Hidden nonlinearities in the Aki-Richards approximation

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ABSTRACT

The Aki-Richards approximation comes in two forms, one involving the incidence angle and the other involving the average of the incidence and transmission angles. The first of these may be straightforwardly derived by expanding a matrix form of the Knott-Zoeppritz equation in series and truncating. The second is formally a linearization but is more reasonably interpreted as being nonlinear, and this can be quantified by expanding the average angle in series about the P-wave velocity perturbation. The Aki-Richards approximation is often discussed in terms of P-wave, S-wave, and density reflectivities. The average angle too may be expressed in terms of the incidence angle and the P-wave reflectivity, with the latter perturbing the former.

INTRODUCTION

The Aki-Richards approximation comes in two forms, one involving the incidence angle and the other involving the average of the incidence and transmission angles. The two behave differently, especially at large angle. The purpose of this paper is to interpret this difference in terms of "hidden" nonlinearities in the second form.

In this paper we:

1. Derive a linearization of the Knott-Zoeppritz equations involving the incidence angle only (the Aki-Richards approximation version I.);
2. State the Aki-Richards approximation in its two most common forms, the first involving the incidence angle (version II.A) and the second involving the average angle (version II. B);
3. Show that version I. is equivalent to version II.A;
4. Demonstrate the nonlinearity of version II.B;
5. Reproduce the increased accuracy provided by the use of the average angle in version II.B, by perturbing the incidence angle with the P-wave reflectivity.

I. A LINEARIZATION OF THE KNOTT-ZOEPPRITZ EQUATIONS

Elsewhere we have expanded the Knott-Zoeppritz equations, both for elastic and anelastic media, in series, and discussed both the inversion of the series, and the importance of second order and higher terms (Innanen, 2011). If we truncate the series for elastic R_{PP} beyond the linear term, we arrive at a straightforward linearization that should be comparable to the Aki-Richards approximation. Briefly, we begin with a portion of the Zoeppritz equations expressed in matrix form:

$$\mathbf{P}\begin{bmatrix} R_{\text{PP}} \\ R_{\text{PS}} \\ T_{\text{PP}} \\ T_{\text{PS}} \end{bmatrix} = \mathbf{b}_P, \tag{1}$$

where

$$\mathbf{P} \equiv \begin{bmatrix} -X & -\Gamma_B(X) & CX & \Gamma_D(X) \\ \Gamma_1(X) & -BX & \Gamma_C(X) & -DX \\ 2B^2X\Gamma_1(X) & B\Gamma^B(X) & 2AD^2X\Gamma_C(X) & AD\Gamma^D(X) \\ -\Gamma^B(X) & 2B^2X\Gamma_B(X) & 2AC\Gamma^D(X) & -2AD^2X\Gamma_D(X) \end{bmatrix},$$

$$\mathbf{b}_P \equiv \begin{bmatrix} X \\ \Gamma_1(X) \\ 2B^2X\Gamma_1(X) \\ \Gamma^B(X) \end{bmatrix},$$

and A–D are ratios in terms of parameters in the incidence medium (0) and the target medium (1):

$$A \equiv \frac{\rho_1}{\rho_0}, \quad B \equiv \frac{V_{S_0}}{V_{P_0}}, \quad C \equiv \frac{V_{P_1}}{V_{P_0}}, \quad D \equiv \frac{V_{S_1}}{V_{P_0}}, \tag{2}$$

and finally

$$\begin{aligned} \Gamma_j(X) &\equiv \sqrt{1 - j^2X^2}, \\ \Gamma^j(X) &\equiv 1 - 2j^2X^2, \end{aligned} \tag{3}$$

and $X \equiv \sin\theta_0$. We also form an auxiliary matrix $\mathbf{P}_p$ by replacing the first column in $\mathbf{P}$ with $\mathbf{b}_P$. We next assign to each of the three parameters which may undergo variation at the boundary a perturbation:

$$a_{VP} = 1 - \frac{V_{P_0}^2}{V_{P_1}^2}, \quad a_{VS} = 1 - \frac{V_{S_0}^2}{V_{S_1}^2}, \quad a_\rho = 1 - \frac{\rho_0}{\rho_1}. \tag{4}$$

We lastly expand the determinants of $\mathbf{P}_p$ and $\mathbf{P}$ in X and in orders $\det\mathbf{P}_p^{(n)}$ and $\det\mathbf{P}^{(n)}$, where n is the combined order in a_{VP}, a_{VS}, and a_ρ. Truncating beyond terms linear in any of these perturbations we have then for R_{PP}:

$$R_{\text{PP}} = \frac{\det\mathbf{P}_p}{\det\mathbf{P}} \approx \frac{\det\mathbf{P}_p^{(1)}}{\det\mathbf{P}^{(0)}}, \tag{5}$$

which when evaluated explicitly is

$$R_{\text{PP}}(\theta) \approx \frac{1}{4}\left(1 + \sin^2\theta\right)\ a_{VP} - 2\frac{V_{S_0}^2}{V_{P_0}^2}\sin^2\theta\ a_{VS} + \frac{1}{2}\left(1 + 4\frac{V_{S_0}^2}{V_{P_0}^2}\sin^2\theta\right)\ a_\rho. \tag{6}$$

II. THE AKI-RICHARDS APPROXIMATION

There are two Aki-Richards approximations. They both look the same, but each has a slightly different definition of *angle* on their respective right-hand sides. One, which we will call version A, uses the incidence angle. The other, which we will call version B, uses an average of the incidence angle and the transmission angle. In Figures 1a and c version A is illustrated in red, for a particular set of elastic parameters against the exact $R_{\rm PP}$ for comparison. In Figures 1b and d version B is illustrated in blue. Let us discuss both in turn.

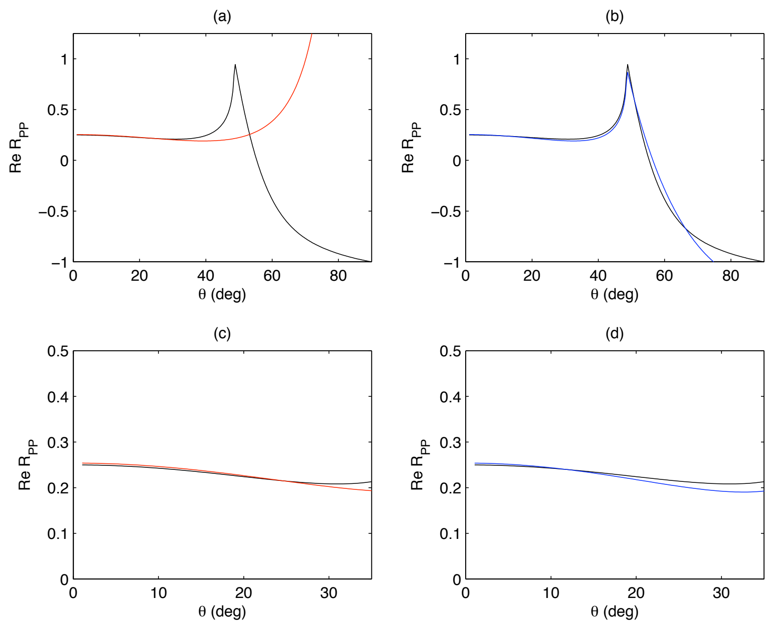

FIG. 1. Numerical behaviour of the two versions of the Aki-Richards approximation. (a), (c) Version A., using incidence angle alone, for angle ranges 0-90° and 0-35° respectively. (b), (d) Version B., using average of incidence and transmission angles, for angle range 0-90° and 0-35° respectively. Elastic incidence parameters: $V_{P_0} = 3000$m/s, $V_{S_0} = 1500$m/s and $\rho_0 = 2.0$gm/cc; target parameters $V_{P_1} = 4000$m/s, $V_{S_1} = 2000$m/s and $\rho_1 = 2.5$gm/cc.

Version A.

Version A of the Aki-Richards approximation—the version actually presented by Aki and Richards (Aki and Richards, 2002)—is given by

$$R_{\rm PP}(\theta) \approx \frac{1}{2}\left(1+\tan^2\theta\right)\,\frac{\Delta V_P}{V_P} - 4\frac{V_S^2}{V_P^2}\sin^2\theta\,\frac{\Delta V_S}{V_S} + \frac{1}{2}\left(1+4\frac{V_S^2}{V_P^2}\sin^2\theta\right)\,\frac{\Delta\rho}{\rho}, \qquad (7)$$

where the Δ's are differences, i.e., $\Delta V_P = V_{P_1} - V_{P_0}$, $\Delta V_S = V_{S_1} - V_{S_0}$ etc., and the parameters are averages, i.e., $V_P = (1/2)(V_{P_1} + V_{P_0})$, $V_S = (1/2)(V_{S_1} + V_{S_0})$ etc., and θ is the angle of incidence.

Version B.

The second version is alluded to, but not derived, by many sources (Shuey, 1985; Castagna and Backus, 1993). In fact, the author of this paper has never found out its exact origin, nor seen a derivation. It is of the same basic form as version A, except it involves a different angle on the right hand side:

$$R_{\rm PP}(\theta) \approx \frac{1}{2}\left(1+\tan^2\theta'\right)\ \frac{\Delta V_P}{V_P} - 4\frac{V_S^2}{V_P^2}\sin^2\theta'\ \frac{\Delta V_S}{V_S} + \frac{1}{2}\left(1+4\frac{V_S^2}{V_P^2}\sin^2\theta'\right)\ \frac{\Delta\rho}{\rho}, \quad (8)$$

where θ' is the average of the incidence and transmission angles:

$$\theta' = \frac{1}{2}\left[\theta + \sin^{-1}\left(\frac{V_{P_1}}{V_{P_0}}\sin\theta\right)\right]. \quad (9)$$

Comparing versions A and B in Figure 1 it is evident that a significant up-tick in accuracy is achieved at large angles by using B.

EQUIVALENCE OF I. AND II.A.

In moving through equations (1)–(6) we have effectively derived version A of the Aki-Richards approximation. To see this, we need to slightly alter both the perturbations and the coefficients in equation (6). Each alteration is allowable assuming that angles and contrasts are small. First we note that for small angles

$$\tan^2\theta = \frac{\sin^2\theta}{1-\sin^2\theta} = \sin^2\theta(1+\sin^2\theta+...) \approx \sin^2\theta. \quad (10)$$

Second, we need to establish the relationship between the perturbations a_{VP} etc., and the perturbations $\Delta V_P/V_P$ etc. First we manipulate the definition of the latter:

$$\frac{\Delta V_P}{V_P} = 2\times\frac{V_{P_1}-V_{P_0}}{V_{P_1}+V_{P_0}} = 2\times\frac{1-(V_{P_0}/V_{P_1})}{1+(V_{P_0}/V_{P_1})}. \quad (11)$$

Then, using the definitions in equation (4), we find the relation

$$\frac{\Delta V_P}{V_P} \approx 2\times\frac{1-(1-a_{VP}/2)}{1+(1-a_{VP}/2)} = 2\times\frac{a_{VP}/4}{1-a_{VP}/4} \approx \frac{1}{2}a_{VP}, \quad (12)$$

and by the same analysis

$$\frac{\Delta V_S}{V_S} \approx \frac{1}{2}a_{VS}, \qquad \frac{\Delta\rho}{\rho} \approx a_\rho. \quad (13)$$

Substituting equations (10) and (20)–(13) into equation (6) we obtain

$$R_{\rm PP}(\theta) \approx \frac{1}{2}\left(1+\tan^2\theta\right)\ \frac{\Delta V_P}{V_P} - 4\frac{V_{S_0}^2}{V_{P_0}^2}\sin^2\theta\ \frac{\Delta V_S}{V_S} + \frac{1}{2}\left(1+4\frac{V_{S_0}^2}{V_{P_0}^2}\sin^2\theta\right)\ \frac{\Delta\rho}{\rho}. \quad (14)$$

This, granting $\frac{V_{S_0}^2}{V_{P_0}^2} \approx \frac{V_S^2}{V_P^2}$, establishes the equivalence of I. and II.A. This last requirement is not strictly true even within the linearization. In fact linearly

$$\frac{V_S^2}{V_P^2} \approx \frac{V_{S_0}^2}{V_P^2}\left[1+\left(\frac{\Delta V_P}{V_P}-\frac{\Delta V_S}{V_S}\right)\right]. \quad (15)$$

Still, assuming roughly comparable P- and S-wave reflectivities this correction may be expected to be quite small.

NONLINEARITY OF II.B.

How can two linearizations of the same equations exhibit such strong differences in accuracy? There are two equally legitimate answers:

1. They are both linear, but they are linear in very different things, which is why they act so differently.
2. They cannot. Version II. A. is linear, and version II. B. is nonlinear.

Consider interpretation (1). Inspection of equation (8) reveals, trivially, that version II. B is linear, on the condition that θ' and the perturbations $\Delta V_P/V_P$, $\Delta V_S/V_S$ and $\Delta\rho/\rho$ are forced to vary independently. Likewise with equation (7), as long as θ and the perturbations vary independently. So, both are linear, but assuming independence of different variables.

Suppose we added a requirement: that the approximation be linear *in target medium properties*. After all, for a geophysicist, a practical $R_{\rm PP}$ approximation answers the question "what happens to $R_{\rm PP}$ when my target medium properties change?"

For version II.A, this requirement changes nothing about the interpretation. The incidence angle θ does not depend on target properties, so equation (7) is linear in any target property information. Not so version II.B. The average angle θ' depends on target properties, because the transmission angle depends on target properties. If $R_{\rm PP}$ is constructed from one quantity that depends on target properties, e.g., $\Delta V_P/V_P$, multiplied by another quantity that also depends on target properties, e.g., $\tan^2\theta'$, then that model for $R_{\rm PP}$ is nonlinear in target property information.

For practical purposes, then, interpretation (2) makes more sense. Version II.B of the Aki-Richards approximation is nonlinear.*

Qualitative nonlinearity

A qualitative clue to the nonlinearity of version II.B comes from an inspection of its behaviour at the critical angle (Figures 2b and d) in contrast to the behaviour of version II.A in the same angle range (Figures 2a and c). In particular, we notice the striking ability of version II.B. to capture the cusp in $R_{\rm PP}$ occurring at the critical angle. Version II.A. does not have a cusp. Rather, it increases rapidly but smoothly as the angle approaches 90°, at which point it is undefined.

But on further consideration, it is difficult to understand how a linear combination of $\sin^2\theta$ and $\tan^2\theta$ could possibly form the blue curve. These are smooth functions in the first quadrant 0-90°. Where is the cusp coming from? In the exact $R_{\rm PP}$ expression its origins are

*In practice average properties and angles can be estimated from a smooth background. This has the effect of reducing the degree of the nonlinearity, a process that is, incidentally, analogous to "making a full waveform inversion problem more linear" by increasing the information content of the reference medium.

not mysterious: it comes because the sine functions are weighted and placed under radical signs (see equation 3), so that functions of angle become imaginary in the first quadrant.

Evidently, the use of the average angle θ' instead of the incidence angle θ under the sine and tangent functions is having a dominant influence on the result. And the angle θ' appears nonlinearly in the approximation. Hence we conclude that not only is version II.B nonlinear, but the nonlinearity is playing a decisive role in its accuracy.

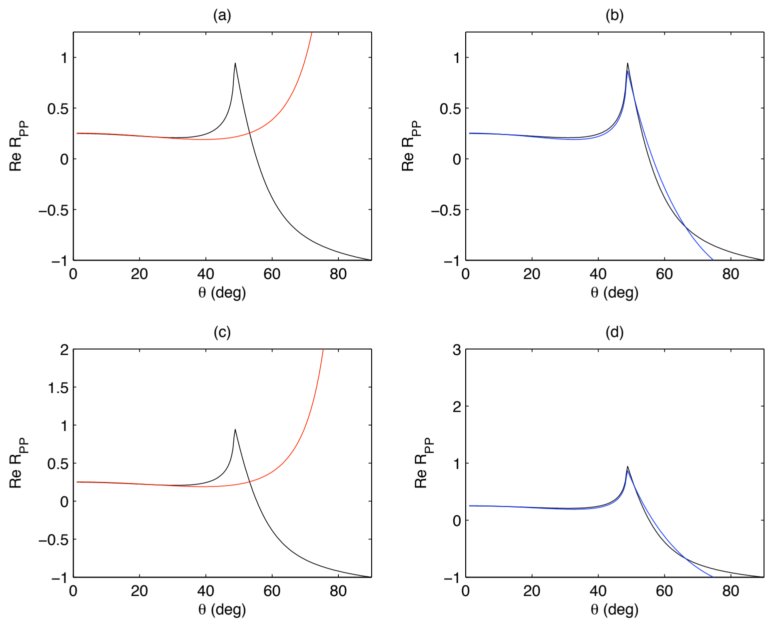

FIG. 2. A further look at R_{PP} approximated with equation (7) in a and c, and with equation (8) in b and d.

Quantitative nonlinearity

The nonlinearity can be discussed formally by expanding θ' in terms of a_{VP}. The average angle is

$$\theta' = \frac{1}{2}\left[\theta + \sin^{-1}\left(\frac{V_{P_1}}{V_{P_0}}\sin\theta\right)\right]. \tag{16}$$

The second term above may be replaced, using equation (4), with

$$\sin^{-1}\left(\frac{V_{P_1}}{V_{P_0}}\sin\theta\right) = \left[\left(1+\frac{a_{VP}}{2}+...\right)\sin\theta\right] + \frac{1}{6}\left[\left(1+\frac{a_{VP}}{2}+...\right)\sin\theta\right]^3 + ... \tag{17}$$

Introducing equation (17) to version II.B of the Aki-Richards approximation, we now have on the right an expression in incidence angle only, but now with series in powers of a_{VP}, or, equivalently, $\Delta V_P/V_P$. This exposes and quantifies the nonlinearity.

PERTURBING θ WITH THE P-WAVE REFLECTIVITY

Shuey (1985), Smith and Gidlow (1987), Foster et al. (2010) and others discuss the Aki-Richards approximation in terms of *reflectivities*, taking advantage of the fact that $\Delta V_P/V_P$

is proportional to the normal incidence reflection coefficient associated with a variation in P-wave velocity only, etc. It turns out that we may treat the average angle part of version II.B of the approximation in terms of incidence angles and reflectivities also.

To see this, we return to the series expansion of part of θ':

$$\begin{aligned}\sin^{-1}\left(\frac{V_{P_1}}{V_{P_0}}\sin\theta\right) = &\left[\sin\theta + \frac{a_{VP}}{2}\sin\theta + ...\right] \\ &+ \left[\frac{1}{6}\sin^3\theta + \frac{a_{VP}}{4}\sin^3\theta + ...\right] \\ &+ ... \ .\end{aligned} \tag{18}$$

Re-arranging, we find a subseries, in θ only, that corresponds to the expansion of a simpler $\arccos$, and a subseries with a_{VP} as a common factor also. Up to first order in a_{VP} we obtain

$$\begin{aligned}\sin^{-1}\left(\frac{V_{P_1}}{V_{P_0}}\sin\theta\right) &= \left[\sin\theta + \frac{1}{6}\sin^3\theta + ...\right] + \frac{a_{VP}}{2}\sin\theta\left[1 + \frac{1}{2}\sin^2\theta + ...\right] \\ &= \sin^{-1}\left[\sin\theta\right] + \frac{a_{VP}}{2}\sin\theta(1 - \sin^2\theta)^{-1/2} + ... \\ &\approx \theta + \frac{a_{VP}}{2}\tan\theta.\end{aligned} \tag{19}$$

Hence for reasonably small contrasts and reasonably small angles, the average angle θ' can be replaced with the incidence angle θ and a correction term in a_{VP}, or, using equation (20), in the P-wave reflectivity $\Delta V_P/V_P$:

$$\begin{aligned}\theta' &\approx \frac{1}{2}\left[\theta + \theta + \frac{a_{VP}}{2}\tan\theta\right] \\ &\approx \theta + \frac{1}{2}\frac{\Delta V_P}{V_P}\tan\theta.\end{aligned} \tag{20}$$

In Figures 3–4 we examine the accuracy of this replacement of θ' with θ perturbed by reflectivity. In Figure 3a–b, three curves are plotted: in black, exact R_{PP}, in blue, version II.A of the Aki-Richards approximation, and in (a) the θ-reflectivity form of version II.B in red, and in (b) the original version II.B in red. In Figure 3c the two instances of version II.B are compared, and in Figure 3d their difference is plotted. The error grows with angle but is quite small.

In Figure 4 we repeat the exercise but adding the next order correction in $(\Delta V_P/V_P)^2$ to θ'. This has the effect of reducing the difference between the two parametrizations of version II.B, though equation (20) alone is already quite accurate for low angles.

CONCLUSIONS

The first of two Aki-Richards approximation forms may be straightforwardly derived by expanding a matrix form of the Knott-Zoeppritz equation in series and truncating. The second is nonlinear, which is quantified by expanding the average angle in series about the P-wave velocity perturbation. The Aki-Richards approximation is often discussed in terms of P-wave, S-wave, and density reflectivities. We have shown that the average angle too may be expressed in terms of the incidence angle and the P-wave reflectivity.

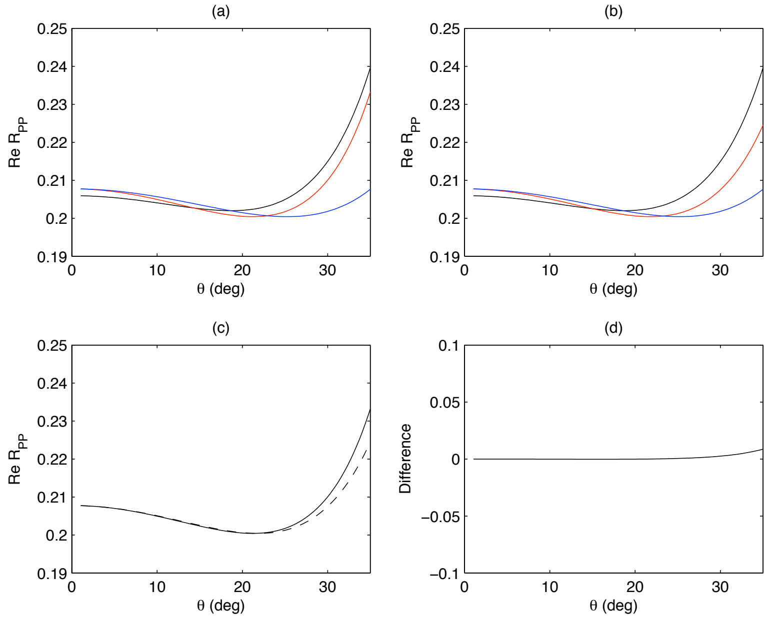

FIG. 3. Aki-Richards version II.B parametrized in two different ways. (a) Three curves are plotted: in black, exact R_{PP}, in blue, version II.A of the Aki-Richards approximation, and the θ-reflectivity form of version II.B in red; (b) as in (a) but with the original version II.B in red. (c) The two instances of version II.B are compared; (d) their difference is plotted.

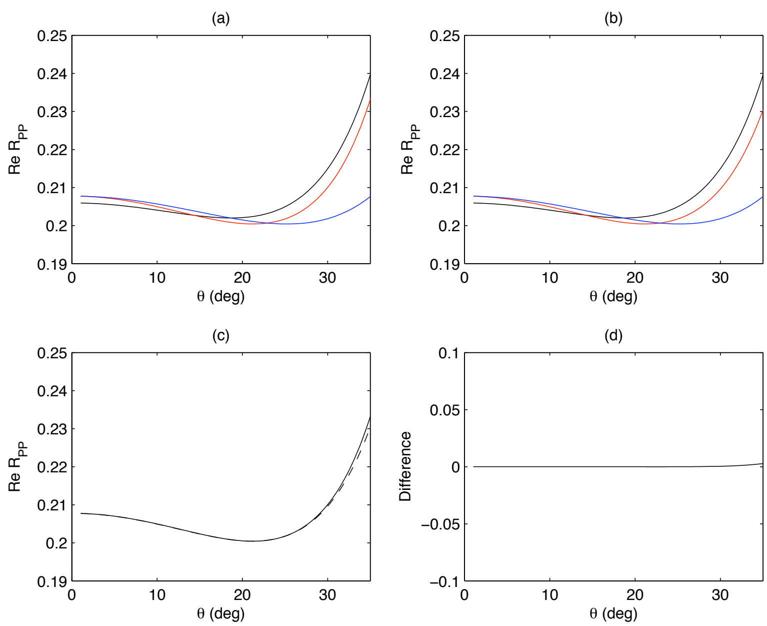

FIG. 4. Aki-Richards version II.B parametrized in two different ways. (a) Three curves are plotted: in black, exact R_{PP}, in blue, version II.A of the Aki-Richards approximation, and the θ-reflectivity form of version II.B in red, here with a second order correction; (b) as in (a) but with the original version II.B in red. (c) The two instances of version II.B are compared; (d) their difference is plotted.

ACKNOWLEDGMENTS

The sponsors of CREWES and NSERC are gratefully acknowledged for their support.

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