Matrix forms for the Knott-Zoeppritz equations

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ABSTRACT

In this note we derive convenient matrix forms for the Knott-Zoeppritz equations. The attempt is to recapture results used (quoted, not derived) by Levin and Keys, and to extend these to include the case of an incident S-wave. In addition to being straightforward to solve and analyze, these forms are used in several other papers in this year's CREWES report.

INTRODUCTION

Our aim is to derive solutions to the Zoeppritz equations for an incident P-wave in a form similar to that quoted by Levin and Keys (Levin, 1986; Keys, 1989), and to extend these to include the results for an incident S-wave. These forms have proven convenient both for the purposes of extension to anelastic media and direct inversion (Innanen, 2011); in this report, they will be used in discussions on reflectivity decomposition of elastic reflection coefficients, AVF analysis of anelastic reflection of shear and converted waves, and in an implementation of the poroelastic AVO theory due to Russell et al. (2011).

BASIC EQUATIONS

We take as our starting point the relations on pg. 140 of Aki and Richards (2002). The assumption of continuity of displacement and traction across an elastic boundary has led to four equations relating all relevant displacement amplitudes above and below the boundary. In our terminology (see Figure 1) these equations relate P_{I}, S_{I}, etc. as follows:

$$\sin\theta_0\left(P_{\mathrm{I}}+P_{\mathrm{R}}\right)+\cos\phi_0\left(S_{\mathrm{I}}+S_{\mathrm{R}}\right)=\sin\phi_1\left(P_{\mathrm{T}}+P_{\mathrm{I}}'\right)+\cos\phi_1\left(S_{\mathrm{T}}+S_{\mathrm{I}}'\right), \tag{1a}$$

$$\cos\theta_0\left(P_{\mathrm{I}}-P_{\mathrm{R}}\right)-\sin\phi_0\left(S_{\mathrm{I}}-S_{\mathrm{R}}\right)=\cos\phi_1\left(P_{\mathrm{T}}-P_{\mathrm{I}}'\right)-\sin\phi_1\left(S_{\mathrm{T}}-S_{\mathrm{I}}'\right), \tag{1b}$$

$$\begin{aligned}&2\rho_0(V_{S_0}^2/V_{P_0})\sin\theta_0\cos\theta_0\left(P_{\mathrm{I}}-P_{\mathrm{R}}\right)-\rho_0V_{S_0}\left[1-2(V_{S_0}^2/V_{P_0}^2)\sin^2\theta_0\right]\left(S_{\mathrm{I}}-S_{\mathrm{R}}\right)\\ =&2\rho_1(V_{S_1}^2/V_{P_0})\sin\theta_0\cos\theta_0\left(P_{\mathrm{T}}-P_{\mathrm{I}}'\right)+\rho_1V_{S_1}\left[1-2(V_{S_1}^2/V_{P_0}^2)\sin^2\theta_0\right]\left(S_{\mathrm{T}}-S_{\mathrm{I}}'\right),\end{aligned} \tag{1c}$$

$$\begin{aligned}&\rho_0V_{P_0}\left[1-2(V_{S_0}^2/V_{P_0}^2)\sin^2\theta_0\right]\left(P_{\mathrm{I}}-P_{\mathrm{R}}\right)-2\rho_0(V_{S_0}^2/V_{P_0})\sin\theta_0\cos\phi_0\left(S_{\mathrm{T}}+S_{\mathrm{I}}'\right)\\ =&\rho_1V_{P_1}\left[1-2(V_{S_1}^2/V_{P_0}^2)\sin^2\theta_0\right]\left(P_{\mathrm{T}}+P_{\mathrm{I}}'\right)-2\rho_1(V_{S_1}^2/V_{P_0})\sin\theta_0\cos\phi_1\left(S_{\mathrm{T}}+S_{\mathrm{I}}'\right).\end{aligned} \tag{1d}$$

We consider several ratios of elastic parameters:

$$A\equiv\frac{\rho_1}{\rho_0},\quad B\equiv\frac{V_{S_0}}{V_{P_0}},\quad B'\equiv\frac{V_{P_0}}{V_{S_0}},\quad C\equiv\frac{V_{P_1}}{V_{P_0}},\quad D\equiv\frac{V_{S_1}}{V_{P_0}},\quad E\equiv\frac{V_{P_1}}{V_{S_0}},\quad F\equiv\frac{V_{S_1}}{V_{S_0}}. \tag{2}$$

INCIDENT P-WAVE: R_{PP} AND R_{PS}

We now limit the number of waves incident on the boundary to one: a P-wave incident from above (see Figure 2). That is, we let

$$S_{\mathrm{I}}=P_{\mathrm{I}}'=S_{\mathrm{I}}'=0. \tag{3}$$

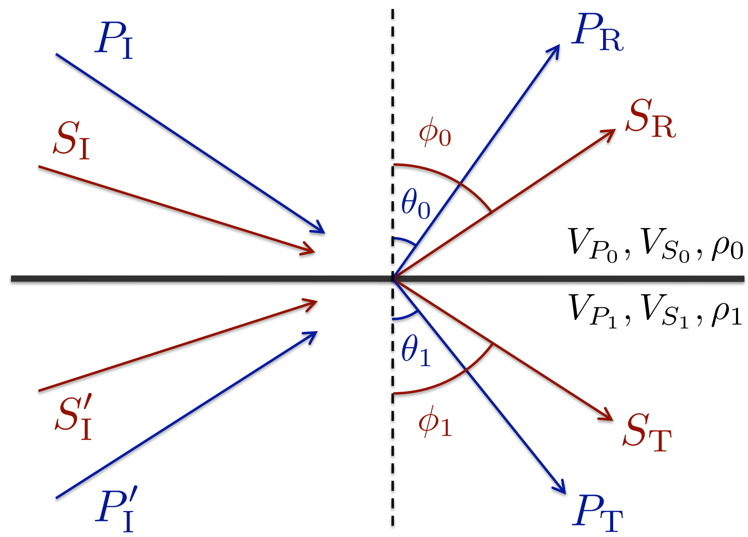

FIG. 1. Displacement amplitudes associated with the Knott-Zoeppritz equations.

We define the displacement coefficients associated with the remaining reflected and transmitted waves to be

$$R_{\mathrm{PP}} \equiv \frac{P_{\mathrm{R}}}{P_{\mathrm{I}}}, \quad R_{\mathrm{PS}} \equiv \frac{S_{\mathrm{R}}}{P_{\mathrm{I}}}, \quad T_{\mathrm{PP}} \equiv \frac{P_{\mathrm{T}}}{P_{\mathrm{I}}}, \quad T_{\mathrm{PS}} \equiv \frac{S_{\mathrm{T}}}{P_{\mathrm{I}}}. \tag{4}$$

We will parametrize the coefficients in terms of the P-wave angle of incidence in the upper medium, θ_0. So beyond substituting the forms in equation (4) into the relations in equations (1), we further transform all angle-dependent factors to expressions in $\sin\theta_0$ using Snell's law, particularly,

$$\begin{aligned} &\cos\theta_0 = \sqrt{1-\sin^2\theta_0}, \quad \sin\phi_0 = \frac{V_{S_0}}{V_{P_0}}\sin\theta_0, \quad \sin\theta_1 = \frac{V_{P_1}}{V_{P_0}}\sin\theta_0, \\ &\cos\theta_1 = \sqrt{1-\frac{V_{P_1}^2}{V_{P_0}^2}\sin^2\theta_0}, \quad \text{and} \quad \cos\phi_1 = \sqrt{1-\frac{V_{S_1}^2}{V_{P_0}^2}\sin^2\theta_0}, \end{aligned} \tag{5}$$

or, setting $X \equiv \sin\theta_0$,

$$\begin{aligned} &\cos\theta_0 = \sqrt{1-X^2}, \quad \sin\phi_0 = BX, \quad \sin\theta_1 = CX, \\ &\cos\theta_1 = \sqrt{1-C^2X^2}, \quad \text{and} \quad \cos\phi_1 = \sqrt{1-D^2X^2}. \end{aligned} \tag{6}$$

Finally, for convenience we define

$$\begin{aligned} \Gamma_j(X) &\equiv \sqrt{1-j^2X^2}, \\ \Gamma^j(X) &\equiv 1-2j^2X^2. \end{aligned} \tag{7}$$

After substitution of equations (2)–(7), equations (1) can be expressed in matrix form as

$$\mathbf{P}\begin{bmatrix} R_{\mathrm{PP}} \\ R_{\mathrm{PS}} \\ T_{\mathrm{PP}} \\ T_{\mathrm{PS}} \end{bmatrix} = \mathbf{b}_P, \tag{8}$$

where

$$\mathbf{P} \equiv \begin{bmatrix} -X & -\Gamma_B(X) & CX & \Gamma_D(X) \\ \Gamma_1(X) & -BX & \Gamma_C(X) & -DX \\ 2B^2X\Gamma_1(X) & B\Gamma^B(X) & 2AD^2X\Gamma_C(X) & AD\Gamma^D(X) \\ -\Gamma^B(X) & 2B^2X\Gamma_B(X) & 2AC\Gamma^D(X) & -2AD^2X\Gamma_D(X) \end{bmatrix},$$

and

$$\mathbf{b}_P \equiv \begin{bmatrix} X \\ \Gamma_1(X) \\ 2B^2X\Gamma_1(X) \\ \Gamma^B(X) \end{bmatrix}.$$

Forming auxiliary matrices $\mathbf{P}_P$ and $\mathbf{P}_S$ by replacing the first and second columns of $\mathbf{P}$ with $\mathbf{b}_P$ respectively, we may use Cramer's rule to determine the displacement reflection coefficients:

$$R_{\text{PP}}(\theta_0) = \frac{\det\mathbf{P}_P}{\det\mathbf{P}}, \quad R_{\text{PS}}(\theta_0) = \frac{\det\mathbf{P}_S}{\det\mathbf{P}}. \tag{9}$$

We thus recover the forms used by Levin and Keys.

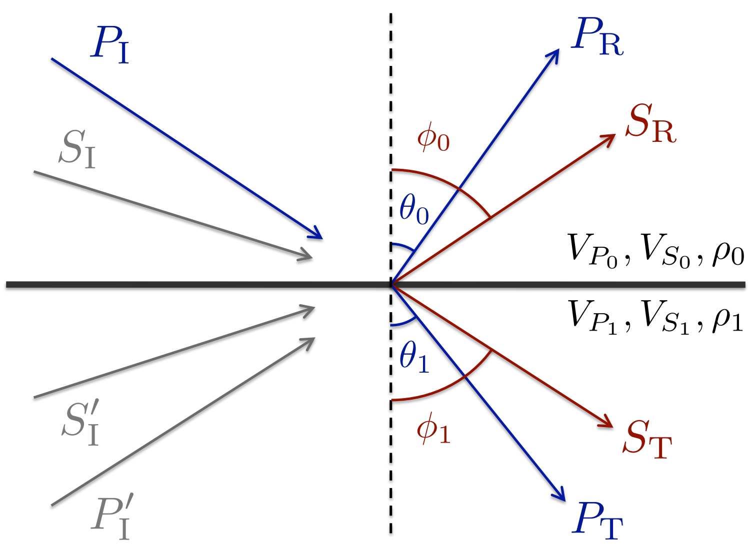

FIG. 2. Displacement amplitudes given an incident P-wave.

INCIDENT S-WAVE: R_{SP} AND R_{SS}

The extension for an incident S-wave is carried out similarly. As illustrated in Figure 3, we disallow all incident fields except an S-wave from above by setting

$$P_{\text{I}} = P'_{\text{I}} = S'_{\text{I}} = 0, \tag{10}$$

and we define displacement coefficients

$$R_{\text{SS}} \equiv \frac{S_{\text{R}}}{S_{\text{I}}}, \quad R_{\text{SP}} \equiv \frac{P_{\text{R}}}{S_{\text{I}}}, \quad T_{\text{SS}} \equiv \frac{S_{\text{T}}}{S_{\text{I}}}, \quad T_{\text{SP}} \equiv \frac{P_{\text{T}}}{S_{\text{I}}}. \tag{11}$$

This time we parametrize in terms of $\sin\phi_0$, i.e., the sine of the S-wave incidence angle. Using equation (7), the other trigonometric functions are expressible in terms of $Y \equiv \sin\phi_0$ as

$$\begin{aligned} &\cos\phi_0 = \Gamma_1(Y), \quad \sin\theta_0 = B'Y, \quad \sin\theta_1 = EY, \quad \sin\phi_1 = FY, \\ &\cos\theta_1 = \Gamma_E(Y), \quad \cos\phi_1 = \Gamma_D(Y), \quad \text{and} \quad \cos\theta_0 = \Gamma_{B'}(Y). \end{aligned} \tag{12}$$

After substitution of equations (10)–(12), equations (1) can also be expressed in matrix form as

$$\mathbf{S}\begin{bmatrix} R_{\text{SS}} \\ R_{\text{SP}} \\ T_{\text{SS}} \\ T_{\text{SP}} \end{bmatrix} = \mathbf{b}_S, \tag{13}$$

where

$$\mathbf{S} \equiv \begin{bmatrix} -\Gamma_1(Y) & -B'Y & \Gamma_F(Y) & EY \\ Y & -\Gamma_{B'}(Y) & FY & -\Gamma_E(Y) \\ \Gamma^1(Y) & 2Y\Gamma_{B'}(Y) & AF\Gamma^F(Y) & 2AF^2Y\Gamma_E(Y) \\ -2Y\Gamma_1(Y) & B'\Gamma^1(Y) & 2AF^2Y\Gamma_F(Y) & -AE\Gamma^F(Y) \end{bmatrix},$$

and

$$\mathbf{b}_S \equiv \begin{bmatrix} \Gamma_1(Y) \\ Y \\ \Gamma_1(Y) \\ 2Y\Gamma_1(Y) \end{bmatrix}.$$

Forming auxiliary matrices $\mathbf{S}_P$ and $\mathbf{S}_S$ by replacing the first and second columns of $\mathbf{S}$ with $\mathbf{b}_S$ respectively, we again use Cramer's rule to determine the displacement reflection coefficients:

$$R_{\text{SS}}(\phi_0) = \frac{\det\mathbf{S}_S}{\det\mathbf{S}}, \quad R_{\text{SP}}(\phi_0) = \frac{\det\mathbf{S}_P}{\det\mathbf{S}}. \tag{14}$$

Thus are generated the incident-S counterparts to the incident-P solutions of Levin and Keys.

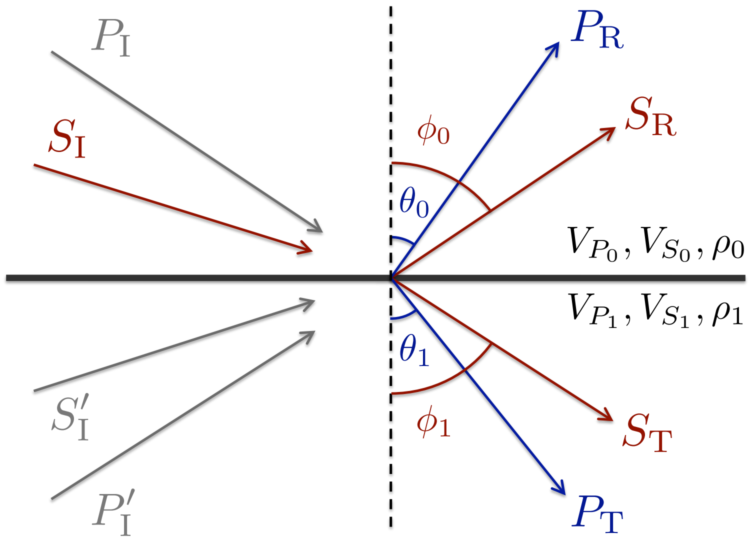

FIG. 3. Displacement amplitudes given an incident S-wave.

CONCLUSIONS

We derive matrix forms for the Knott-Zoeppritz equations. The attempt is to recapture results of Levin and Keys, show how they are arrived at, and to extend these to include the case of an incident S-wave. We have made particular use of these forms in several other papers in this year's CREWES report.

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