modeling of shear (S_H) wave propagation in a three dimensional structure using the potential scalar wave equation is considered. A combination of finite difference and finite integral transform methods are employed for this purpose. It is assumed that the variation in the medium parameters is smoothly varying in one of the spatial dimensions, and as such can be removed from the finite difference problem through the use of a finite integral transform. Although what will be considered is not a true $3D$ model it does allow for some inhomogeneity in all spatial dimensions and as a consequence provides an intermediate point between a $2.5D$ and a truly $3D$ situation that provides a computational basis that can be expanded upon to address a general fully inhomogeneous $3D$ problem.

INTRODUCTION

A sampling of references in the geophysical literature on the topic of using a combination of finite integral transforms and finite difference method are Alekseev and Mikhailenko (1980), Kosloff and Baysal (1982), and Mikhailenko (1985), where additional references may be found.. However, much of the relevant literature, to this solution method is relatively inaccessible in that they appear in journals and texts written in Russian and is presented at a fairly mathematically intense level (Citations of these are not included here.) Papers in English by Russian scientists engaged in this research area are most often found in mathematical journals and contain little in the way of numerical implementation. For these reason it was deemed appropriate to investigate a simple problem in this area and deal with the more important numerical considerations that arise.

A combination of finite difference and finite integral transform techniques is employed to obtain a numerical solution to the S_H scalar potential equation in a media type approaching that which could be classified as three dimensional. Its initial development was to compensate for hardware limitations, requiring less physical computer resources, with the trade off of a more complex algorithm but similar run times for modeling complex geological structures by conventional finite difference methods. Consideration of this simple problem within the context of hybrid solution techniques, has the ability to conveniently both introduce the method's concept and present computational measures that maximize the algorithm's efficiency, topics not usually dealt with in the literature. The transformed problem is highly parallel and amenable to vectorization techniques.

The use of variations of source and receiver patterns can enhance the $3D$ modeling capabilities of what is essentially a $2.5D+$ solution with $3D$ kinematics and dynamics (geometrical spreading). As several lines of receivers are recorded, a type of rebinning, using traces from individual lines may be used to produce receiver lines oriented at specific angles across the $3D$ array of surface receivers, within a single run.

BASIC THEORY

Assume a $3D$ elastic medium in a Cartesian coordinate system that is generally inhomogeneous in two spatial dimensions and what will be termed *mildly* inhomogeneous in the third spatial dimension, say the $y-$ direction. The general $S_H -$wave potential equation for some scalar amplitude, $\phi = \phi\left(x, y, z, t\right)$, is given by

$$\nabla \cdot \left[\beta^2\left(\mathbf{x}\right)\nabla\phi\left(\mathbf{x},t\right)\right] - \partial_t^2\phi = \delta\left(\mathbf{x}-\mathbf{x}_0\right) f\left(t\right) \tag{1}$$

where $\beta\left(\mathbf{x}\right)$ is the shear wave velocity defined in terms of the Lame parameter and volume density as $\beta^2\left(\mathbf{x}\right) = \mu\left(\mathbf{x}\right)/\rho\left(\mathbf{x}\right)$, $\delta\left(\mathbf{x}\right)-$ indicates a point source of S_H waves and $f\left(t\right)$ is a band limited wavelet, which will be discussed in more detail later. The unphysical, but mathematically convenient assumption that $\rho\left(\mathbf{x}\right) \equiv 1$ mass unit per volume unit in the entire model is made. Assume further that that the velocity may be specified in the form

$$\beta^2\left(x, y, z\right) = \beta^2\left(x, z\right) \pm \frac{\Delta\beta^2\left(x, z\right)}{2}\left(1 - \cos\frac{\ell\pi y}{c}\right), \quad \text{for some integer } \ell . \tag{2}$$

which is the simplest form of the velocity specification

$$\beta^2\left(x, y, z\right) = \beta_0^2\left(x, z\right) + \sum_{\ell=0}^{\infty}\Delta\beta_\ell^2\left(x, z\right)\left(1 - \cos\frac{\ell\pi y}{c}\right). \tag{3}$$

Only one term in the cosine series will be used in to introduce the method numerically with minimal complexity. Let

$$\beta^2\left(x, y, z\right) = A\left(x, z\right) - B\left(x, z\right)\cos\ell\pi y/c . \tag{4}$$

so that when compared with (2) the quantities $A\left(x, z\right)$ and $B\left(x, z\right)$ are explicitly defined as

$$A\left(x, z\right) = \beta^2\left(x, z\right) + \frac{\Delta\beta^2\left(x, z\right)}{2} \quad \text{and} \quad B\left(x, z\right) = \frac{\Delta\beta^2\left(x, z\right)}{2} \tag{5}$$

The $3D$ coordinate system is chosen so that the vertical coordinate, *z*, is positive downwards. The initial value problem is fully specified by introducing the conditions

$$\phi\big|_{t=0} = \partial_t\phi\big|_{t=0} = 0 . \tag{6}$$

Wave propagation, due to a point source excitation of $S_H -$waves, will be assumed to be confined to the spatial volume $\left(0 \le x \le a ; 0 \le z \le b ; 0 \le y \le c\right)$. Only four of these six boundaries will initially be taken to be perfectly reflecting, $\left(x = 0 \text{ and } a ; z = 0 \text{ and } b\right)$ as the finite integral transform may require that other conditions be specified at $\left(y = 0 \text{ and } c\right)$. These preliminary boundary conditions require that some measures such as absorbing boundaries (Clayton and Enquist, 1977 and Reynolds, 1978) or attenuating boundaries (for example, Cerjan et al., 1985) be incorporated in the solution method so that spurious reflections from them will not contaminate, to any significant extent, the wave field propagating within the spatial volume and recorded at the receivers. The y spatial dimension is to be temporarily removed employing a finite cosine transform, which requires the specification of distinct boundary conditions at $y = 0$ and $y = c$ in the transform procedure

Spurious reflections from the boundaries at $y = 0$ and $y = c$ can be eliminated by setting the source at a sufficient distance from distance from both of these boundaries so that reflections from them do not arrive at the receivers within the specified time window of the synthetic trace at any given receiver.

Finite cosine transform:

If the function $\phi(y)$ satisfies the Dirichlet conditions in the interval $(0,c)$ and if in this interval the relation

$$\Phi(n) = \int_0^c \phi(y)\cos\left(\frac{n\pi y}{c}\right)dy \tag{7}$$

is valid at all points in the interval $(0,c)$, where the function $\phi(y)$ is continuous, the following equality

$$\phi(y) = \frac{\Phi(0)}{c} + \frac{2}{c}\sum_{n=1}^{\infty}\Phi(n)\cos\left(\frac{n\pi y}{c}\right) \equiv \frac{2}{c}\sum_{n=0}^{\infty}\Phi(n)\cos\left(\frac{n\pi y}{c}\right) \tag{8}$$

holds.

It is understood that the $n = 0$ term has been included in the summation in the last term in equation (8) for convenience of notation. Some upper bound on the summation must be determined that adequately approximates the infinite series. This number has a linear dependence on the distance c and is dependent on the spectral content of the source wavelet, which as previously stated has been assumed to be bad limited for this exercise.

Applying the finite cosine transform to equation (1) leads to the intermediate finite difference analogue

$$
\begin{aligned}
&\Phi^{p+1}(n) = -\Phi^{p-1}(n) + 2\Phi^{p}(n) + \\
&(\Delta t)^2 \Bigg\{ \partial_x \left[A(x,z)\partial_x \Phi(n) \right] - \frac{1}{2}\Big(\partial_x \left[B(x,z)\partial_x \Phi(n+\ell) \right] + \partial_x \left[B(x,z)\partial_x \Phi(n-\ell) \right] \Big) - \\
&\frac{n^2\pi^2}{c^2} A(x,z)\Phi(n) + B(x,z)\frac{\ell\pi^2}{2c^2}\left[(n+\ell)\Phi(n+\ell) + (n-\ell)\Phi(n-\ell) \right] + \\
&\partial_z \left[A(x,z)\partial_z \Phi(n) \right] - \frac{1}{2}\Big(\partial_z \left[B(x,z)\partial_z \Phi(n+\ell) \right] + \partial_z \left[B(x,z)\partial_z \Phi(n-\ell) \right] \Big) - \\
&\delta(x-x_0)\cos\left(\frac{n\pi y_0}{c} \right)\delta(z-z_0) f(t) \Bigg\}^{p}
\end{aligned}
\tag{9}
$$

with the abbreviated notation $\Phi(n) = \Phi(x,n,z,p\Delta t)$ being used in the above equation.

The spatial sampling rate in the (x,z) directions was taken to be h and the time step, was set as Δt. Some stability condition, which is required for the finite difference part of the problem, may be shown to be of the form

$$
\Delta t \le \sqrt{\frac{3}{8}} \frac{h}{\beta_{\max}\left[1 + n_{\max}^2 \pi^2 h^2 / c^2 \right]^{1/2}}
\tag{10}
$$

where $\beta_{\max}$ is the maximum value of shear wave velocity encountered in the entire $3D$ space and $n_{\max}$ the number of terms in the inverse cosine series which adequately approximates the infinite series. Both the stability condition and number of terms in the inverse series summation are treated in detail in (Daley et al., 2008) and consequently will not be addressed here.

The quantities $A_{p,q}$ and $B_{p,q}$ are the harmonic averages used for any of the (x,z) spatially dependent parameters in equation (9) with $A_{p,q}$ given by

$$
A_{p,q} = \frac{2\mu_{p,q}\mu_{p-1,q}}{\mu_{p,q} + \mu_{p-1,q}}
\tag{11}
$$

used in derivatives involving the x dimension as

$$
\frac{\partial}{\partial x}\left[\mu(x,z)\frac{\partial \phi}{\partial x} \right] \approx \frac{A_{p,q}\phi_{p+1,q}^{\ell} - \left(A_{p,q} + A_{p-1,q} \right)\phi_{p,q}^{\ell} + A_{p-1,q}\phi_{p-1,q}^{\ell}}{h^2}
\tag{12}
$$

and $B_{p,q}$ specified by

$$B_{p,q} = \frac{2\mu_{p,q}\mu_{p,q-1}}{\mu_{p,q} + \mu_{p,q-1}} \tag{13}$$

used in derivatives in z in a manner similar to equation (12). This leads to the extension of equation (9) to the more complete finite difference analogue

$$\begin{aligned}
&\Phi^{p+1}(n) = -\Phi^{p-1}_{i,j}(n) + 2\Phi^{p}_{i,j}(n) + \\
&\frac{(\Delta t)^2}{h^2}\Big\{\Big[A_{i,j}\Phi^{p}_{i+1,j}(n) - \big(A_{i,j} + A_{i-1,j}\big)\Phi^{p}_{i,j}(n) + A_{i-1,j}\Phi^{p}_{i-1,j}(n)\Big] - \\
&\frac{1}{2}\Big[B_{i,j}\Phi^{p}_{i+1,j}(n+\ell) - \big(B_{i,j} + B_{i-1,j}\big)\Phi^{p}_{i,j}(n+\ell) + B_{i-1,j}\Phi^{p}_{i-1,j}(n+\ell)\Big] - \\
&\frac{1}{2}\Big[B_{i,j}\Phi^{p}_{i+1,j}(n-\ell) - \big(B_{i,j} + B_{i-1,j}\big)\Phi^{p}_{i,j}(n-\ell) + B_{i-1,j}\Phi^{p}_{i-1,j}(n-\ell)\Big] - \\
&\frac{h^2 n^2 \pi^2}{c^2} A_{ij}\Phi^{p}_{i,j}(n) + \\
&\frac{h^2 \ell^2 \pi^2}{c^2} B_{i,j}\Big[(n+\ell)\Phi^{p}_{i,j}(n+\ell) + (n-\ell)\Phi^{p}_{i,j}(n-\ell)\Big] + \\
&\Big[A_{i,j}\Phi^{p}_{i,j+1}(n) - \big(A_{i,j} + A_{i,j-1}\big)\Phi^{p}_{i,j}(n) + A_{i,j-1}\Phi^{p}_{i,j-1}(n)\Big] - \\
&\frac{1}{2}\Big[B_{i,j}\Phi^{p}_{i,j+1}(n+\ell) - \big(B_{i,j} + B_{i,j-1}\big)\Phi^{p}_{i,j}(n+\ell) + B_{i,j-1}\Phi^{p}_{i,j-1}(n+\ell)\Big] - \\
&\frac{1}{2}\Big[B_{i,j}\Phi^{p}_{i,j+1}(n-\ell) - \big(B_{i,j} + B_{i,j-1}\big)\Phi^{p}_{i,j}(n-\ell) + B_{i,j-1}\Phi^{p}_{i,j-1}(n-\ell)\Big] - \\
&\delta(x - x_0)\cos\left(\frac{n\pi y_0}{c}\right)\delta(z - z_0) f(t)\Big\}
\end{aligned} \tag{14}$$

It is implicit in the above equation that both the terms $A_{m,n}$ and $B_{m,n}$ represent the harmonic averages of these quantities and not merely their values at the (x,z) grid point (i,j) (see equations (11) and (13)). The time step is Δt and the spatial step, $h = \Delta x = \Delta z$.

The boundary at the surface of the model, $z = 0$, is assumed to be perfectly reflecting, a not unrealistic assumption, based on the condition that $\phi(x,y,z,t)\big|_{z=0}$ be continuous, and the half space $z < 0$ is taken to be a vacuum, making the model boundary perfectly reflecting there. If so required it may also be made absorbing.

The saving in space, having to use only $2D$ arrays to specify the elastic parameter and density, requires an expenditure of computational time when compared to a $2D$ finite difference algorithm as equation (15) must be solved for those values of n, $(0 \le n \le n_{\max})$, where $n_{\max}$ is the number of terms in the inverse cosine series that reasonably approximates the infinite series. However, it is comparable in run time to a truly $3D$ finite difference algorithm as the $2D$ finite difference computations must be undertaken $n_{\max}+1$ times. The exterior loop is over n followed by the time loop and then the loops over the spatial dimensions x and z.

DISCUSSION AND CONCLUSIONS

The problem of an S_H wave propagating in a medium, which has been termed *almost* three dimensional, is presented. This preliminary problem where the S_H (velocity)^2 in the $y-$direction in a $3D$ Cartesian is approximated by

$$\beta^2(x,y,z)=\beta^2(x,z)\pm\frac{\Delta\beta^2(x,z)}{2}\left(1-\cos\frac{\ell\pi y}{c}\right),\quad \text{for some integer } \ell$$

presents the basic formulation for the use of hybrid methods (finite integral transforms combined with finite differences) in a truly $3D$ geometrical structure. Formally, this is accomplished by using equation (3 rather than (2) to specify the velocity variation in the $y-$direction. As with other chapters in this report by the author, software was written and tested for this problem in the early 1980s. However, as in the other cases, all source code and ancillary documentation, has been lost. It should be mentioned that software related to this report has been rewritten in a generic form, as the programming language that will be most useful is yet to determined. The use of the hybrid method provides a significant saving in memory with computational time comparable to a similar geological structure described by a $3D$ gridded model. The hybrid method is also highly parallel.

A schematic of a simple type of the problems that may be solved using this method is shown in Figure 1, while a selection of receiver line options is given in Figure 2. It has been assumed that the source lies at or near the surface as are the receivers. It may be seen that the receiver spacing is not constant for all of the lines.

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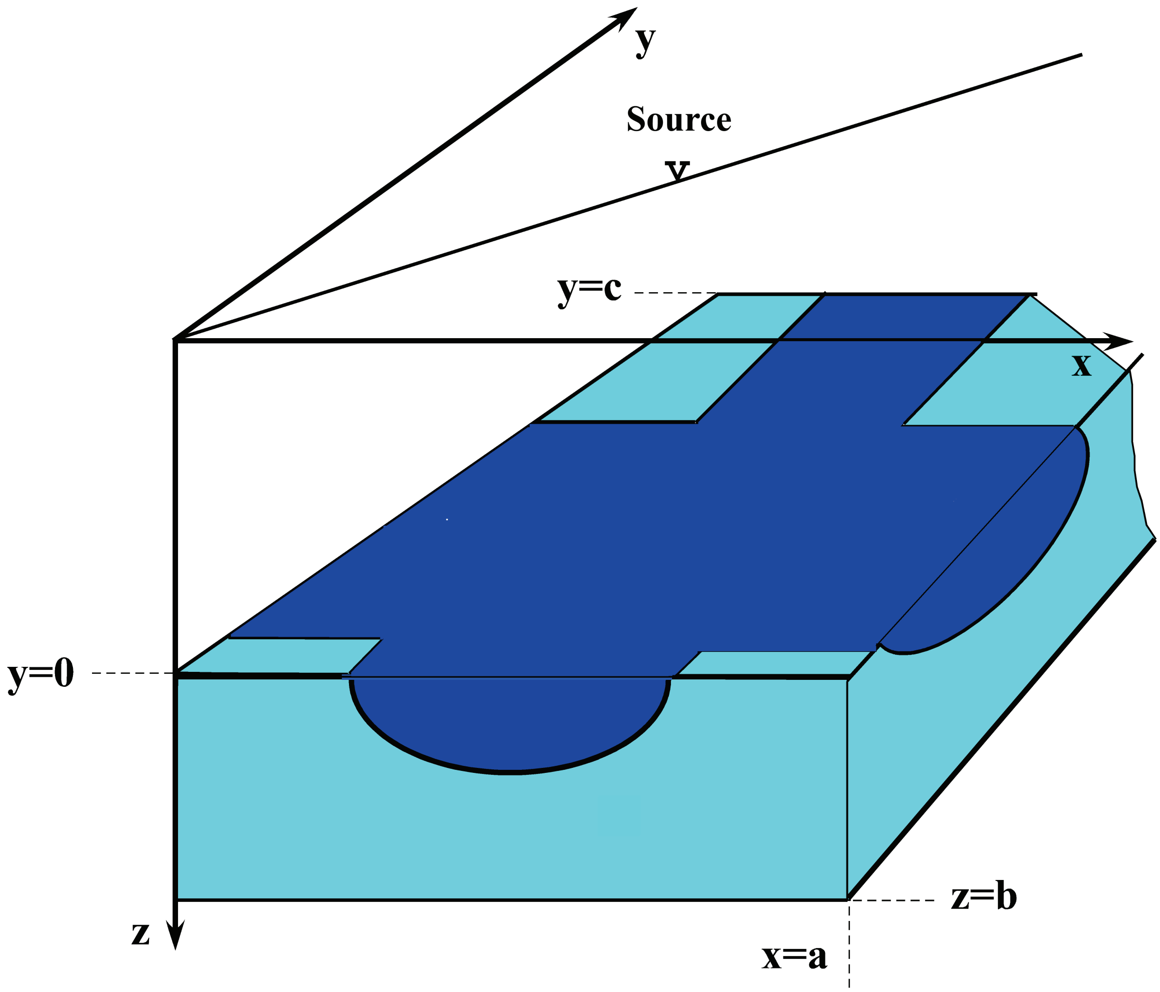

Fig.1. Schematic of a model that may be treated using the method described here. The computational model employed is similar with a source located at $z_S = 0$, $0 < x_S < a$, $0 < y_S < c$.

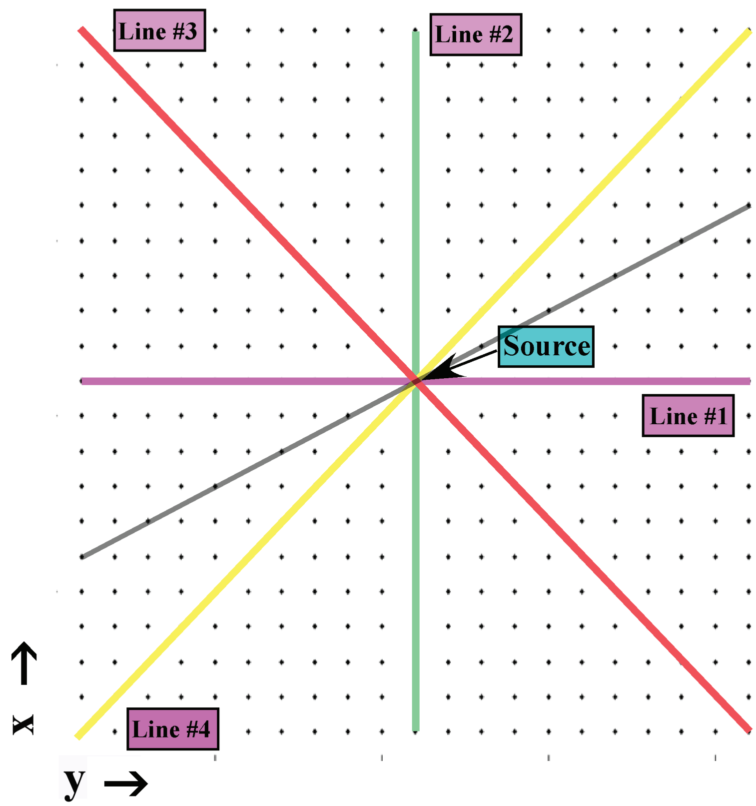

Fig. 2. Schematic of shooting geometry. A single source as indicated and an $n \times n$ surface array of receivers with $\Delta x = \Delta y = h$. Other possible receiver lines are shown. Note that the receiver lines at an angle to the (x, y) spatial points have different spatial increments between receivers.