The relationship between Lipschitz exponents and Q: synthetic data tests

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ABSTRACT

In seismic signal analysis, points of sharp variation classified as "edges" contain a considerable amount of a signal's information, thus making edge detection and the study of a signal's local properties an appropriate mechanism for obtaining information from seismic data. Several important physical processes can in principle affect the local regularity of a reflected event in a seismic trace: processes of absorption and wave attenuation. The local regularity of a given signal is characterised by the continuous wavelet transform and subsequently measured by its corresponding Lipschitz exponent(s). For a single seismic event resembling a delta type function, a linear model can be used in order to estimate the associated Lipschitz regularity, however for practical settings a non-linear objective function would have to be minimised in order to estimate the associated regularity. A robust estimation of a functions local properties and differentiability from seismic data, alongside prior geological information, could potentially lead to processing and inversion algorithms able to discern and characterise such targets.

INTRODUCTION

In seismic signal analysis, regions of abrupt change, often considered expressions of underlying singularities within a given function contain considerable amount of a signal's information (Innanen, 2003). The Wavelet transform closely related to multi-scale edge detection, characterises the local regularity of a signal by decomposing signals into fundamental building blocks localised in space and frequency. Applying advanced mathematical techniques namely continuous wavelet transform enables us to obtain the modulus maxima from seismic data and estimate the Lipschitz exponents which in turn allows us to measure the local regularity of functions and differentiate the intensity profile of different edges (Mallat and Zhong, 1992). Several important physical processes can in principle affect the local regularity of a reflected event in a seismic trace: processes of absorption/wave attenuation, and reflections from targets composed of thin (sub-wavelength) layers. The study and analysis of characteristic singularities of a seismic trace and in particular the effects of absorption on uniform regularities of a seismic trace could potentially lead to extraction of useful information (Innanen, 2003).

It is generally understood that due to absorption, the energy of seismic waves propagating through an anelastic medium would dissipate over a given distance. As a result, transient waveforms are distorted as they propagate through such media; progressive loss of amplitudes and changes of phase are typically encountered (Kjartansson, 1979; Zhang, 2008). The overall effect of seismic attenuation is described by the dimensionless quality factor Q, with studies in seismic data processing concentrating either on modelling, estimation or compensation (Innanen, 2003). In practical terms, estimation and compensation can hope to dramatically enhance the resolving power of seismic data.

In this paper we discuss numerical implementation of the continuous wavelet transform and estimation of the associated Lipschitz exponent (α) and the possibility of establishing an empirical relation between a function's regularity and the quality factor Q. *

NUMERICAL IMPLEMENTATION

I. Implementation of the CWT and extraction of modulus maxima

Applying continuous wavelet transform to a given function say f and obtaining the modulus maxima at each scale s would lead to the following relation (Mallat and Zhong, 1992),

$$|W_s f(x)| \leq A s^{\alpha}, \tag{1}$$

where $|W_s f(x)|$ is the modulus maxima of the function $f(x)$ at various scales $s = 2^j$ for $j \epsilon \mathbb{Z}$.

In seismic signal analysis, we are mainly interested in Lipschitz regularities ranging from -1 to 1. Thus, our preferred wavelet may be first derivative of a Gaussian function with a single vanishing moment. Linearising equation (1) by taking logarithm of both sides provides the following,

$$\log_2 |W_s f(x)| \leq \log_2 A + \alpha \log_2(s). \tag{2}$$

Theoretically, based on equation (2), one would be able to obtain the modulus maxima of a given signal and subsequently estimate the corresponding Lipschitz exponent and the associated constant A. As a result, one would expect a relatively straightforward implementation of the algorithm and subsequently detection of singularities within a signal. However, we are confronted with several difficulties such as the associated constant A, whether it could be estimated or pre-determined prior to the implementation of the algorithm. Furthermore, in practical applications, most signal structures could be described as smoothed functions with an underlying singularity (Mallat and Zhong, 1992; Innanen, 2003). Such a function would have to be modelled as a delta function convolved or smoothed by a Gaussian with variance σ^2. As a result, one would have to minimise the following non-linear objective function

$$\phi(A, \alpha, \sigma) = \sum_{i,j=1}^{n} [\log_2 |a_i| - \log_2(A) - j + \frac{\alpha - 1}{2} (\log_2(\sigma^2 + 2^{2j})]^2. \tag{3}$$

It is clearly evident that minimising the objective function above requires a relatively expensive computational algorithm such as the conjugate gradient method or the steepest descent.

II. Estimation of the Lipschitz exponent: Linear model

In order to test and verify the accuracy of our linear model based on equation (2), we start with a single event, represented by a delta function as our input signal where

*In an earlier paper in this report, we discussed the theory of continuous wavelet transform and Lipschitz exponents, and algorithms for estimation of the latter (Izadi et al., 2011). Hence we consider practical uses and implementation of these ideas.

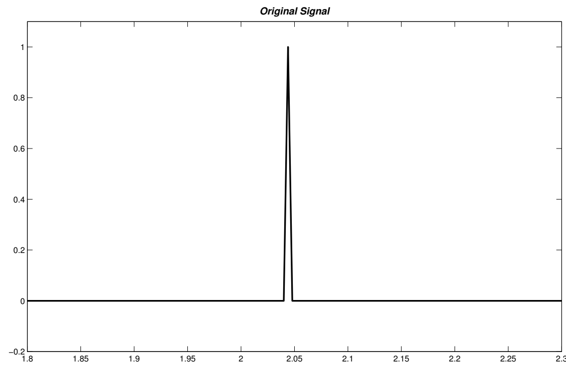


FIG. 1. Original signal.

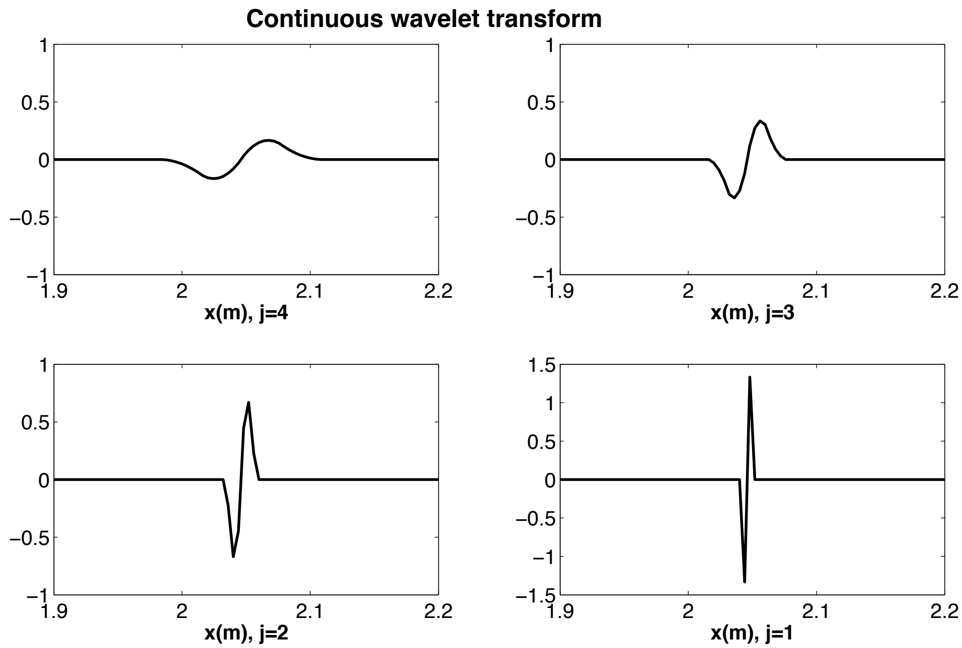


FIG. 2. Corresponding wavelet transform.

the Lipschitz regularity of such an event is equal to -1. Figures 1,2 and 3 represent the input function and its corresponding continuous wavelet transform and modulus maxima respectively.

As expected for a delta function the modulus decreases with increasing scale. It should be noted that the scale ranges from $j = 1,,5$. Plotting the logarithm of the modulus maxima against the scale and computing the slope yields an estimate for α which is equal to -0.999, a relatively close value to the theoretical value of α which is equal to -1. The intercept in figure 5 (a) produces the value for A, which is equal to 2.33 (we have to add 1 to the initial value since the x-axis starts at 1).

j	1	2	3	4	5
$\|a_i\|$	1.333	0.669	0.333	0.166	0.083
$\log_2 \|a_i\|$	0.415	-0.578	-1.583	-2.588	-3.582

Table 1. Corresponding maxima modulus values at each scale for delta function.

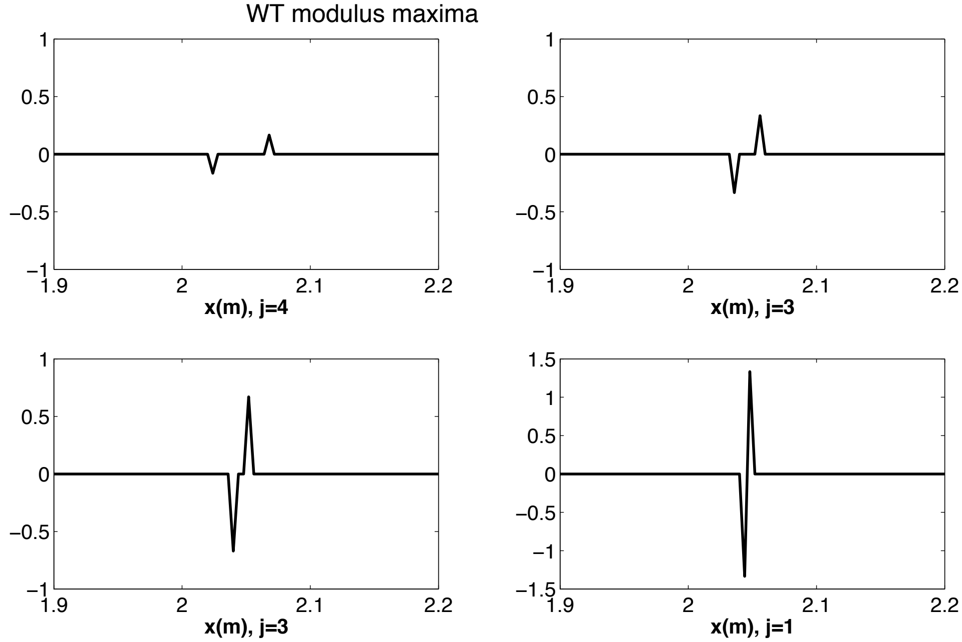


FIG. 3. Corresponding modulus maxima of original signal at each scale

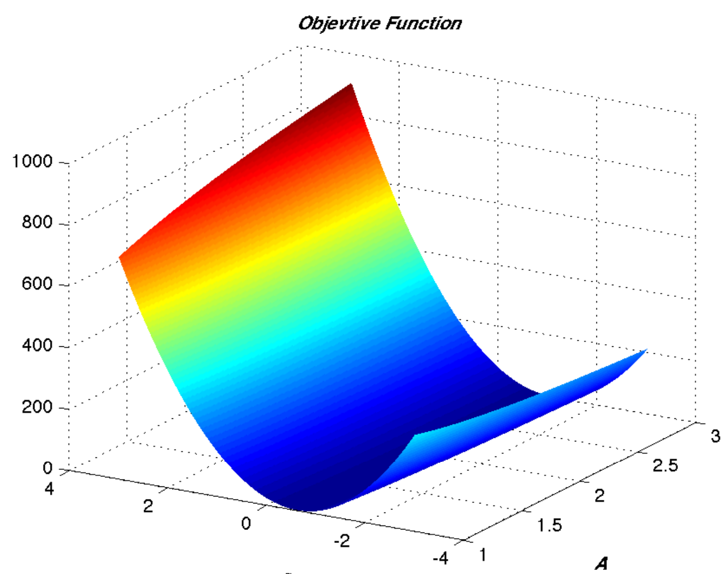
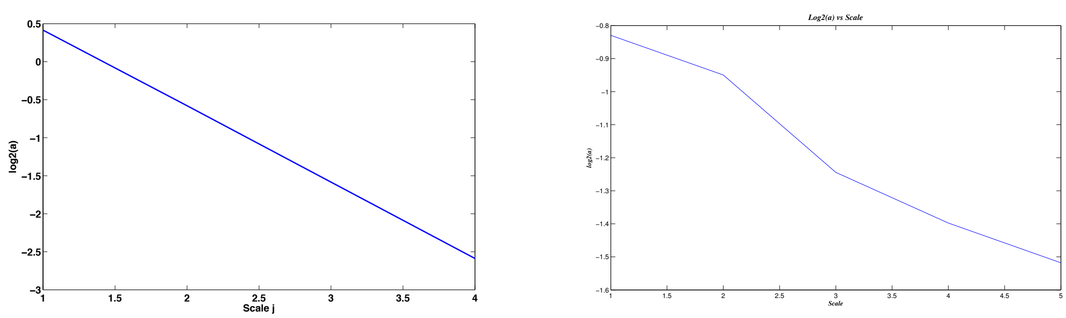


FIG. 4. Two parameter objective function. The x, y axes represent the values for α and A respectively.

j	1	2	3	4
$\lvert a_i \rvert$	1.333	1.339	1.335	1.330
$\log_2 \lvert a_i \rvert$	0.415	-0.421	0.416	0.411

Table 2. Corresponding maxima modulus values at each scale for Heaviside function.

A surprising result in regards to the plot in figure 5 (a) is related to the fact that a perfect line is produced with no observed errors. A possible explanation could be related to the dyadic sampling which might reduce the errors associated with computing the continuous wavelet transform of a signal. Using a continuous scale and computing the continuous wavelet coefficients of the delta function does produce errors as illustrated in figure 5 (b). However, we can not state with any certainty whether dyadic sampling reduces or eliminates the errors associated with the continuous wavelet transform of a delta function without further analysis.Additionally one could estimate α and A by forming the objective function (illustrated in figure 4) and subsequently minimising in order to estimate A and α. Using this method, we obtain a slightly higher value of 2.44 for A and a slightly lower value of -1.0004 for α.

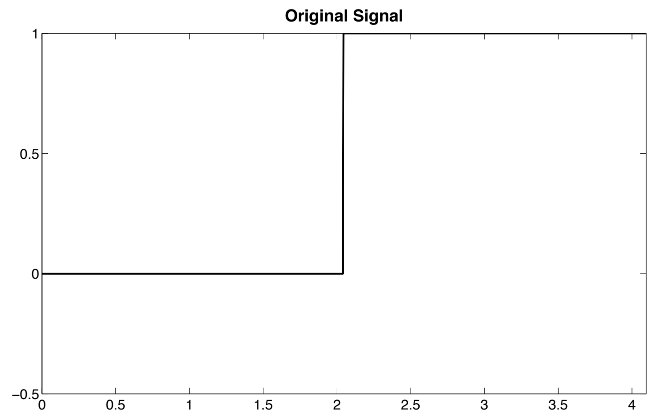


(a) (b)

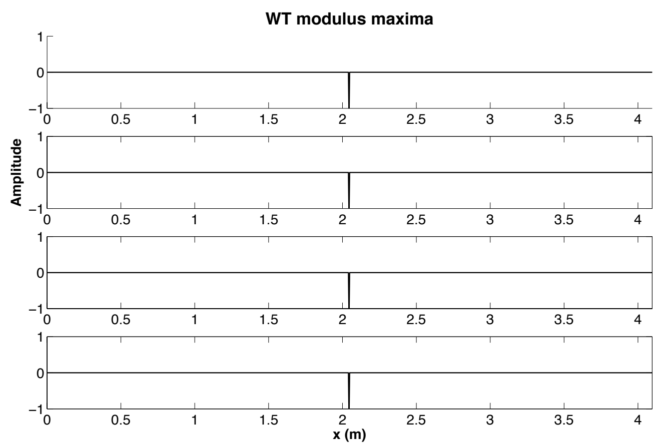
FIG. 5. (a) $\log_2(a)$ vs scale for delta function (b) $\log_2(a)$ vs scale for delta function using continuous scale.

By integrating the delta function, one would obtain a Heaviside function. Based on the properties of Lipschitz regularity, integration should increase α by 1, therefore we expect a value of $\alpha = 0$ for a Heaviside function. By minimising or finding the intercept we would obtain the corresponding A value for the Heaviside function and gain a relatively deeper insight into the behaviour of the constant A, thus we would have to consider whether A has a fixed value regardless of our input function or it varies for each function, hence requiring a time consuming calibration scheme in order to obtain the correct value.

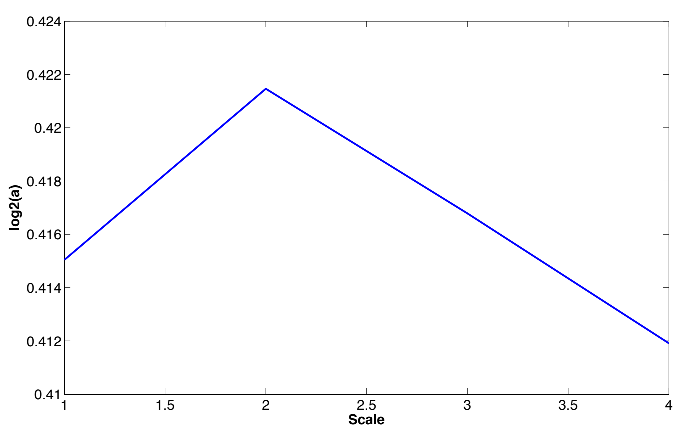
Based on figure 6 (c), we observe that a relatively small degree of error is introduced into the plot. Nevertheless, the corresponding values are equal to 0.0014 and 1.4198 for α and A respectively. As expected the Lipschitz exponent is equal to 1, however the value of A seemingly depends on the input function and may be the measure of the energy at the lowest scale. This tends to be problematic for the following reason (it should be noted that integrating the Heaviside function further increased the α by 1).



(a)



(b)



(c)

FIG. 6. (a) Original signal (Heaviside Function) (b) corresponding modulus maxima (c) Plot of $\log_2(a)$ vs scale for Heaviside function ($j = 1, 2, 3, 4$).

III. Estimation of the Lipschitz regularity: Non-linear model

In order to asses the effects of absorption on a function's regularity and establish an empirical relation between the Lipschitz exponent (α) and Q, we apply the continuous wavelet transform on a seismic trace and subsequently form and minimise the objective function given in equation (3). For a single layer $z = 800m$ in depth with a single Q value and a velocity $v = 1500m/s$, the trace as illustrated in figure 6 is obtained by plotting amplitude (direct arrivals only) for series of receivers vs arrival time (a total of 41 receivers spaced $20m$ apart along the vertical or $z - axis$).

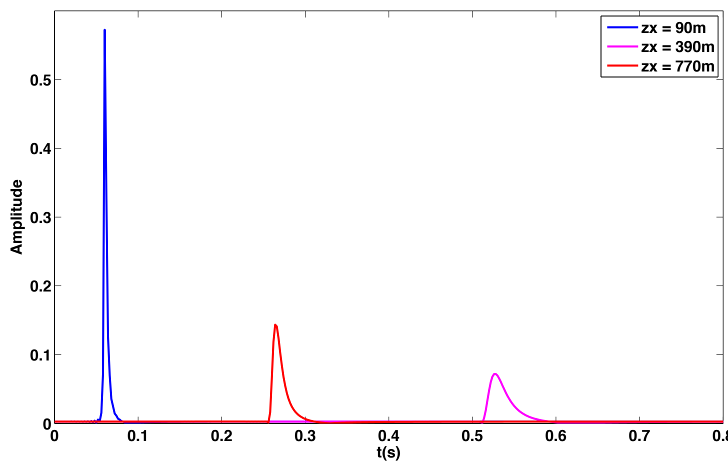


FIG. 7. Amplitude of direct arrivals vs time for three receivers located at depth $z_1 = 90m$, $z_2 = 390m$ and $z_3 = 770m$ with $Q = 50$.

As expected and illustrated figure 7, for a highly absorptive medium, $Q = 50$, the pulse starts to broaden and lose amplitude, hence obtaining spectral characteristics of a Gaussian. Given the resemblance of the first arrival to a delta function, we expect a Lipschitz value close to -1. Using the steepest descent method, regardless of our initial guess we obtain relatively stable results for A and α as given in table 3. However, the estimated value for σ is dependent on our initial guess. This could be related to the topography of our objective function such that a global minimum may not exits in the $\sigma - axis$ direction (could be a flat surface in the σ direction). For the third direct arrival, corresponding to a receiver located at $z_3 = 770m$, the Lipschitz exponent decreases as expected (should move towards 0). However, for both A and σ, the estimated values (given in table 4) are relatively unstable and seemingly dependent on the initial guess. Similar to the first arrival, a global minimum may not exist in the $\sigma - axis$ direction, however the uncertainty in regards to the estimated values of A needs to be further examined.

	Guess 1	Estimate	Guess 2	Estimate	Guess 3	Estimate
α	-2	-0.3500	-1	-0.3547	0	-0.3724
A	1	2.0860	2	2.1105	3	2.2306
σ	0.003	0.0003	0.01	0.0095	0.1	-0.0294
∇g		2.8735		2.8725		2.8723

Table 3. Corresponding Estimated values using steepest descent for fist arrival in figure 7 ($z_1 = 90m$).

	Guess 1	Estimate	Guess 2	Estimate	Guess 3	Estimate
α	-1	-0.2036	0	-0.2704	1	-0.3150
A	1	1.7427	2	2.1554	3	2.4792
σ	0.1	0.0850	1	0.9009	1	1.1805
∇g		0.0497		0.0325		0.0428

Table 4. Corresponding Estimated values using steepest descent for third arrival in figure 7 ($z_3 = 770m$).

IV. Scale cut-off and a new apporach

Given the non-linear nature of our model, the steepest descent is not only inefficient in terms of estimating two of the three variables, but also time consuming with the gradient (∇_g) serving as our only guide. As an alternative to the steepest descent, one may consider linearising the problem by analysing the dominant behaviour of α by imposing a threshold on the scales. Doing so would eliminate two of the variables namely A and σ and reduce the problem to the sole task of estimating α.

By taking continuous wavelet transform of the trace given in figure 7 and subsequently calculating the corresponding modulus maxima values of each event ($|a_i|$, $i = 1, 2, 3, ..., 7$) and plotting $\log_2 |a_i|$ values vs scale (for $j = 1, 2, 3, .., 7$) a dominant behaviour in terms of regularity is observed for each arrival (illustrated in figure 8). Hence, calculating the slope from $j = 3, ..., 6$ for each event or from $j = 2, ..., 6$ for the first arrival and $j = 3, ..., 7$ for the second and third arrival yields an estimate for the corresponding Lipschitz exponent. Thus a value of $\alpha = -0.8666$, $\alpha = -0.8104$ and $\alpha = -0.6342$ is obtained for the first, second and third arrival respectively.

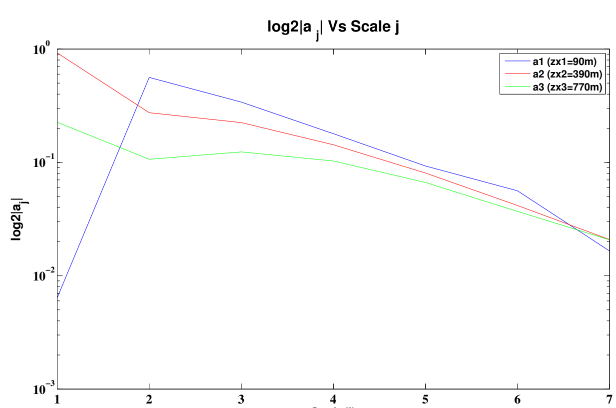


FIG. 8. Plot of $\log_2(a)$ vs scale for each arrival corresponding to receivers located at depth $z_1 = 90m$, $z_2 = 390m$ and $z_3 = 770m$ respectively.

Estimating α (from the slope) for each each arrival and plotting the values versus corresponding receiver depth for various values of Q provides an insight into the relation between a functions regularity and absorption. Figure 9 and 10 clearly illustrate a proportional dependency or relation between α and depth and an inverse relation between α and Q.

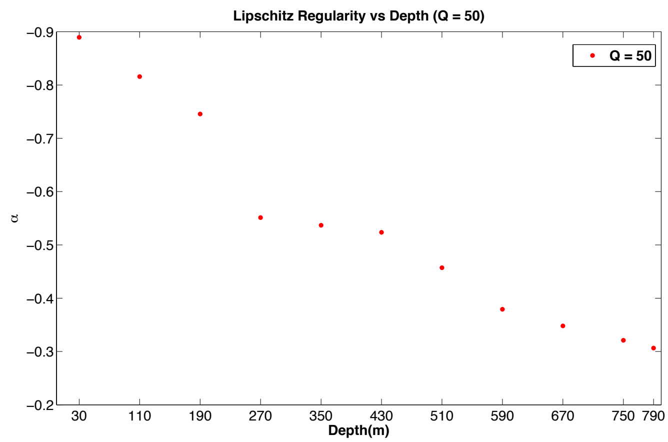


FIG. 9. Plot of α vs depth for $Q = 50$.

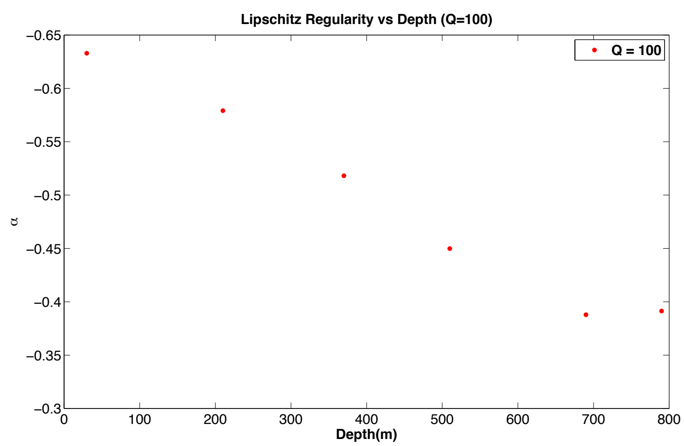


FIG. 10. Plot of α vs depth $Q = 100$.

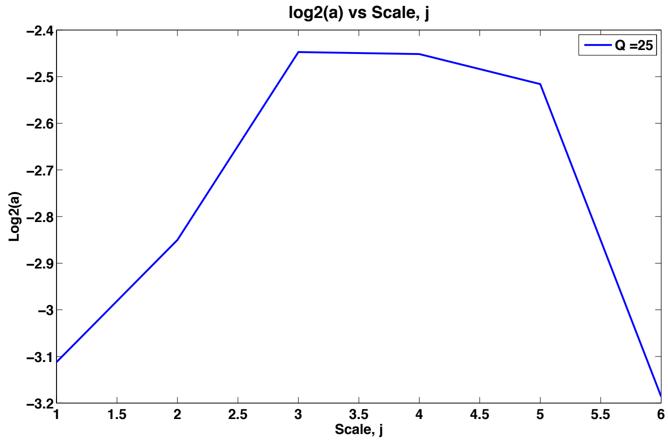


FIG. 11. Plot of $\log 2|a_i|$ vs scale j for $Q = 25$.

The clear advantage based on thresholding as opposed to the steepest descent method lies in the linearisation and subsequently simplification of the problem. However one should approach this method with caution. The problem of visual bias, accuracy of thresholding and subsequent estimations of α complicates this approach. In addition, this approach should be expected to encounter significant accuracy issues for combination of low Q and large propagation distances, e.g. $Q \sim 25$ and $z \sim 700m$ is an example of a limiting pair. For a medium with $Q = 25$, a dominant behaviour can not be observed in regards to the regularity of direct arrival corresponding to a receiver located at $z = 690m$ (illustrated in figure 11).

CLOSELY SPACED EVENTS

An additional difficulty in regards to the continuous wavelet model is the presence of two closely spaced events which further complicates our analysis and estimation of the corresponding Lipschitz exponent(s). For two closely placed events, the Modulus maxima values would start to merge with increasing scale, hence represent a single event (illustrated in figure 13) and rendering any form of distinction between two events almost impossible. One possible idea would be imposing some sort of thresholding on the scale. However such a procedure becomes complicated due to behaviour of a Gaussian function. A Gaussian with small σ is dominated by a delta type function regularity whereas a Gaussian with a relatively large σ value is dominated by a ramp function type regularity. Figure 14, illustrates the fact that unfortunately we do not observe a trend or pattern in order to impose a threshold on the scale. For two Gaussians with $\sigma = 0.01$ and $\sigma = 0.05$, it is unclear how one could restrict the scale without the corresponding modulus values merging (at a certain scale) in order to differentiate between the two events.

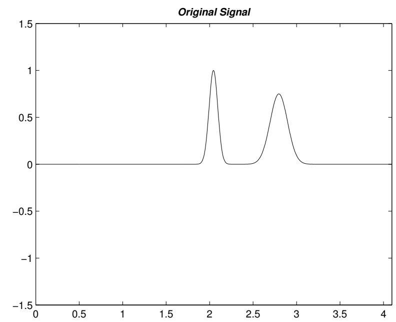


FIG. 12. Two closely spaced Gaussians representing the original input signal.

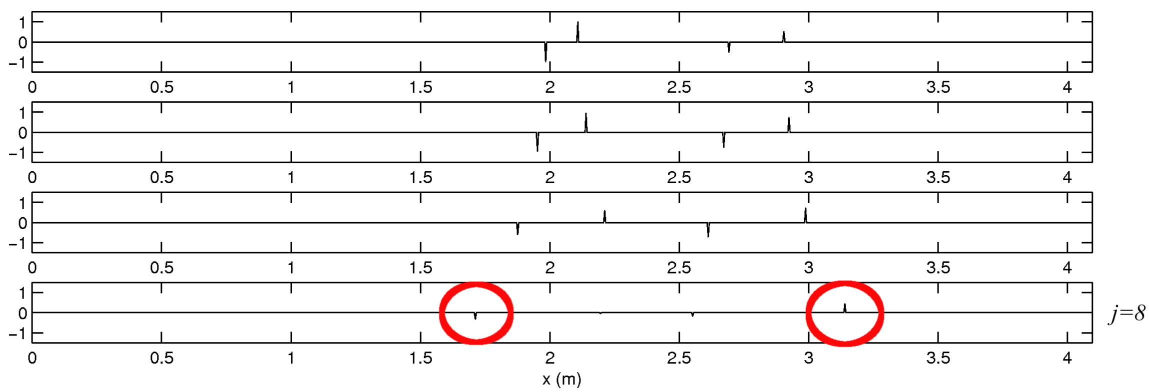


FIG. 13. Corresponding modulus maxima. From $j = 5, .., 8$ we have two max values corresponding to two distinct events. However at $j = 8$ the two max values merge, thus representing a single event .

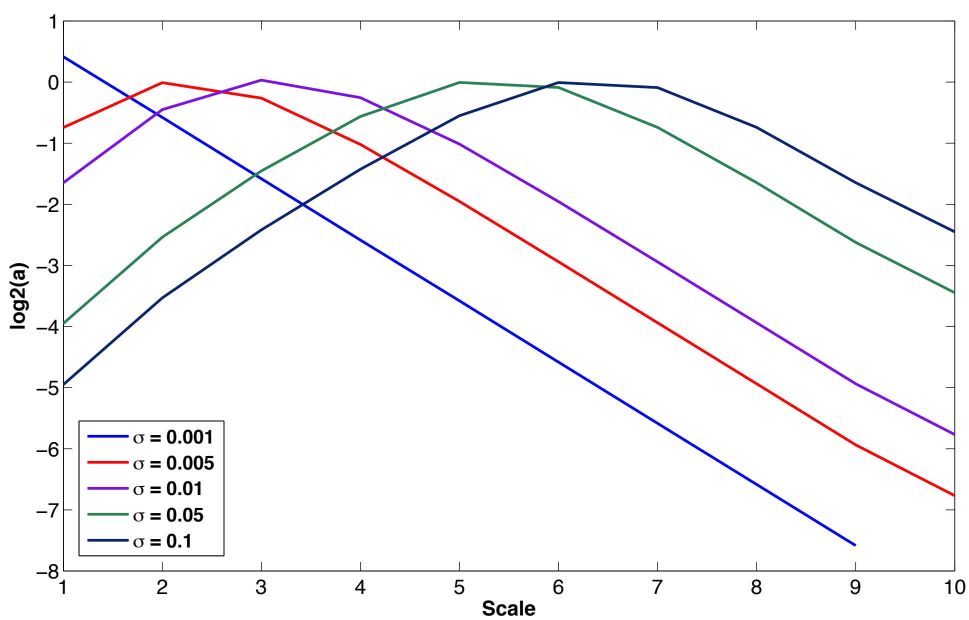


FIG. 14. $\log 2(a)$ vs scale for Gaussians with increasing σ.

CONCLUSION

An accurate estimation of the Lipschitz regularity of a seismic trace is regarded as a highly desirable goal, thus the principle aim of this paper has been to utilise the continuous wavelet transform in order to analyse and study the effects of absorption on a given pulse and subsequently establish an empirical relation between a functions regularity and the loss factor Q. For a single event, resembling a delta function or a Heaviside function, one could use a linear model, based on a functions smoothness at each scale and estimate the Lipschitz exponent by finding the slope or by forming and minimising an objective function. However due to absorption, one would have to model a given pulse as delta function convolved with a Gaussian which in effect leads to a non-linear model. Thus, in order to estimate the Lipschitz exponent, one could use the steepest descent or some sort of thresholding method in order to simplify the problem.

Using the steepest descent method and for a certain range Q values ($50 < Q < 150$), a relatively stable value for the Lipschitz exponent can be obtained. However, the difficulty relates to the estimation of the two additional parameters (the amplitude A and pulse width σ) and also time consuming nature of the steepest descent method for this specific problem.

By imposing some sort of thresholding, one could linearise the problem, eliminate two of the parameters (A and σ) and solely focus on estimating the Lipschitz exponent α. However this method is prone to introduction of errors and visual bias.

The results illustrate a relation between absorption and a functions decay, which could be used to establish an empirical relation between the associated Lipschitz exponent and Q. Moving forward, it would be of particular interest to test the model on field data and analyse the presence of noise, primaries, multiple layers and velocities with varying Q values on the non-linear model.

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