Potentials for anelastic scattering

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ABSTRACT

Previous efforts to characterize the scattering problem for anelastic waves have been carried out in the anacoustic regime, and in the full anelastic regime but for single reflectors (i.e., the anelastic AVO problem). Here we begin to frame the full anelastic scattering problem, focusing on some key issues: transformation of anelastic scattering potentials to the P-, Sv-, and Sh- potential domain, and the consequences to that transformation of moving from an elastic reference/anelastic perturbation model to an anelastic reference/anelastic perturbation model. We use the scattering potentials thus derived to produce sensitivity kernels for full waveform inversion iterates wherein V_P, V_S, ρ updates are carried out in elastic target determination, and Q_P and Q_S updates are added in anelastic target determination.

INTRODUCTION

The linear and nonlinear "anacoustic" scattering problem, involving a single scalar velocity and quality factor Q, has received some focused attention in recent years (Innanen and Weglein, 2003, 2007; Mulder and Hak, 2009; Innanen and Lira, 2010; Hak and Mulder, 2010), in particular with regard to the inverse problem. A 2-parameter, single component SH model has also been considered (Ribodetti and Virieux, 1998). A major conclusion of those studies concerned the importance of dispersion in the backscattered amplitudes to the estimation of target Q and P-wave velocities.

With this foreknowledge, a subsequent effort has since been made to characterize dispersive backscatter in the full anelastic regime. As of this writing, anelastic backscatter has been more or less fully characterized in a simple prototype environment, involving a single specular reflection generated at a planar interface between two anelastic half-spaces. These studies, of what is essentially anelastic amplitude-variation-with-offset (AVO) or frequency (AVF) modelling (i.e., an extension to attenuating media of elastic AVO methods, e.g., Castagna and Backus, 1993; Foster et al., 2010), further confirmed the dominant role dispersion plays in producing interpretable amplitude signatures. The resulting communications (Innanen, 2011; Bird et al., 2011) join a growing set of discussions of the practical applicability of dispersive AVO/AVF analysis to exploration and monitoring seismology (e.g., Odebeatu et al., 2006; Chapman et al., 2006; Quintal et al., 2009; Lines et al., 2008, 2012). Furthermore, since the AVO/AVF framework is a special case of the layered-medium wave problem, these results are connected with the full viscoelastic theory of Borcherdt (2009).

In this paper we will build on the conclusions drawn from this simple anelastic reflection problem, formulating instead the full anelastic scattering problem for waves in 3D interacting with arbitrary heterogeneous perturbations. This is expected to act as a foundation for both linear and nonlinear inverse scattering algorithms, and for the analysis of gradients and sensitivities in anelastic seismic inversion of the "full waveform" type. We will adopt the methodology recently published for elastic scattering by Stolt and Weglein (2012), in particular the approach to rotating the scattering operator such that its elements are interpretable in terms of the potentials of a volume element to scatter waves from P-P, P-Sv, Sh-Sh, etc. That framework will be adapted to allow reference and/or perturbed media to take on finite Q_P and Q_S values. We will formulate the problem such that in addition to driving inverse procedures, we ultimately may be able to extend many of Borcherdt's results, concerning reflection and conversion of homogeneous and inhomogeneous anelastic waves in layered media (Borcherdt, 2009), to scattering from arbitrary anelastic heterogeneities.

In the main portion of the paper, we formulate anelastic wave equations in terms of familiar versions of Q_P and Q_S, show that these are consistent with viscoelastic/continuum models and macroscopice/exploration seismic models, and discuss transformations of these equations to P- and S-potential domains. Scattering potentials and their role in Born approximate data models are discussed next; these are seen to be matrices with elements interpretable in terms of P-P, P-Sv, Sv-P, Sv-Sv, and Sh-Sh conversions. The case of an elastic reference medium leads to relatively straightforward extensions of the elastic derivations of Stolt and Weglein (2012), whereas the case of an anelastic reference medium produces some significantly altered behaviour which varies depending on the degree of homogeneity and inhomogeneity of the incoming and outgoing waves. Finally, we use the framework to derive sensitivities associated with multicomponent anelastic data and variations in the five anelastic parameters V_P, V_S, ρ, Q_P, and Q_S, which will form the basis for study of the anelastic full waveform inversion problem.

1. ANELASTIC MODELS AND WAVE EQUATIONS

1a. Anelasticity

Linear anelasticity is formulated via a change in the constitutive (stress-strain) relations. Whereas for the elastic case the stress $\sigma_{ij}(\mathbf{r}, t)$ and the strain $e_{kl}(\mathbf{r}, t)^*$ are instantaneously related by $\sigma_{ij} = c_{ijkl}e_{kl}$, for the anelastic case we require a more general relationship:

$$\sigma_{ij}(t) = \int_{-\infty}^{\infty} d\tau \; r_{ijkl}(t-\tau)e_{kl}(\tau). \tag{1}$$

This permits the current value of the stress to depend on the current and past values of the strain, and vice versa, a necessity for anelastic models. For isotropic media, the tensor r_{ijkl} simplifies to

$$r_{ijkl}(t) = \frac{1}{3}\left[r_K(t) - r_S(t)\right]\delta_{ij}\delta_{kl} + \frac{1}{2}r_S(t)\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right), \tag{2}$$

in analogy to the elastic problem. Rather than the Lamé parameters we have instead their counterparts the bulk and shear relaxation functions r_K and r_S, which are defined such that

*We will from now on avoid the variables $\mathbf{r}$ and t wherever possible, but they are always implied.

in the Fourier domain they satisfy

$$\begin{aligned} K(\omega) &= \frac{1}{3} i\omega R_K(\omega), \qquad \text{and} \\ \mu(\omega) &= \frac{1}{2} i\omega R_S(\omega). \end{aligned} \tag{3}$$

Here $K(\omega)$ and $\mu(\omega)$ are the complex, frequency-dependent generalizations of the bulk and shear moduli.

1b. Anelastic wave equations

Anelastic disturbances propagate through a 3D continuum according to an equation of the form

$$\mathcal{L}_A \mathbf{u}(\mathbf{r}, \omega) = \mathbf{f}(\mathbf{r}, \omega), \tag{4}$$

where $\mathbf{f}(\mathbf{r}, \omega)$ is a distribution of sources, $\mathbf{u}$ is the displacement from equilibrium, $\mathbf{r} = (x, y, z)$ and ω is the angular frequency. The operator $\mathcal{L}_A$, whose various possible forms in "displacement space" are given in Appendix A, is in this paper primarily considered in "potential space", i.e.,

$$\mathcal{L}_A = \Pi^{-1} \begin{bmatrix} \mathcal{L}_P & 0 & 0 & 0 \\ 0 & \mathcal{L}_S & 0 & 0 \\ 0 & 0 & \mathcal{L}_S & 0 \\ 0 & 0 & 0 & \mathcal{L}_S \end{bmatrix} \Pi \tag{5}$$

where

$$\Pi = \begin{bmatrix} \partial_x & \partial_y & \partial_z \\ 0 & -\partial_z & \partial_y \\ \partial_z & 0 & -\partial_x \\ -\partial_y & \partial_x & 0 \end{bmatrix} \tag{6}$$

transforms 3-length vectors into 4-length vectors as follows:

$$\begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix} \rightarrow \begin{bmatrix} \nabla \cdot \mathbf{u} \\ (\nabla \times \mathbf{u})_x \\ (\nabla \times \mathbf{u})_y \\ (\nabla \times \mathbf{u})_z \end{bmatrix}, \tag{7}$$

and (for all wave-like displacements) has the inverse

$$\Pi^{-1} = \nabla^{-2} \Pi^T. \tag{8}$$

Under the transform being inverted in equation (5), the anelastic wave operator has the elements

$$\mathcal{L}_P = \rho^2 \tilde{V}_P \left[\nabla^2 + \left(\frac{\omega}{\tilde{V}_P} \right)^2 \right], \tag{9}$$

and

$$\mathcal{L}_S = \rho^2 \tilde{V}_S \left[\nabla^2 + \left(\frac{\omega}{\tilde{V}_S} \right)^2 \right]. \tag{10}$$

The velocities $\tilde{V}_P$ and $\tilde{V}_S$ contain the anelastic moduli:

$$\begin{aligned} \tilde{V}_P^2 &= \frac{K(\omega) + (4/3)\mu(\omega)}{\rho} = \frac{\gamma(\omega)}{\rho}, \\ \tilde{V}_S^2 &= \frac{\mu(\omega)}{\rho}. \end{aligned} \tag{11}$$

This technically does enough to bring anelasticity into the formalism leading to the scattering description, but we will take the extra step to parametrize in terms of "nearly constant" Q_P and Q_S models (NCQ) before continuing. Standard NCQ models (Aki and Richards, 2002) are framed in terms of the propagation constants

$$\begin{aligned} k_P &= \frac{\omega}{V_P} \left[1 + Q_P^{-1} F_P(\omega) \right], \\ k_S &= \frac{\omega}{V_S} \left[1 + Q_S^{-1} F_S(\omega) \right], \end{aligned} \tag{12}$$

where, following "anacoustic" scattering theory (Innanen and Weglein, 2007), we have sequestered the attenuation and dispersion terms in the functions

$$F_P(\omega) = \frac{i}{2} - \frac{1}{\pi} \log \left(\frac{\omega}{\omega_P} \right), \quad F_S(\omega) = \frac{i}{2} - \frac{1}{\pi} \log \left(\frac{\omega}{\omega_S} \right). \tag{13}$$

Equations (12) imply

$$\tilde{V}_P = V_P \left[1 + Q_P^{-1} F_P(\omega) \right]^{-1}, \quad \tilde{V}_S = V_S \left[1 + Q_S^{-1} F_S(\omega) \right]^{-1}, \tag{14}$$

and hence in terms of P-wave and S-wave quality factors the elements of the potential-space anelastic wave operator are

$$\mathcal{L}_P = \rho^2 V_P \left[1 + Q_P^{-1} F_P(\omega) \right]^{-1} \left[\nabla^2 + \left(\frac{\omega}{V_P} \right)^2 \left[1 + Q_P^{-1} F_P(\omega) \right]^2 \right] \tag{15}$$

and

$$\mathcal{L}_S = \rho^2 V_S \left[1 + Q_S^{-1} F_S(\omega) \right]^{-1} \left[\nabla^2 + \left(\frac{\omega}{V_S} \right)^2 \left[1 + Q_S^{-1} F_S(\omega) \right]^2 \right]. \tag{16}$$

The real-valued constants V_P and V_S are the phase velocities with which the P- and S-waves propagate at the reference frequencies ω_P and ω_S respectively.

1c. Q_P and Q_S

To finish this section, let us compare the NCQ forms quoted above with the quality factors of continuum mechanics, which are commonly defined as the ratios of the imaginary and real parts of the two moduli (White, 1983). Combining equations (11) and (14) and neglecting nonlinear terms in Q_S^{-1} we have

$$\mu(\omega) \approx \rho V_P^2 \left[1 - iQ_S^{-1} + \frac{1}{\pi} Q_S^{-1} \log\left(\frac{\omega}{\omega_S}\right)\right]. \tag{17}$$

Equating the real and imaginary parts of the right-hand side with $\mu(\omega) = \mu_R(\omega) + i\mu_I(\omega)$, we obtain

$$\begin{aligned} \mu_R(\omega) &\approx \rho V_P^2 \left[1 + \frac{1}{\pi} Q_S^{-1} \log\left(\frac{\omega}{\omega_S}\right)\right] \\ \mu_I(\omega) &\approx -\rho V_P^2 Q_S^{-1}, \end{aligned} \tag{18}$$

and from this, again neglecting Q_S^{-2}, we have

$$\frac{\mu_I}{\mu_R} \approx -Q_S^{-1} \left[1 - \frac{1}{\pi} Q_S^{-1} \log\left(\frac{\omega}{\omega_S}\right)\right] \approx -Q_S^{-1}. \tag{19}$$

Similarly

$$\frac{\gamma_I}{\gamma_R} \approx -Q_P^{-1}. \tag{20}$$

Hence indeed the NCQ models we have adopted have quality factors which agree with the standard continuum-mechanical definitions. Thus comforted, we next use the anelastic wave formulation to define scattering quantities, knowing that the formal results will be consistent with both the macroscopic Q models common to exploration seismology (e.g., Hargreaves and Calvert, 1991) and those of viscoelasticity and continuum mechanics (Borcherdt, 2009).

2. BORN-APPROXIMATE MODELLING OF SEISMIC DATA

The Born approximation as a forward model for seismic primaries dates to the 1970s (Bleistein, 1979; Bleistein et al., 2000). Subsequently it has been used as the basis for seismic imaging and inversion in multiparameter problems (the literature is to vast to cite, but Clayton and Stolt, 1981; Beylkin, 1985, are classical examples). Here we apply the particular approach used by Stolt and Weglein (2012) to treat the 3D elastic problem because of its appropriateness for, and proximity to, the anelastic case of interest.

In the model, any particular scalar component of the measured data (e.g., P-P, or P-S) is related to all contributing medium perturbations $\Delta a_i / a_i$ by

$$\begin{aligned} D(\mathbf{r}, \mathbf{r}_s, \omega) &\approx \int_V d\mathbf{r}' G_L(\mathbf{r}, \mathbf{r}', \omega) \mathcal{V}(\mathbf{r}', \omega) G_R(\mathbf{r}', \mathbf{r}_s, \omega) \\ &\approx \int_V d\mathbf{r}' G_L(\mathbf{r}, \mathbf{r}', \omega) \left[\sum_i \mathcal{A}_i^L(\mathbf{r}', \omega) \frac{\Delta a_i}{a_i}(\mathbf{r}') \mathcal{A}_i^R(\mathbf{r}', \omega)\right] G_R(\mathbf{r}', \mathbf{r}_s, \omega), \end{aligned} \tag{21}$$

where G_L and G_R are appropriate Green's functions. A key result for the approach of Stolt and Weglein (2012) is that the weighting operators $\mathcal{A}_i^L$ and $\mathcal{A}_i^R$, which generally involve both multiplicative and gradient operations, can be altered through integration by parts such that they act on the left and right Green's functions instead:

$$D(\mathbf{r},\mathbf{r}_s,\omega) \approx \sum_i \int_V d\mathbf{r}' \left[\tilde{\mathcal{A}}_i^L(\mathbf{r}',\omega) G_L(\mathbf{r},\mathbf{r}',\omega)\right] \frac{\Delta a_i}{a_i}(\mathbf{r}') \left[\tilde{\mathcal{A}}_i^R(\mathbf{r}',\omega) G_R(\mathbf{r}',\mathbf{r}_s,\omega)\right]. \tag{22}$$

This will lead to considerable simplification. For instance, as long as the Green's function represents plane wave propagation in a homogeneous or smoothly varying medium, any gradient operations in the scattering potential, now acting solely on the Green's functions, will appear in the form

$$\begin{aligned} \nabla_{\mathbf{r}'} G_R(\mathbf{r}',\mathbf{r}_s,\omega) &= i\mathbf{k}_R(\mathbf{r}') \\ \nabla_{\mathbf{r}'} G_L(\mathbf{r},\mathbf{r}',\omega) &= -i\mathbf{k}_L(\mathbf{r}') \end{aligned} \tag{23}$$

where $\mathbf{k}_R$ is the vector propagation constant, pointing in the direction of propagation, for the right hand Green's function, which has the interpretation of a wave incident on the volume scattering element, and $\mathbf{k}_L$ is the propagation constant for the left hand Green's function, the wave propagating away from the scattering element after the interaction.

3. THE ANELASTIC SCATTERING POTENTIALS $\mathcal{V}^{\mathrm{AA}}$ AND $\mathcal{V}^{\mathrm{AE}}$

We next prepare the wave operators for anelastic propagation such that a volume scattering element involving perturbations in Q_P and Q_S can be included in the linear formulation of equations (21)–(22). We will cast the elastic components of the scattering problem as

$$\frac{\Delta V_P}{V_P} = 2\frac{V_{P_1} - V_{P_0}}{V_{P_1} + V_{P_0}}, \ \frac{\Delta V_S}{V_S} = 2\frac{V_{S_1} - V_{S_0}}{V_{S_1} + V_{S_0}}, \ \frac{\Delta\rho}{\rho} = 2\frac{\rho_1 - \rho_0}{\rho_1 + \rho_0}, \tag{24}$$

and to these will add, for anelastic reference media and anelastic perturbations,

$$\frac{\Delta Q_P}{Q_P} = 2\frac{Q_{P_1} - Q_{P_0}}{Q_{P_1} + Q_{P_0}}, \ \frac{\Delta Q_S}{Q_S} = 2\frac{Q_{S_1} - Q_{S_0}}{Q_{S_1} + Q_{S_0}}, \tag{25}$$

and for elastic reference media and anelastic perturbations,

$$\frac{\Delta Q_P}{Q_P} = Q_{P_1}^{-1}, \ \frac{\Delta Q_S}{V_S} = Q_{S_0}^{-1}. \tag{26}$$

The elastic reference definitions are consistent with those studied in the anacoustic scattering problem (Innanen and Weglein, 2007).

3a. $\mathcal{V}^{\mathrm{AA}}$ and $\mathcal{V}^{\mathrm{AE}}$ in terms of displacement

It is possible to pose elastic and anelastic scattering problems in displacement space and in potential (P- and S-wave) space. The value of the latter is that the reference medium propagations can be exactly or approximately expressed in terms of independently propagating waves, and the scattering operator can be expressed as a matrix whose elements are interpretable in terms of conversions P-P, P-Sv, Sv-Sv, etc. We will begin with the displacement space formulation, and, following Stolt and Weglein (2012), transform to the P- and S-domain.

Scattering operators for anelastic reference media

The scattering operator is defined as the difference between a perturbed and a reference wave operator. In Appendix A (and above) we formulate the displacement-space anelastic wave operator $\mathcal{L}_A$ and quote the form of the elastic wave operator $\mathcal{L}_E$. The displacement space scattering operator is, then, for anelastic reference media, of the form

$$\mathcal{V}^{\mathrm{AA}} = \mathcal{L}_A - \mathcal{L}_{A0}. \tag{27}$$

Substituting the definitions of Appendix A, equations (64)–(66), into equation (27), we obtain

$$\mathcal{V}^{\mathrm{AA}}(\mathbf{r},\omega) = \begin{bmatrix} \mathcal{V}^{\mathrm{AA}}_{xx}(\mathbf{r},\omega) & \mathcal{V}^{\mathrm{AA}}_{xy}(\mathbf{r},\omega) & \mathcal{V}^{\mathrm{AA}}_{xz}(\mathbf{r},\omega) \\ \mathcal{V}^{\mathrm{AA}}_{yx}(\mathbf{r},\omega) & \mathcal{V}^{\mathrm{AA}}_{yy}(\mathbf{r},\omega) & \mathcal{V}^{\mathrm{AA}}_{yz}(\mathbf{r},\omega) \\ \mathcal{V}^{\mathrm{AA}}_{zx}(\mathbf{r},\omega) & \mathcal{V}^{\mathrm{AA}}_{zy}(\mathbf{r},\omega) & \mathcal{V}^{\mathrm{AA}}_{zz}(\mathbf{r},\omega) \end{bmatrix}, \tag{28}$$

where the diagonal elements are

$$\begin{aligned} \mathcal{V}^{\mathrm{AA}}_{ii} =& \rho_0 \left\{ V_{P_0}^2 \left[\partial_i \frac{\Delta\rho}{\rho} \partial_i + 2\left(\partial_i \frac{\Delta V_P}{V_P} \partial_i - Q_{P_0}^{-1} F_P(\omega) \partial_i \frac{\Delta Q_P}{Q_P} \partial_i \right) \right] \right. \\ & \left. + V_{S_0}^2 \sum_{j\neq i} \left[\partial_j \frac{\Delta\rho}{\rho} \partial_j + 2\left(\partial_j \frac{\Delta V_S}{V_S} \partial_j - Q_{S_0}^{-1} F_S(\omega) \partial_j \frac{\Delta Q_S}{Q_S} \partial_j \right) \right] + \omega^2 \frac{\Delta\rho}{\rho} \right\} \end{aligned} \tag{29}$$

for all $i, j = x, y, z$, and the off-diagonal elements are

$$\begin{aligned} \mathcal{V}^{\mathrm{AA}}_{ij} =& \rho_0 \left[\left(V_{P_0}^2 - 2V_{S_0}^2 \right) \partial_i \frac{\Delta\rho}{\rho} \partial_j + 2V_{P_0}^2 \left(\partial_i \frac{\Delta V_P}{V_P} \partial_j - Q_{P_0}^{-1} F_P(\omega) \partial_i \frac{\Delta Q_P}{Q_P} \partial_j \right) \right. \\ & - 4V_{S_0}^2 \left(\partial_i \frac{\Delta V_S}{V_S} \partial_j + Q_{S_0}^{-1} F_S(\omega) \partial_i \frac{\Delta Q_S}{Q_S} \partial_j \right) \\ & \left. + V_{S_0}^2 \left(\partial_j \frac{\Delta\rho}{\rho} \partial_i + 2\partial_j \frac{\Delta V_S}{V_S} \partial_i - 2Q_{S_0}^{-1} F_S(\omega) \partial_j \frac{\Delta Q_S}{Q_S} \partial_i \right) \right], \end{aligned} \tag{30}$$

where $i \neq j$.

Scattering operators for elastic reference media

Continuing, for an elastic reference medium, making use of equations (64)–(69), we determine a scattering operator of the form

$$\mathcal{V}^{\mathrm{AE}} = \mathcal{L}_A - \mathcal{L}_{E0}, \tag{31}$$

which has elements

$$\mathcal{V}^{\mathrm{AE}}(\mathbf{r},\omega) = \begin{bmatrix} \mathcal{V}^{\mathrm{AE}}_{xx}(\mathbf{r},\omega) & \mathcal{V}^{\mathrm{AE}}_{xy}(\mathbf{r},\omega) & \mathcal{V}^{\mathrm{AE}}_{xz}(\mathbf{r},\omega) \\ \mathcal{V}^{\mathrm{AE}}_{yx}(\mathbf{r},\omega) & \mathcal{V}^{\mathrm{AE}}_{yy}(\mathbf{r},\omega) & \mathcal{V}^{\mathrm{AE}}_{yz}(\mathbf{r},\omega) \\ \mathcal{V}^{\mathrm{AE}}_{zx}(\mathbf{r},\omega) & \mathcal{V}^{\mathrm{AE}}_{zy}(\mathbf{r},\omega) & \mathcal{V}^{\mathrm{AE}}_{zz}(\mathbf{r},\omega) \end{bmatrix}, \tag{32}$$

where the diagonal elements are

$$\begin{aligned}\mathcal{V}_{ii}^{\mathrm{AE}} =&\rho_0 \left\{ V_{P_0}^2 \left[\partial_i \frac{\Delta\rho}{\rho}\partial_i + 2\left(\partial_i \frac{\Delta V_P}{V_P}\partial_i - F_P(\omega)\partial_i \frac{\Delta Q_P}{Q_P}\partial_i \right)\right]\right. \\ &\left. + V_{S_0}^2 \sum_{j\neq i} \left[\partial_j \frac{\Delta\rho}{\rho}\partial_j + 2\left(\partial_j \frac{\Delta V_S}{V_S}\partial_j - F_S(\omega)\partial_j \frac{\Delta Q_S}{Q_S}\partial_j \right)\right] + \omega^2 \frac{\Delta\rho}{\rho} \right\}\end{aligned} \tag{33}$$

for all $i, j = x, y, z$ and the off-diagonal elements are

$$\begin{aligned}\mathcal{V}_{ij}^{\mathrm{AE}} =&\rho_0 \left[\left(V_{P_0}^2 - 2V_{S_0}^2 \right) \partial_i \frac{\Delta\rho}{\rho}\partial_j + 2V_{P_0}^2 \left(\partial_i \frac{\Delta V_P}{V_P}\partial_j - F_P(\omega)\partial_i \frac{\Delta Q_P}{Q_P}\partial_j \right)\right. \\ & - 4V_{S_0}^2 \left(\partial_i \frac{\Delta V_S}{V_S}\partial_j + F_S(\omega)\partial_i \frac{\Delta Q_S}{Q_S}\partial_j \right) \\ &\left. + V_{S_0}^2 \left(\partial_j \frac{\Delta\rho}{\rho}\partial_i + 2\partial_j \frac{\Delta V_S}{V_S}\partial_i - 2F_S(\omega)\partial_j \frac{\Delta Q_S}{Q_S}\partial_i \right)\right]\end{aligned} \tag{34}$$

where $i \neq j$. Although the anelastic reference case and the elastic reference case look quite similar, it is important to remember that this is partly because the differences are hidden—the two cases have significantly different definitions for $\frac{\Delta Q_P}{Q_P}$ and $\frac{\Delta Q_S}{Q_S}$.

3b. Scattering operators in terms of P- and S-potentials

We next map the displacement space scattering operators in equations (28) and (32) to potentials, wherein we will perform our analysis and consider sensitivities for inversion. This occurs through an extension of the transformations of Stolt and Weglein (2012) to anelastic reference media. We envision four different possible scattering scenarios: an incoming P-wave and outgoing P-wave (Figure 1a), an incoming P-wave and outgoing S-wave (Figure 1b), an incoming S-wave and outgoing P-wave (Figure 1c), and an incoming S-wave and outgoing S-wave (Figure 1d).

The left (L) and right (R) Green's functions in the Born model above involve: incoming P- or S-waves with directions $\mathbf{k}_P^L$ or $\mathbf{k}_S^L$, and outgoing P- or S-waves with directions $\mathbf{k}_P^R$ or $\mathbf{k}_S^R$, respectively. Since the Π operators discussed above, through the integration by parts that changes equation (21) to equation (22), act on the left and right Green's functions, all of the partial derivatives therein appear in the form of products of i times components of these vectors. Specifically, the Π operator acting from the left has the form

$$\Pi_L = i \begin{bmatrix} \mathbf{k}_P^L \cdot \\ \mathbf{k}_S^L \times \end{bmatrix}, \tag{35}$$

and the Π operator acting from the right has the form

$$\Pi_R^{-1} = -\frac{i}{\omega^2} \begin{bmatrix} V_{P_0}^2 \mathbf{k}_P^{R} \cdot^H & V_{S_0}^2 \mathbf{k}_S^R \times^H \end{bmatrix}. \tag{36}$$

The result of applying these operators, for instance $\Pi_l \mathcal{V}^{AA} \Pi_R^{-1}$, is a scattering operator in terms of P- and S-waves, consistent with Lamé's theorem (Aki and Richards, 2002,

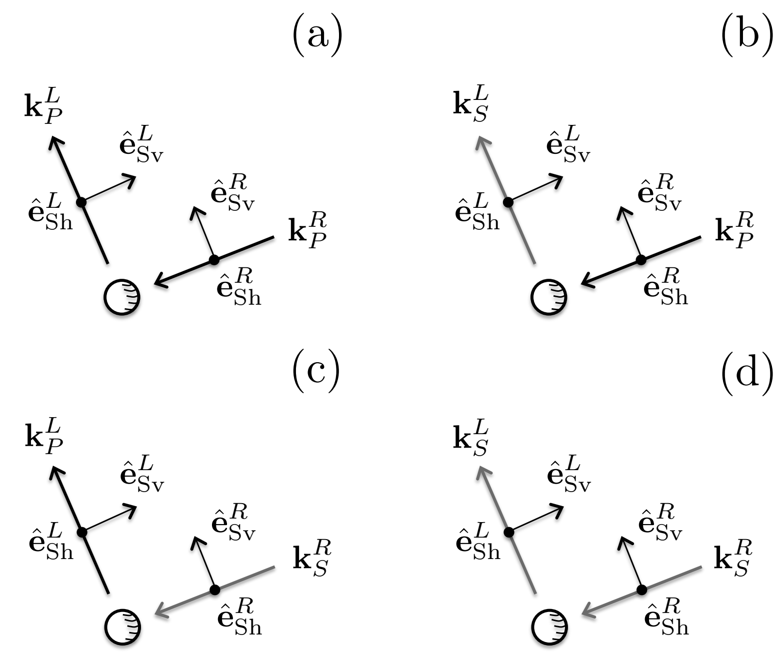

FIG. 1. Anelastic scattering scenarios: (a) P-P, (b) P-S, (c) S-P, and (d) S-S. The plane defined by the incoming and outgoing wave vectors contains the Sv waves, and the coordinate direction perpendicular to that plane (i.e., out of the page) contains the Sh waves. All of these vectors are potentially complex for anelastic waves.

pg. 67). Continuing in the vein of Stolt and Weglein (2012), we will next add a set of transformations that further subdivide the wave types into P-, Sv, and Sh. We formulate rotation matrices

$$\mathbf{E}_L = \begin{bmatrix} 1 & \mathbf{0}^T \\ 0 & \hat{\mathbf{e}}_{\text{Sv}}^{L\;T} \\ 0 & -\hat{\mathbf{e}}_{\text{Sh}}^{L\;T} \end{bmatrix}, \tag{37}$$

and

$$\mathbf{E}_R = \begin{bmatrix} 1 & \mathbf{0}^T \\ 0 & \hat{\mathbf{e}}_{\text{Sv}}^{R\;T} \\ 0 & -\hat{\mathbf{e}}_{\text{Sh}}^{R\;T} \end{bmatrix}, \tag{38}$$

such that the full action of Π and $\mathbf{E}$ on the displacement space scattering operators is

$$\mathrm{V}^{AA} = \mathbf{E}_L \Pi_L \left(\mathcal{V}^{\text{AA}}\right) \Pi_R^{-1} \mathbf{E}_R, \tag{39}$$

for scattering within an anelastic reference medium, and

$$\mathrm{V}^{AE} = \mathbf{E}_L \Pi_L \left(\mathcal{V}^{\text{AE}}\right) \Pi_R^{-1} \mathbf{E}_R, \tag{40}$$

for scattering in an elastic reference medium. Under this transformation, the elements

$$\mathrm{V}^{AA} = \begin{bmatrix} \mathrm{V}_{\text{PP}}^{AA} & \mathrm{V}_{\text{PSh}}^{AA} & \mathrm{V}_{\text{PSv}}^{AA} \\ \mathrm{V}_{\text{ShP}}^{AA} & \mathrm{V}_{\text{ShSh}}^{AA} & \mathrm{V}_{\text{ShSv}}^{AA} \\ \mathrm{V}_{\text{SvP}}^{AA} & \mathrm{V}_{\text{SvSh}}^{AA} & \mathrm{V}_{\text{SvSv}}^{AA} \end{bmatrix} = \begin{bmatrix} \mathrm{V}_{\text{PP}}^{AA} & 0 & \mathrm{V}_{\text{PSv}}^{AA} \\ 0 & \mathrm{V}_{\text{ShSh}}^{AA} & 0 \\ \mathrm{V}_{\text{SvP}}^{AA} & 0 & \mathrm{V}_{\text{SvSv}}^{AA} \end{bmatrix}, \tag{41}$$

and

$$\mathrm{V}^{AE} = \begin{bmatrix} \mathrm{V}^{AE}_{\mathrm{PP}} & \mathrm{V}^{AE}_{\mathrm{PSh}} & \mathrm{V}^{AE}_{\mathrm{PSv}} \\ \mathrm{V}^{AE}_{\mathrm{ShP}} & \mathrm{V}^{AE}_{\mathrm{ShSh}} & \mathrm{V}^{AE}_{\mathrm{ShSv}} \\ \mathrm{V}^{AE}_{\mathrm{SvP}} & \mathrm{V}^{AE}_{\mathrm{SvSh}} & \mathrm{V}^{AE}_{\mathrm{SvSv}} \end{bmatrix} = \begin{bmatrix} \mathrm{V}^{AE}_{\mathrm{PP}} & 0 & \mathrm{V}^{AE}_{\mathrm{PSv}} \\ 0 & \mathrm{V}^{AE}_{\mathrm{ShSh}} & 0 \\ \mathrm{V}^{AE}_{\mathrm{SvP}} & 0 & \mathrm{V}^{AE}_{\mathrm{SvSv}} \end{bmatrix}, \tag{42}$$

are directly interpretable as *the potentials for incoming waves of type P, Sv and/or Sh to scatter into waves of type P, Sv, and/or Sh.* Returning to Figure 1, we see that, by these rotations, Sv waves involve displacements in the plane formed by the incoming and outgoing wave vectors, and Sh waves involve displacements perpendicular to this plane. This is a deliberate rotation to a coordinate system in which no conversion from P- or Sv- to Sh-waves occur, nor any conversion from Sh- to P- or Sv-waves, which explains the zeros in the right-hand parts of equations (41) and (42).

3c. Anelastic scattering potentials given elastic reference media

If the reference medium is elastic, then the geometric framework developed by Stolt and Weglein (2012) can be used without alteration for the anelastic problem—we simply add the appropriate scattering contributions from the new Q_P and Q_S changes to those already in existence for V_P, V_S and ρ (or γ, μ, and ρ, which is Stolt's parametrization).

In it, we treat each element of V^{AE} separately. Here as an example we will express explicitly the P-P scattering potential; the other four nonzero potentials expand similarly. Evaluating the row-column multiplications in equation (40) that give rise to the $(1, 1)$ element of equation (42), we have

$$\mathrm{V}^{AE}_{\mathrm{PP}}(\mathbf{r},\omega) = \left(\frac{V_{P_0}}{\omega}\right)^2 \mathbf{k}_P^{L\,T} \cdot \mathcal{V}^{\mathrm{AE}} \cdot \mathbf{k}_P^R\,, \tag{43}$$

where $\mathbf{k}_P^L$ and $\mathbf{k}_P^R$ are real vectors of size ω/V_{P_0} with directions aligned with the outgoing and incoming propagation directions respectively. Taking the matrix $\mathcal{V}^{\mathrm{AE}}$ in equation (32), replacing the left and right derivative operators with $i\mathbf{k}_P^L$ and $i\mathbf{k}_P^R$ respectively, and evaluating the inner products in equation (43), we obtain

$$\mathrm{V}^{AE}_{\mathrm{PP}}(\mathbf{r},\omega) = \mathrm{V}^{AE}_{\mathrm{PPvp}}(\mathbf{r},\omega) + \mathrm{V}^{AE}_{\mathrm{PPvs}}(\mathbf{r},\omega) + \mathrm{V}^{AE}_{\mathrm{PP}\rho}(\mathbf{r},\omega) + \mathrm{V}^{AE}_{\mathrm{PPqp}}(\mathbf{r},\omega) + \mathrm{V}^{AE}_{\mathrm{PPqs}}(\mathbf{r},\omega), \tag{44}$$

where each perturbation (in V_P, V_S, ρ, Q_P and Q_S) is a possible contributor: for V_P we have

$$\mathrm{V}^{AE}_{\mathrm{PPvp}}(\mathbf{r},\omega) = -2\left(\frac{\rho_0 V_{P_0}^2}{\omega}\right)^2 \frac{\Delta V_P}{V_P} |\mathbf{k}_P^L|^2\,|\mathbf{k}_P^R|^2, \tag{45}$$

for V_S we have

$$\mathrm{V}^{AE}_{\mathrm{PPvs}}(\mathbf{r},\omega) = 2\left(\frac{\rho_0 V_{P_0}}{\omega}\right)^2 V_{S_0}^2 \frac{\Delta V_S}{V_S} |\mathbf{k}_P^L \times \mathbf{k}_P^R|^2, \tag{46}$$

for ρ we have

$$\mathrm{V}^{AE}_{\mathrm{PP}\rho}(\mathbf{r},\omega) = \rho_0 V_{P_0}^2 \frac{\Delta\rho}{\rho}\left[\mathbf{k}_P^L \cdot \mathbf{k}_P^R - \left(\frac{V_{P_0}}{\omega}\right)^2 |\mathbf{k}_P^L|^2\,|\mathbf{k}_P^R|^2\right], \tag{47}$$

for Q_P we have

$$V^{AE}_{\text{PPqp}}(\mathbf{r},\omega) = 2F_P(\omega)\left(\frac{\rho_0 V_{P_0}^2}{\omega}\right)^2 \frac{\Delta Q_P}{Q_P}|\mathbf{k}_P^L|^2\,|\mathbf{k}_P^R|^2, \tag{48}$$

and for Q_S we have

$$V^{AE}_{\text{PPqs}}(\mathbf{r},\omega) = -2F_S(\omega)\left(\frac{\rho_0 V_{P_0}}{\omega}\right)^2 V_{S_0}^2\frac{\Delta Q_S}{Q_S}|\mathbf{k}_P^L \times \mathbf{k}_P^R|^2. \tag{49}$$

Each of these scattering potentials can be inserted into a Born integral to determine the contribution to the reflection primary data due to variations in that parameter. We will return to these presently, and use them to calculate sensitivities for anelastic full waveform inversion.

SCATTERING COORDINATES FOR ANELASTIC REFERENCE MEDIA

For the case of anelastic reference media, the situation becomes significantly altered. In particular, the geometry and interpretation of the transformations from displacement to P- and S- potential space must be revisited. This is because the four propagation vectors for incoming and outgoing P- and S-waves are now complex:

$$\begin{aligned}
\mathbf{k}_P^L &= \mathbf{k}_{Pr}^L + i\mathbf{k}_{Pi}^L,\\
\mathbf{k}_P^R &= \mathbf{k}_{Pr}^R + i\mathbf{k}_{Pi}^R,\\
\mathbf{k}_S^L &= \mathbf{k}_{Sr}^L + i\mathbf{k}_{Si}^L,\\
\mathbf{k}_S^R &= \mathbf{k}_{Sr}^R + i\mathbf{k}_{Si}^R,
\end{aligned} \tag{50}$$

specifically having the magnitudes

$$\begin{aligned}
|\mathbf{k}_P^L| &= \frac{\omega}{V_{P_0}}\left[1 + Q_{P_0}^{-1}F_P(\omega)\right],\\
|\mathbf{k}_P^R| &= \frac{\omega}{V_{P_0}}\left[1 + Q_{P_0}^{-1}F_P(\omega)\right],\\
|\mathbf{k}_S^L| &= \frac{\omega}{V_{S_0}}\left[1 + Q_{S_0}^{-1}F_S(\omega)\right],\\
|\mathbf{k}_S^R| &= \frac{\omega}{V_{S_0}}\left[1 + Q_{S_0}^{-1}F_S(\omega)\right],
\end{aligned} \tag{51}$$

as discussed above. Following Borcherdt (2009), we recall that plane wave solutions for anelastic P- and S-waves are of the form

$$G \sim G_0 e^{-\mathbf{k}_{Pi}^L\cdot\mathbf{r}} e^{i\mathbf{k}_{Pr}^L\cdot\mathbf{r}}, \tag{52}$$

and

$$\mathbf{G} \sim \mathbf{G}_0 e^{-\mathbf{k}_{Si}^L\cdot\mathbf{r}} e^{i\mathbf{k}_{Sr}^L\cdot\mathbf{r}}, \tag{53}$$

respectively. In other words, the real and imaginary parts comprise the *propagation vector* and the *attenuation vector* respectively, and they may not be parallel. A P-wave for which

$\mathbf{k}_{Pr}^{L} \parallel \mathbf{k}_{Pr}^{L}$ is said to be *homogeneous*, and a P-wave for which $\mathbf{k}_{Pr}^{L} \nparallel \mathbf{k}_{Pr}^{L}$ is said to be *inhomogeneous*, and likewise with the S-wave.

In either case, if we follow Stolt's prescription, beginning with the incoming and outgoing wave vectors, then defining the Sh direction to be:

$$\hat{\mathbf{e}}_{\text{Sh}}^{L} = \hat{\mathbf{e}}_{\text{Sh}}^{R} = \hat{\mathbf{e}}_{\text{Sh}} = \frac{\mathbf{k}_{S}^{L} \times \mathbf{k}_{S}^{R}}{|\mathbf{k}_{S}^{L} \times \mathbf{k}_{S}^{R}|}, \tag{54}$$

and the incoming and outgoing Sv directions to be

$$\begin{aligned} \hat{\mathbf{e}}_{\text{Sv}}^{L} &= \frac{V_{S_0}}{\omega}\mathbf{k}_{S}^{L} \times \hat{\mathbf{e}}_{\text{Sh}}, \\ \hat{\mathbf{e}}_{\text{Sv}}^{R} &= \frac{V_{S_0}}{\omega}\mathbf{k}_{S}^{R} \times \hat{\mathbf{e}}_{\text{Sh}}, \end{aligned} \tag{55}$$

then we must grapple with the fact that all of these vectors are complex. The consequences of the complexity range from no consequence, to perhaps significant consequence, depending on whether the waves in question are homogeneous or inhomogeneous. In Figure 2 we illustrate schematically the four possible cases, which we will discuss briefly in turn.

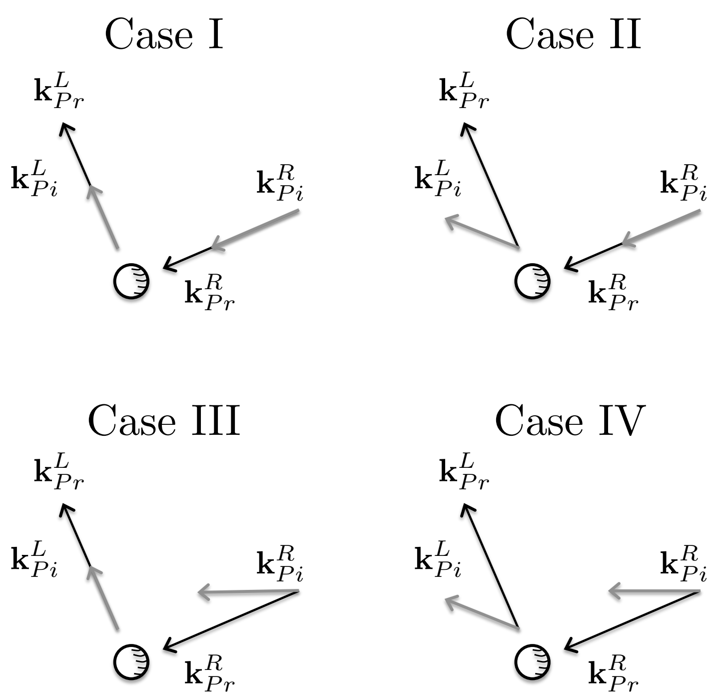

FIG. 2. Schematic diagram for incoming and outgoing waves in anelastic reference media. Case I: homogeneous incoming and outgoing waves. Case II: homogeneous incoming and inhomogeneous outgoing waves. Case III: inhomogeneous incoming and homogeneous outgoing waves. Case IV: inhomogeneous incoming and outgoing waves.

Case I: homogeneous incoming and outgoing waves

In this simplest case we can remain with the precise geometrical interpretation of the P, Sv, Sh decomposition of the elastic problem, because, since the attenuation vector and the propagation vector are parallel before and after scattering, the direction of the anelastic $\hat{\mathrm{e}}_{\mathrm{Sh}}$ is the same as that of the elastic case:

$$\hat{\mathrm{e}}_{\mathrm{Sh}} = \frac{\mathbf{k}^L_{Pr} \times \mathbf{k}^R_{Pr}}{|\mathbf{k}^L_{Pr} \times \mathbf{k}^R_{Pr}|}. \tag{56}$$

Case II: homogeneous incoming and inhomogeneous outgoing waves

If the incoming wave is homogeneous, then the real and imaginary parts of the right-hand $\mathbf{k}^R$ are collinear, and

$$\hat{\mathrm{e}}_{\mathrm{Sh}} \parallel \left[\left(\mathbf{k}^L_{Pr} - \mathbf{k}^L_{Pi}\right) + i\left(\mathbf{k}^L_{Pr} + \mathbf{k}^L_{Pi}\right)\right] \times \mathbf{k}^R_{Pr}. \tag{57}$$

Case III: inhomogeneous incoming and homogeneous outgoing waves

If the opposite is true, then the real and imaginary parts of the left-hand $\mathbf{k}^L$ are collinear, but the right-hand parts are not, and

$$\hat{\mathrm{e}}_{\mathrm{Sh}} \parallel \mathbf{k}^L_{Pr} \times \left[\left(\mathbf{k}^R_{Pr} - \mathbf{k}^L_{Pi}\right) + i\left(\mathbf{k}^R_{Pi} + \mathbf{k}^R_{Pr}\right)\right]. \tag{58}$$

Case IV: inhomogeneous incoming and outgoing waves

If both incoming and outgoing waves are inhomogeneous, then the Sh direction is maximally complicated and has real and imaginary parts

$$\hat{\mathrm{e}}_{\mathrm{Sh}} = |\mathbf{k}^L_P \times \mathbf{k}^R_P|^{-1}\left[\left(\mathbf{k}^L_{Pr} \times \mathbf{k}^R_{Pr} - \mathbf{k}^L_{Pi} \times \mathbf{k}^R_{Pi}\right) + i\left(\mathbf{k}^L_{Pr} \times \mathbf{k}^R_{Pi} + \mathbf{k}^L_{Pi} \times \mathbf{k}^R_{Pr}\right)\right]. \tag{59}$$

ELASTIC AND ANELASTIC SENSITIVITIES

With the anelastic scattering potentials derived it is relatively straightforward to compute the corresponding sensitivities.

Sensitivity of the P-P field to V_P

To demonstrate, we begin by replacing the left and right Green's functions by P-wave Green's functions, and forming the scattered field. If all perturbations except that associated with V_P are set to zero, we refer to the resulting scattered field as $\delta\mathrm{PP}_{\mathrm{vp}}$, which is given by

$$\begin{aligned}\delta\mathrm{PP}_{\mathrm{vp}}(\mathbf{r}_g, \mathbf{r}_s, \omega) = &\int_V d\mathbf{r}' G_P(\mathbf{r}_g, \mathbf{r}', \omega) \mathrm{V}^{AE}_{\mathrm{PPvp}}(\mathbf{r}, \omega) G_P(\mathbf{r}', \mathbf{r}_s, \omega) \\ &- 2\rho_0 \left(\frac{\omega}{V_{P_0}}\right)^2 \int_V d\mathbf{r}' G_P(\mathbf{r}_g, \mathbf{r}', \omega) \delta V_P(\mathbf{r}') G_P(\mathbf{r}', \mathbf{r}_s, \omega),\end{aligned} \tag{60}$$

where for notational convenience we have set $\delta V_P = \Delta V_P / V_P$. Following standard methods, we construct the sensitivities by focusing on the response of the scattered field to a small change at a fixed point $\mathbf{r}$. We do this by setting $\delta V_P(\mathbf{r}') = \delta V_P \delta(\mathbf{r} - \mathbf{r}')$, whereupon the volume integration is carried out analytically. Dividing through by δV_P and taking the limit as this perturbation goes to zero, we form the derivative

$$\frac{\partial \mathrm{PP}_{\mathrm{vp}}(\mathbf{r}_g, \mathbf{r}_s, \omega)}{\partial V_P(\mathbf{r})} = -2\rho_0 \left(\frac{\omega}{V_{P_0}}\right)^2 G_P(\mathbf{r}_g, \mathbf{r}, \omega) G_P(\mathbf{r}, \mathbf{r}_s, \omega), \tag{61}$$

which is identifiable as the P-P sensitivity to V_P changes.

Sensitivity of the P-P field to Q_P

Of more interest in the current paper is the sensitivity of the P-P data to local Q_P variations. Following exactly the same procedure, we begin with the variation in the P-P data from a variation in Q_P only:

$$\begin{aligned} \delta \mathrm{PP}_{\mathrm{qp}}(\mathbf{r}_g, \mathbf{r}_s, \omega) &= \int_V d\mathbf{r}' G_P(\mathbf{r}_g, \mathbf{r}', \omega) \mathrm{V}^{AE}_{\mathrm{PPqp}}(\mathbf{r}, \omega) G_P(\mathbf{r}', \mathbf{r}_s, \omega) \\ &= 2F_P(\omega)\rho_0 \left(\frac{\omega}{V_{P_0}}\right)^2 \int_V d\mathbf{r}' G_P(\mathbf{r}_g, \mathbf{r}', \omega) \delta Q_P(\mathbf{r}') G_P(\mathbf{r}', \mathbf{r}_s, \omega), \end{aligned} \tag{62}$$

where $\delta Q_P = \Delta Q_P / Q_P$; setting $\delta Q_P(\mathbf{r}') = \delta Q_P \delta(\mathbf{r} - \mathbf{r}')$, we finally have

$$\frac{\partial \mathrm{PP}_{\mathrm{qp}}(\mathbf{r}_g, \mathbf{r}_s, \omega)}{\partial Q_P(\mathbf{r})} = 2F_P(\omega)\rho_0 \left(\frac{\omega}{V_{P_0}}\right)^2 G_P(\mathbf{r}_g, \mathbf{r}, \omega) G_P(\mathbf{r}, \mathbf{r}_s, \omega). \tag{63}$$

All other sensitivities follow immediately from similar analysis.

CONCLUSIONS

Characterization of the scattering problem for anelastic waves has been carried out in full anelastic regime, focusing on some key issues: transformation of anelastic scattering potentials to the P-, Sv-, and Sh- potential domain, and the consequences to that transformation of moving from an elastic reference/anelastic perturbation model to an anelastic reference/anelastic perturbation model. We use the scattering potentials thus derived to produce sensitivity kernels for full waveform inversion iterates wherein V_P, V_S, ρ updates are carried out in elastic target determination, and Q_P and Q_S updates are added in anelastic target determination.

Key next steps are: (1) use the anelastic reference medium framework to further understand the ability of homogeneous waves to scattering into inhomogeneous waves, (2) use the same framework to discuss scattering of Borcherdt's Type-I and II anelastic S-waves, (3) use the anelastic sensitivities to analyze potential gradient-based and Newton iterates of full waveform inversion based thereupon.

ACKNOWLEDGMENTS

This work was funded by CREWES. CREWES sponsors and research personnel are gratefully acknowledged.

APPENDIX A - WAVE OPERATORS IN DISPLACEMENT SPACE

In displacement space we consider two types of wave operator. First, the anelastic case:

$$\mathcal{L}_A(\mathbf{r},\omega) = \begin{bmatrix} \mathcal{L}_{A11} & \mathcal{L}_{A13} & \mathcal{L}_{A14} \\ \mathcal{L}_{A21} & \mathcal{L}_{A22} & \mathcal{L}_{A23} \\ \mathcal{L}_{A31} & \mathcal{L}_{A32} & \mathcal{L}_{A33} \end{bmatrix}, \tag{64}$$

where the diagonal elements are given by

$$\begin{aligned}\mathcal{L}_{Aii} =& \partial_i \left[\rho V_P^2 \left(1 - 2Q_P^{-1} F_P(\omega)\right)\right] \partial_i \\ &+ \sum_{j\neq i} \partial_j \left[\rho V_S^2 \left(1 - 2Q_S^{-1} F_S(\omega)\right)\right] \partial_j + \omega^2 \rho,\end{aligned} \tag{65}$$

for $i, j = x, y, z$, and the off-diagonal elements by

$$\begin{aligned}\mathcal{L}_{Aij} =& \partial_i \left[\rho V_P^2 \left(1 - 2Q_P^{-1} F_P(\omega)\right) - 2\rho V_S^2 \left(1 - 2Q_S^{-1} F_S(\omega)\right)\right] \partial_j \\ &+ \partial_j \left[\rho V_S^2 \left(1 - 2Q_S^{-1} F_S(\omega)\right)\right] \partial_i\end{aligned} \tag{66}$$

for $i \neq j$. In general the parameters $V_P = V_P(\mathbf{r})$, $V_S = V_S(\mathbf{r})$, $\rho = \rho(\mathbf{r})$, $Q_P = Q_P(\mathbf{r})$ and $Q_S = Q_S(\mathbf{r})$ are all functions of space. And second, the elastic case, which is re-parametrization of the operator discussed by Stolt and Weglein (2012):

$$\mathcal{L}_E = \begin{bmatrix} \mathcal{L}_{E11} & \mathcal{L}_{E13} & \mathcal{L}_{E14} \\ \mathcal{L}_{E21} & \mathcal{L}_{E22} & \mathcal{L}_{E23} \\ \mathcal{L}_{E31} & \mathcal{L}_{E32} & \mathcal{L}_{E33} \end{bmatrix}, \tag{67}$$

where the diagonal elements are

$$\mathcal{L}_{Eii} = \partial_i \left(\rho V_P^2\right) \partial_i + \sum_{j\neq i} \partial_j \left(\rho V_S^2\right) \partial_j + \omega^2 \rho \tag{68}$$

as before for $i, j = x, y, z$, and the off-diagonal elements are

$$\mathcal{L}_{Eij} = \partial_i \left(\rho V_P^2 - 2\rho V_S^2\right) \partial_j + \partial_j \left(\rho V_S^2\right) \partial_i \tag{69}$$

for $i \neq j$. We will also discuss “reference” and “perturbed” versions of these operators, indicating the former with a subscript 0.

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