Higher order theory for analytic saddle point approximations to the $\grave{P}\acute{P}$ and $\grave{P}\acute{S}$ reflected arrivals at a solid/solid interface

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ABSTRACT

The high frequency solution to the problem of a $\grave{P}\acute{P}$ and $\grave{P}\acute{S}$ reflected wave at a plane interface between two isotropic homogeneous halfspaces at near grazing incidence is developed. In reflection seismology sub-critical reflections are often all that is required in data processing. A distorted (compared to the zero order geometrical optics approximation to the arrival) wavelet is seen on the traces in the area of grazing incidence. For a proper velocity distribution, there may an area in the vicinity of the critical point which also requires special treatment. This results in the zero order geometrical optics approximation being supplemented by a term involving Parabolic Cylinder Function of the order $1/2$. Grazing incidence may occur in cross hole tomography or for a point source in a thin surface layer with a large velocity contrast at the interface with the subsequent layer.

INTRODUCTION

$\grave{P}\acute{P}$ reflected wave, Červený and Ravindra (1970) and Brekhovskikh (1980). In each of these works a different conformal mapping is introduced so as to incorporate a real parameter in the solutions by high frequency methods involving the evaluation of integrals of the Sommerfeld type. After stating this, it should be noted that the motivation for considering the $\grave{P}\acute{S}$ reflected arrival is partly due to the fact that in the $\grave{P}\acute{P}$ reflected case the saddle point location may be obtained analytically, while in the $\grave{P}\acute{S}$ case this must be done numerically.

Another reason for considering this problem type is to introduce, in as simple a manner as possible, software for displaying the effects described above, which may be useful in the area of geophysical interpretation. To obtain analytical expressions for wave types such as the reflected $\grave{P}\acute{P}$ and $\grave{P}\acute{S}$ a thorough understanding of the formalism required to obtain solutions for these problems in the elastic case is an important and useful precondition. In the elastic case the saddle points and branch point lie on the real axis of the complex p-plane.

Higher order approximations are required when the spherical nature of the incident (P – wave) must be considered. This occurs for shallow (less than about 1/4 of a wavelength associated with the predominant frequency of the source wavelet) point sources of P – wave. This may cause erroneous results at near vertical and at grazing incidence.

What is considered here does not address those conditions of incidence but rather reflected arrivals that lie within a range of incident angles and as a result in an offset range that is considered in reflection seismology analysis.

Although plane wave reflection coefficients are what are of interest here, some aspects of spherically wave incidence must be introduced. Again, to keep the implementation of the theory as simple as possible, only 3 additional parameters, beyond those for the computation of plane wave reflection coefficients are required. These are the distance of both the source and receiver above the reflecting interface and some reference frequency, usually associated with the source wavelet being used in the acquisition of the seismic data being processed.

THEORY

Following the method presented in Aki and Richards (1980) for the analysis of the reflected potential due to a point source of $P-$ wave incidence at a free surface, the following presentation of the underlying theory will be followed for consistency. In the case investigated here two elastic medium are considered. The parameters describing the two media are the $P-$wave velocities, α_1 and α_2, the $S-$wave velocities, β_1 and β_2 and the densities, ρ_1 and ρ_2. A $P-$ wave source is located a positive distance h above the interface and a receiver is a positive distance z above the interface. The horizontal distance between the source and receiver is r. In terms of potentials the $\grave{P}\acute{P}$ reflected arrival is given by

$$\phi_{PP}\left(r,h_i,t\right)=i\omega\frac{e^{-i\omega t}}{2}\int_{-\infty}^{\infty}R_{PP}\left(p\right)H_0^{(1)}\left(\omega pr\right)\exp\left[i\omega\left(h+z\right)\xi_1\right]\frac{pdp}{\xi_1} \tag{1}$$

The horizontal and vertical components of displacement are obtained from

$$\mathbf{u}_{PP}=\left(u_r,u_z\right)=\left(\partial_r\phi_{PP},\partial_z\phi_{PP}\right) \tag{2}$$

to yield

$$u_r\left(r,h_i,t\right)=-i\omega^2\frac{e^{-i\omega t}}{2}\int_{-\infty}^{\infty}R_{PP}\left(p\right)H_1^{(1)}\left(\omega pr\right)\exp\left[i\omega\left(h+z\right)\xi_1\right]\frac{p^2dp}{\xi_1} \tag{3}$$

and

$$u_z\left(r,z,t\right)=-\omega^2\frac{e^{-i\omega t}}{2}\int_{-\infty}^{\infty}R_{PP}\left(p\right)H_0^{(1)}\left(\omega pr\right)\exp\left[i\omega\left(h+z\right)\xi_1\right]pdp \tag{4}$$

For the $\grave{P}\acute{S}$ problem, the potential is given by

$$\phi_{PS}\left(r,h_i,t\right)=i\omega\frac{e^{-i\omega t}}{2}\int_{-\infty}^{\infty}\left(\frac{\beta_1 R_{PP}\left(p\right)}{i\omega p\alpha_1}\right)H_0^{(1)}\left(\omega pr\right)\exp\left[i\omega\left(h\xi_1+z\eta_1\right)\right]\frac{pdp}{\xi_1} \tag{5}$$

so that with

$$\mathbf{w}_{PS}=\left(w_r,w_z\right)=\left(\partial_{rz}^2\phi_{PS},-r^{-1}\partial_r\left(r\partial_r\phi_{PS}\right)\right) \tag{6}$$

the resulting displacement components are

$$w_r\left(r,z,t\right)=\omega^3\frac{e^{-i\omega t}}{2}\int_{-\infty}^{\infty}\left(\frac{\beta_1 R_{PP}\left(p\right)}{i\omega p\alpha_1}\right)H_1^{(1)}\left(\omega pr\right)\exp\left[i\omega\left(h\xi_1+z\eta_1\right)\right]\frac{p^2\eta_1 dp}{\xi_1} \quad (7)$$

and

$$w_z\left(r,h_i,t\right)=i\omega^3\frac{e^{-i\omega t}}{2}\int_{-\infty}^{\infty}\left(\frac{\beta_1 R_{PP}\left(p\right)}{i\omega p\alpha_1}\right)H_0^{(1)}\left(\omega pr\right)\exp\left[i\omega\left(h\xi_1+z\eta_1\right)\right]\frac{p^3 dp}{\xi_1} \quad (8)$$

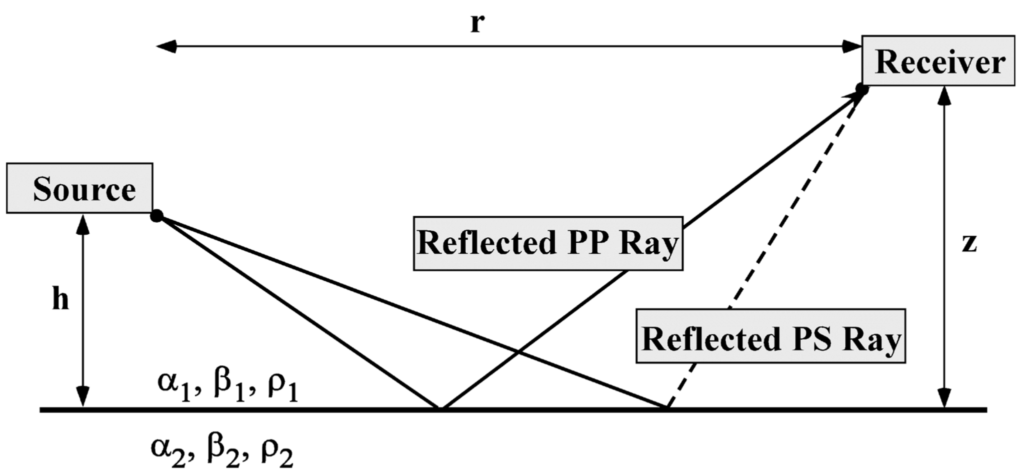

Fig. 1. Schematic of the incident rays due to reflection from a solid/solid interface between two elastic media in welded contact.

$\grave{P}\acute{S}$ REFLECTED PARTICLE DISPLACEMENT VECTOR COMPONENTS

Consider only the vertical component of $\grave{P}\acute{S}$ reflection, as all of the other three components of displacement may be solved for in a similar fashion.

$$w_z\left(r,h_i,t\right)=i\omega^3\frac{e^{-i\omega t}}{2}\int_{-\infty}^{\infty}\left(\frac{\beta_1 R_{PS}\left(p\right)}{i\omega p\alpha_1}\right)H_0^{(1)}\left(\omega pr\right)\exp\left[i\omega\left(h\xi_1+z\eta_1\right)\right]\frac{p^3 dp}{\xi_1} \quad (9)$$

$$w_z\left(r,h_i,t\right)=\omega^2\frac{e^{-i\omega t}}{2}\left(\frac{\beta_1}{\alpha_1}\right)\int_{-\infty}^{\infty}R_{PS}\left(p\right)H_0^{(1)}\left(\omega pr\right)\exp\left[i\omega\left(h\xi_1+z\eta_1\right)\right]\frac{p^2 dp}{\xi_1} \quad (10)$$

Introducing the large order expansion for the Hankel function (Abramowitz and Stegun, 1980).

$$H_0^{(1)}\left(\omega pr\right)=\left(2/\pi\omega pr\right)^{1/2}\left[1+1/8i\omega pr\right]e^{i\omega pr-i\pi/4} \quad (11)$$

results in

$$w_z\left(r,h_i,t\right)=\omega^2\frac{e^{-i\omega t}}{2}\left(\frac{\beta_1}{\alpha_1}\right)\int_{-\infty}^{\infty}R_{PS}\left(p\right)\left(1+\frac{1}{8i\omega pr}\right)e^{i\omega pr-i\pi/4}\sqrt{\frac{2}{\pi\omega pr}}\times \\ \exp\left[i\omega\left(h\xi_1+z\eta_1\right)\right]\frac{p^2dp}{\xi_1} \tag{12}$$

or expanding has

$$\begin{aligned} w_z\left(r,h_i,t\right)=\omega^2e^{-i\omega t-i\pi/4}\left(\frac{\beta_1}{\alpha_1}\right)\sqrt{\frac{1}{2\pi\omega}}\Bigg[\int_{-\infty}^{\infty}R_{PS}\left(p\right)\sqrt{\frac{1}{pr}}\times \\ \exp\left[i\omega\left(pr+h\xi_1+z\eta_1\right)\right]\frac{p^2dp}{\xi_1}+ \\ \int_{-\infty}^{\infty}R_{PS}\left(p\right)\left(\frac{1}{8i\omega pr}\right)\sqrt{\frac{1}{pr}}\times \\ \exp\left[i\omega\left(pr+h\xi_1+z\eta_1\right)\right]\frac{p^2dp}{\xi_1}\Bigg] \end{aligned} \tag{13}$$

which leads to

$$\begin{aligned} w_z\left(r,h_i,t\right)=\omega^2e^{-i\omega t-i\pi/4}\left(\frac{\beta_1}{\alpha_1}\right)\sqrt{\frac{1}{2\pi\omega}}\Bigg[\int_{-\infty}^{\infty}\frac{R_{PS}\left(p\right)}{\sqrt{r/p}}\times \\ \exp\left[i\omega\left(pr+h\xi_1+z\eta_1\right)\right]\frac{pdp}{\xi_1}+ \\ \left(\frac{1}{8i\omega pr}\right)\frac{1}{\sqrt{r/p}}\int_{-\infty}^{\infty}R_{PS}\left(p\right)\times \\ \exp\left[i\omega\left(pr+h\xi_1+z\eta_1\right)\right]\frac{pdp}{\xi_1}\Bigg] \end{aligned} \tag{14}$$

Note that terms in $\sqrt{r/p}$ have been left under the integral sign in the first integral. It is taken outside in the second integral as it is already a first order term, when compared to the zero order term in the first integral (in the geometrical optics sense, as it is of the order of $\left(i\omega\right)^{-1}$). Define the exponential term as

$$f\left(p\right)=rp+h\xi_1+z\eta_1 \tag{15}$$

so that the saddle point is given by the (numerical) solution of

$$f'(p)\Big|_{p=p_0} = r - \frac{hp}{\xi_1} - \frac{zp}{\eta_1}\Bigg|_{p=p_0} = 0 = r - \frac{hp}{\hat{\xi}_1} - \frac{zp}{\hat{\eta}_1} \tag{16}$$

and the second derivative, $f''(p)\big|_{p=p_0}$ is

$$f''(p)\Big|_{p=p_0} = -\left(\frac{hp_1^2}{\xi_1^3} + \frac{zp_3^2}{\eta_1^3}\right)\Bigg|_{p=p_0} = -\left(\frac{hp_1^2}{\hat{\xi}_1^3} + \frac{zp_3^2}{\hat{\eta}_1^3}\right). \tag{17}$$

Here

$$\xi_i = \left(\alpha_i^{-2} - p^2\right)^{1/2} = \left(p_i^2 - p^2\right)^{1/2} \quad , \quad p_i = \alpha_i^{-1} \quad (i = 1,2) \tag{18}$$

and

$$\eta_i = \left(\beta_i^{-2} - p^2\right)^{1/2} = \left(p_j^2 - p^2\right)^{1/2} \quad , \quad p_{j-2} = \beta_i^{-1} \quad (i = 1,2 : j = 3,4)\,. \tag{19}$$

where a circumflex over a parameter indicates that it is evaluated at the saddle point.

It will be assumed that in the vicinity of the saddle point

$$K(p) = \left(\frac{r}{p}\right)^{1/2} = \left(\frac{h}{\xi_1} + \frac{z}{\eta_1}\right)^{1/2}. \tag{20}$$

At this point it is convenient to introduce the change of variable in terms of the real quantity y as (Červený and Ravindra, 1970)

$$\left(p_1^2 - p^2\right)^{1/2} = \left(p_1^2 - p_0^2\right)^{1/2} - ye^{-i\pi/4} \quad (-\infty < y < \infty) \tag{21}$$

(Figure 2). It might be more correct to make y dimensionless by the replacement $y \to p_0 y$. However, the present definition is mathematically acceptable. Required formulae from (21) are

$$\frac{dp}{dy} = \frac{\left(p_1^2 - p^2\right)^{1/2}}{p} e^{-i\pi/4} \tag{22}$$

$$dp = \frac{\xi_1}{p} e^{-i\pi/4} dy \tag{23}$$

$$\frac{dp}{dy} = \frac{\xi_1}{p} e^{-i\pi/4} \quad \to \quad dp = \frac{\xi_1}{p} e^{-i\pi/4} dy \tag{24}$$

$$\frac{d^2 p}{dy^2} = \left[\frac{d}{dp}\left(\frac{dp}{dy}\right)\right]\left(\frac{dp}{dy}\right) = \frac{ip_1^2}{p^3} \tag{25}$$

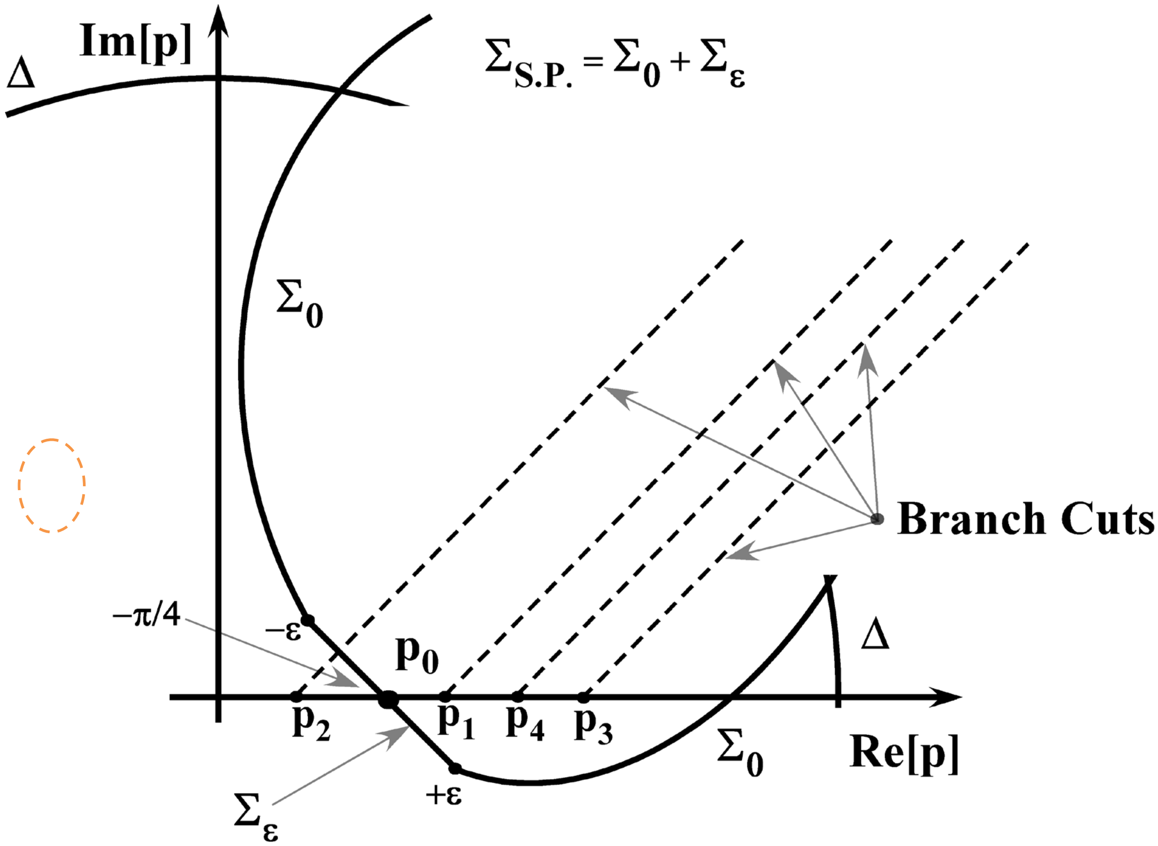

Fig. 2. Parameterized saddle point contour given by equation (21) in the first quadrant of the complex p-plane. The portion of the contour Σε has been slightly modified to show that in the vicinity of the saddle point p0 the saddle point contour passes through the saddle point at an angle of –π/4.

The quantities $R_{PP}(p)$ and $R_{PS}(p)$ are the displacement reflection coefficients specified in Aki and Richards (1980) and given in Appendices A and B together with their first and second derivatives with respect to p .

In the integrals in equation (14), define

$$Q_{PS}(p) = \frac{R_{PS}(p)}{(r/p)^{1/2}} \approx R_{PS}(p)\left(\frac{h}{\xi_1} + \frac{z}{\eta_1}\right)^{-1/2} = \frac{R_{PS}(p)}{K_{PS}(p)} \tag{26}$$

so that with

$$K_{PS}(p) = \left(\frac{r}{p}\right)^{1/2} = \left(\frac{h}{\xi_1} + \frac{z}{\eta_1}\right)^{1/2} \tag{27}$$

the following may be obtained

$$K'_{PS}(p)^2 = \frac{hp}{\xi_1^3} + \frac{zp}{\eta_1^3} \tag{28}$$

together with

$$K''_{PS}(p)^2 = \frac{h\left(p_1^2 + 2p^2\right)}{\xi_1^5} + \frac{z\left(p_3^2 + 2p^2\right)}{\eta_1^5} \tag{29}$$

leads to

$$K'_{PS}(p) = \frac{K'_{PS}(p)^2}{2K_{PS}(p)} \tag{30}$$

and

$$K''_{PS}(p) = \frac{1}{2}\left[\frac{2K'_{PS}(p)K''_{PS}(p) - K'_{PS}(p)^2 K_{PS}(p)}{K_{PS}(p)^2}\right] \tag{31}$$

or equivalently

$$K''_{PS}(p) = -\frac{K'_{PS}(p)^2}{4K_{PS}(p)^3} + \frac{K''_{PS}(p)^2}{2K_{PS}(p)}\,. \tag{32}$$

With equation (26)

$$Q_{PS}(p) = R_{PS}(p)/K_{PS}(p) \tag{26}$$

it may be determined that

$$Q'_{PS}(p) = \frac{R'_{PS}(p)}{K_{PS}(p)} - \frac{R_{PS}(p)K'_{PS}(p)}{K_{PS}^2(p)} \tag{33}$$

and

$$\begin{aligned} Q''_{PS}(p) = {} & \frac{R''_{PS}(p)}{K_{PS}(p)} - \frac{R'_{PS}(p)K'_{PS}(p)}{K_{PS}^2(p)} - \\ & -\left[\frac{R'_{PS}(p)K'_{PS}(p) + R_{PS}(p)K''(p)}{K_{PS}^2(p)} - \frac{2R_{PS}(p)\left(K'_{PS}(p)\right)^2}{K_{PS}^3(p)}\right] \end{aligned} \tag{34}$$

or equivalently

$$Q''_{PS}(p) = \frac{R''_{PS}(p)}{K_{PS}(p)} - \frac{\left[2R'_{PS}(p)K'(p) + R_{PS}(p)K'_{PS}(p)\right]}{K^2_{PS}(p)} + \frac{2R_{PS}(p)\left(K'_{PS}(p)\right)^2}{K^3_{PS}(p)} \tag{35}$$

From this is follows, using equations (22) and (25) with the above that

$$\frac{dQ_{PS}}{dy} = Q'_{PS}(p)\left(\frac{dp}{dy}\right) \tag{36}$$

and

$$\frac{d^2Q_{PS}}{dy^2} = \left[Q''_{PS}(p)\left(\frac{dp}{dy}\right)^2 + Q'_{PS}(p)\left(\frac{d^2p}{dy^2}\right)\right] \tag{37}$$

or

$$\frac{d^2Q_{PS}}{dy^2} = -\frac{i}{p_0^2}\left[Q''_{PS}(p_0)\hat{\xi}_1^2 - Q'_{PS}(p_0)\left(\frac{p_1^2}{p_0}\right)\right]. \tag{38}$$

Expanding $Q_{PS}(y)$ in a Taylor series about $p = p_0 \quad (y=0)$ has

$$Q_{PS}(y) \approx Q_{PS}(p_0) + Q'_{PS}(p_0)(p-p_0) + \frac{1}{2}\frac{d^2Q_{PS}}{dy^2}y^2 + \cdots \tag{39}$$

The second term in the above expansion will not contribute to the integral, leaving only the first and third terms. Thus equation (14) may be approximated as

$$w_z(r,h,t) = \omega^2 e^{-i\omega t - i\pi/4}\left(\frac{\beta_1^2}{2\pi\omega\alpha_1^2}\right)^{1/2}\left[Q_{PS}(p_0)\int_{-\infty}^{\infty} e^{i\omega\left[rp+(h\xi_1+z\eta_1)\right]}\frac{pdp}{\xi_1} + \frac{1}{2}\frac{d^2Q_{PS}}{dy^2}\int_{-\infty}^{\infty} y^2 e^{i\omega\left[rp+(h\xi_1+z\eta_1)\right]}\frac{pdp}{\xi_1} + \left(\frac{Q_{PS}(p_0)}{8i\omega p_0 r}\right)\int_{-\infty}^{\infty} e^{i\omega\left[rp+(h\xi_1+z\eta_1)\right]}\frac{pdp}{\xi_1}\right] \tag{40}$$

A Taylor series expansion of $f(p)$ in terms of y becomes

$$i\omega f\left(p\right)\approx i\omega f\left(p_0\right)+\frac{i\omega}{2}\left[f''\left(p\right)\left(\frac{dp}{dy}\right)^2\right]_{p_0}y^2=i\omega f\left(p_0\right)-\frac{\omega a^2}{2}y^2 \tag{41}$$

leading to

$$f''\left(p\right)\Big|_{p=p_0}=-\left(\frac{hp_1^2}{\hat{\xi}_1^3}+\frac{zp_3^2}{\hat{\eta}_1^3}\right)\Bigg|_{p=p_0}=-\left(\frac{hp_1^2}{\hat{\xi}_1^3}+\frac{zp_3^2}{\hat{\eta}_1^3}\right) \tag{42}$$

and finally to

$$\frac{i\omega}{2}\left[f''\left(p\right)\left(\frac{dp}{dy}\right)^2\right]_{p_0}y^2=-\frac{\omega}{2}\left(\frac{hp_1^2}{\hat{\xi}_1^3}+\frac{zp_3^2}{\hat{\eta}_1^3}\right)\left(\frac{\hat{\xi}_1}{p_0}\right)^2y^2=-\frac{\omega}{2}a_{PS}^2y^2 \tag{43}$$

where

$$a_{PS}^2=\left(\frac{hp_1^2}{\hat{\xi}_1^3}+\frac{zp_3^2}{\hat{\eta}_1^3}\right)\left(\frac{\hat{\xi}_1}{p_0}\right)^2. \tag{44}$$

The above expansion has taken the truncated form because $f'\left(p_0\right)=0$. The following sequence of steps (equations (45) – (49)) for the determination of $w_z\left(r,h,t\right)$ may then be taken

$$\begin{aligned}w_z\left(r,h,t\right)=\omega^2e^{-i\omega t-i\pi/2+\tau_{PS}}\left(\frac{\beta_1^2}{2\pi\omega\alpha_1^2}\right)^{1/2}\Bigg[&Q_{PS}\left(p_0\right)\int_{-\infty}^{\infty}e^{-\omega a^2y^2/2}dy+\\&\frac{1}{2}\frac{d^2Q_{PS}}{dy^2}\int_{-\infty}^{\infty}y^2e^{-\omega a^2y^2/2}dy+\\&\left(\frac{Q_{PS}\left(p_0\right)}{8i\omega p_0r}\right)\int_{-\infty}^{\infty}e^{-\omega a^2y^2/2}dy\Bigg]\end{aligned} \tag{45}$$

$$\begin{aligned}w_z\left(r,h,t\right)=\omega^2e^{-i\omega t-i\pi/2+i\tau_{PS}}\left(\frac{\beta_1^2}{2\pi\omega\alpha_1^2}\right)^{1/2}\Bigg[&Q_{PS}\left(p_0\right)\left(\frac{2\pi}{\omega a^2}\right)^{1/2}+\\&\frac{\sqrt{\pi}}{4}\left(\frac{2}{\omega a_{PS}^2}\right)^{3/2}\frac{d^2Q_{PS}}{dy^2}+\\&\left(\frac{Q_{PS}\left(p_0\right)}{8i\omega p_0r}\right)\left(\frac{2\pi}{\omega a_{PS}^2}\right)^{1/2}\Bigg]\end{aligned} \tag{46}$$

$$w_z\left(r,h,t\right)=\frac{\omega e^{-i\omega t-i\pi/2+i\tau_{PS}}}{a_{PS}}\left(\frac{\beta_1^2}{\alpha_1^2}\right)^{1/2}\times$$
$$\left[Q_{PS}\left(p_0\right)+\frac{1}{2}\left(\frac{1}{\omega a_{PS}^2}\right)\frac{d^2Q_{PS}}{dy^2}+\left(\frac{Q_{PS}\left(p_0\right)}{8i\omega p_0 r}\right)\right] \tag{47}$$

$$w_z\left(r,h,t\right)=\frac{-i\omega e^{-i\omega t+i\tau_{PS}}}{a_{PS}K_{PS}\left(p_0\right)}\frac{\beta_1}{\alpha_1}\times$$
$$\left[R_{PS}\left(p_0\right)+\frac{K_{PS}\left(p_0\right)}{2}\left(\frac{1}{\omega a_{PS}^2}\right)\frac{d^2Q_{PS}}{dy^2}+\frac{R_{PS}\left(p_0\right)}{8i\omega p_0 r}\right] \tag{48}$$

$$w_z\left(r,h,t\right)=\frac{-i\omega e^{-i\omega t+i\tau_{PS}}}{a_{PS}K_{PS}\left(p_0\right)}\frac{\beta_1}{\alpha_1}\times$$
$$\left[R_{PS}\left(p_0\right)+\frac{K_{PS}\left(p_0\right)}{2}\left(\frac{1}{\omega a_{PS}^2}\right)\frac{d^2Q_{PS}}{dy^2}+\frac{R_{PS}\left(p_0\right)}{8i\omega p_0 r}\right] \tag{49}$$

The third term is finite if $K\left(p_0\right)$ is used to define r. Based on the above derivation, the $\grave{P}\acute{S}$ radial vector component of displacement may be determined to be

$$w_r\left(r,z,t\right)=\frac{i\omega e^{-i\omega t+i\omega\tau_{PS}}}{K_{PS}\left(p_0\right)}\left(\frac{\beta_1\hat{\eta}_1}{\alpha_1 p_0 a_{PS}}\right)\left[R_{PS}\left(p_0\right)+\right.$$
$$\left.\left(\frac{K\left(p_0\right)}{2\omega a_{PS}^2}\right)\frac{d^2Q_{PS}}{dy^2}-\frac{3R_{PS}\left(p_0\right)}{8i\omega p_0 r}\right] \tag{50}$$

where

$$r=p_0\left(\frac{h}{\hat{\xi}_1}+\frac{z}{\hat{\eta}_1}\right) \tag{51}$$

Using a similar derivation as was used to obtain (50), the horizontal component of $\grave{P}\acute{S}$ is obtained as

$$w_r\left(r,z,t\right)=\frac{i\omega e^{-i\omega t+i\omega\tau_{PS}}}{K_{PS}\left(p_0\right)}\left(\frac{\beta_1\hat{\eta}_1}{\alpha_1 p_0 a_{PS}}\right)\left[R_{PS}\left(p_0\right)+\right.$$
$$\left.\left(\frac{K\left(p_0\right)}{2\omega a_{PS}^2}\right)\frac{d^2Q_{PS}}{dy^2}-\frac{3R_{PS}\left(p_0\right)}{8i\omega p_0 r}\right] \tag{52}$$

$\grave{P}\acute{P}$ REFLECTED PARTICLE DISPLACEMENT VECTOR COMPONENTS

Consider now the case of $\grave{P}\acute{P}$ reflection at an interface between two elastic isotropic solid media. The expressions for the radial and vertical components of particle displacement, $\mathbf{u}_{PP} = (u_r, u_z)$, are given by equations (3) and (4). If the same method as was used to obtain a modified saddle point approximation for the $\grave{P}\acute{S}$ case, the resultant expressions may be obtained as follows

$$u_r(r,z,t) = i\omega \frac{e^{-i\omega t + i\omega\tau_{PP}}}{a_{PP}K_{PP}(p_0)}\Big[R_{PP}(p_0) + \left(\frac{K_{PP}(p_0)}{2\omega a_{PP}^2}\right)\frac{d^2Q_{PP}}{dy^2} - \left(\frac{3R_{PP}(p_0)}{8i\omega p_0 r}\right)\Big] \tag{53}$$

$$u_z(r,z,t) = i\omega e^{-i\omega t - i\omega\tau_{PP}}\left(\frac{\hat{\xi}_1}{p_0}\right)\frac{1}{a_{PP}K_{PP}(p_0)}\Big[R_{PP}(p_0) + \frac{1}{2}\left(\frac{K_{PP}(p_0)}{\omega a_{PP}^2}\right)\frac{d^2Q_{PP}}{dy^2} + \left(\frac{R_{PP}(p_0)}{8i\omega p_0 r}\right)\Big] \tag{54}$$

In this instance, some function values have different forms such as

$$f(p) = rp + (h+z)\xi_1 . \tag{55}$$

The derivative of $f(p)$ with respect to p is equal to zero at $p = p_0$, which defines the saddle point. Also, for this problem

$$K_{PP}(p) = \left(\frac{r}{p}\right)^{1/2} = \left(\frac{h+z}{\xi_1}\right)^{1/2} \tag{56}$$

and the quantities $K'_{PP}(p)$ and $K''_{PP}(p)$ follow with minimal derivation. The term a_{PP}^2 has the form

$$a_{PP}^2 = \left(\frac{(h+z)p_1^2}{\hat{\xi}_1^3}\right)\left(\frac{\hat{\xi}_1}{p_0}\right)^2 . \tag{57}$$

Thus all quantites required to evaluate equation (53) and (54), except for the reflection coefficient $R_{PP}(p)$ and its first and second derivatives which may be found in the Appendices.

DISCUSSION AND CONCLUSIONS

Plane wave reflection coefficients together with correction terms as a consequence of the saddle point related to a specific reflected ray being close to grazing incidence for $\grave{P}\acute{P}$ and $\grave{P}\acute{S}$ plane wave reflections at an interface between two elastic media have been presented The addition of the correction term is similar, but not equivalent to obtaining higher order terms in the infinite asymptotic series which describes a given reflection or transmission between two elastic media.

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APPENDIX A: DISPLACEMENT REFLECTION COEFFICIENTS $\grave{P}\acute{P}$ AND $\grave{P}\acute{S}$ AND THEIR FIRST AND SECOND DERIVATIVES WITH RESPECT TO HORIZONTAL SLOWNESS

Two elastic media are assumed to be in welded contact. The parameters describing the two media are the $P-$wave velocities, $\alpha_j\ (j=1,2)$, the $S-$wave velocities, $\beta_j\ (j=1,2)$ and the densities $\rho_j\ (j=1,2)$. The upper medium is denoted as "1" and the lower as "2".

$\grave{P}\acute{P}$ Coefficient:

The $\grave{P}\acute{P}$ reflection coefficient, $R_{PP}(p)$, may be written as (Aki and Richards, 1980)

$$R_{PP}(p)=\frac{\tilde{E}F-\tilde{G}Hp^2}{D} \tag{A.1}$$

$\grave{P}\acute{S}$ Coefficient:

The plane wave particle displacement $\grave{P}\acute{S}$ reflection coefficient, $R_{PS}(p)$, at an interface between two elastic media has the form (Aki and Richards, 1980)

$$R_{PS}(p)=-\frac{2\alpha_1\xi_1(ab+cd\xi_2\eta_2)p}{\beta_1 D}=-2\left(\frac{\alpha_1}{\beta_1}\right)\left(\frac{\xi_1(ab+cd\xi_2\eta_2)p}{D}\right) \tag{A.2}$$

$$R_{PS}(p)=-2\left(\frac{\alpha_1}{\beta_1}\right)\left(\frac{U}{D}\right) \tag{A.3}$$

$$D=EF+GHp^2 \tag{A.4}$$

$$R'_{PS}\left(p\right)=-2\left(\frac{\alpha_1}{\beta_1}\right)\left(\frac{U'}{D}-\frac{UD'}{D^2}\right) \tag{A.5}$$

$$R''_{PS}\left(p\right)=-2\left(\frac{\alpha_1}{\beta_1}\right)\left(\frac{U''}{D}-\frac{\left(2U'D'+UD''\right)}{D^2}+\frac{2U\left(D'\right)^2}{D^3}\right) \tag{A.6}$$

$$U=\xi_1\left(ab+cd\xi_2\eta_2\right)p \tag{A.7}$$

$$\begin{aligned}U'=&\left(\xi_1'p+\xi_1\right)\left(ab+cd\xi_2\eta_2\right)+\\&\xi_1 p\left(a'b+ab'+c'd\xi_2\eta_2+cd\xi_2'\eta_2+cd\xi_2\eta_2'\right)\end{aligned} \tag{A.8}$$

$$\begin{aligned}U''=&\left(\xi_1''p+2\xi'\right)\left(ab+cd\xi_2\eta_2\right)+\\&2\left(\xi_1'p+\xi_1\right)\left[a'b+ab'+c'd\xi_2\eta_2+cd\left(\xi_2'\eta_2+\xi_2\eta_2'\right)\right]+\\&\xi_1 p\left[a''b+2a'b'+ab''+c''d\xi_2\eta_2+2c'd\left(\xi_2'\eta_2+\xi_2\eta_2'\right)\right]+\\&\xi_1 pcd\left(\xi_2''\eta_2+2\xi_2'\eta'+\xi_2\eta_2''\right)\end{aligned} \tag{A.9}$$

$$D=EF+GHp^2 \tag{A.10}$$

$$D'=E'F+EF'+\left(G'H+GH'\right)p^2+2GHp \tag{A.11}$$

$$\begin{aligned}D''=&E''F+2E'F'+EF''+\\&\left(G''H+2G'H'+GH''\right)p^2+4p\left(G'H+GH'\right)+2GH\end{aligned} \tag{A.12}$$

$$R_{PP}\left(p\right)=\frac{\tilde{E}F-\tilde{G}Hp^2}{D} \tag{A.13}$$

$$R_{PP}\left(p\right)=\frac{U}{D} \tag{A.14}$$

$$U=\tilde{E}F-\tilde{G}Hp^2 \tag{A.15}$$

$$D=EF+GHp^2 \tag{A.16}$$

$$R'_{PP}\left(p\right)=\left(\frac{U'}{D}-\frac{UD'}{D^2}\right) \tag{A.17}$$

$$R''_{PP}\left(p\right)=\left(\frac{U''}{D}-\frac{\left(2U'D'+UD''\right)}{D^2}+\frac{2U\left(D'\right)^2}{D^3}\right) \tag{A.18}$$

$$U'=\tilde{E}'F+\tilde{E}F'-\left(\tilde{G}'H+\tilde{G}H'\right)p^2-2\tilde{G}Hp \tag{A.19}$$

$$\begin{aligned}U''=&\tilde{E}''F+2\tilde{E}'F'+\tilde{E}F''-\\&\left(\tilde{G}''H+2\tilde{G}'H'+\tilde{G}H''\right)p^2-4p\left(\tilde{G}'H+\tilde{G}H'\right)-2\tilde{G}H\end{aligned} \tag{A.20}$$

APPENDIX B: REQUIRED QUANTITIES FOR APPENDIX A

The quantities requiring definition in the above reflection coefficients and their first and second derivatives with respect to p are given by:

$$\xi_i = \left(p_i^{-2} - p^2\right)^{1/2},\ i = 1,2. \quad \left(p_i = 1/\alpha_i\right) \tag{B.1}$$

$$\xi_i' = -\left(p/\xi_i\right),\ i = 1,2. \tag{B.2}$$

$$\xi_i'' = -\left(p_i^2/\xi_i^3\right),\ i = 1,2. \tag{B.3}$$

$$\eta_i = \left(p_j^{-2} - p^2\right)^{1/2},\ i = 1,2. \quad j = 3,4. \quad \left(p_j = 1/\beta_{j-2}\right) \tag{B.4}$$

$$\eta_i' = -\left(p/\eta_i\right),\ i = 1,2. \tag{B.5}$$

$$\eta_i'' = -\left(p_j^2/\eta_i^3\right),\ i = 1,2. \quad j = 3,4. \tag{B.6}$$

$$E = b\xi_1 + c\xi_2 \tag{B.7}$$

$$E' = b'\xi_1 + b\xi_1' + c'\xi_2 + c\xi_2' \tag{B.8}$$

$$E'' = b''\xi_1 + 2b'\xi_1' + b\xi_1'' + c''\xi_2 + 2c'\xi_2' + c\xi_2'' \tag{B.9}$$

$$\tilde{E} = b\xi_1 - c\xi_2 \tag{B.10}$$

$$\tilde{E}' = b'\xi_1 + b\xi_1' - c'\xi_2 - c\xi_2' \tag{B.11}$$

$$\tilde{E}'' = b''\xi_1 + 2b'\xi_1' + b\xi_1'' - c''\xi_2 - 2c'\xi_2' - c\xi_2'' \tag{B.12}$$

$$F = b\eta_1 + c\eta_2 \tag{B.13}$$

$$F' = b'\eta_1 + b\eta_1' + c'\eta_2 + c\eta_2' \tag{B.14}$$

$$F'' = b''\eta_1 + 2b'\eta_1' + b\eta_1'' + c''\eta_2 + 2c'\eta_2' + c\eta_2'' \tag{B.15}$$

$$G = a - d\xi_1\eta_2 \tag{B.16}$$

$$G' = a' - d\xi_1'\eta_2 - d\xi_1\eta_2' \tag{B.17}$$

$$G' = a'' - d\xi_1''\eta_2 - 2d\xi_1'\eta_2' - d\xi_1\eta'' \tag{B.18}$$

$$\tilde{G} = a + d\xi_1\eta_2 \tag{B.19}$$

$$\tilde{G}' = a' + d\xi_1'\eta_2 + d\xi_1\eta_2' \tag{B.20}$$

$$\tilde{G}'' = a'' + d\xi_1''\eta_2 + 2d\xi_1'\eta_2' + d\xi_1\eta'' + 2d\xi_1'\eta_2' \tag{B.21}$$

$$H = a - d\xi_2\eta_1 \tag{B.22}$$

$$H' = a' - d\xi_2'\eta_1 - d\xi_2\eta_1' \tag{B.23}$$

$$H'' = a'' - d\xi_2''\eta_1 - 2d\xi_2'\eta_1' - d\xi_2\eta_1'' \tag{B.24}$$

with

$$a = \rho_2\left(1-2\beta_2^2p^2\right) - \rho_1\left(1-2\beta_1^2p^2\right) \tag{B.25}$$

$$a' = -4p\left(\rho_2\beta_2^2 - \rho_1\beta_1^2\right) \tag{B.26}$$

$$a'' = -4\left(\rho_2\beta_2^2 - \rho_1\beta_1^2\right) \tag{B.27}$$

$$b = \rho_2\left(1-2\beta_2^2p^2\right) + 2\rho_1\beta_1^2p^2 \tag{B.28}$$

$$b' = -4p\left(\rho_2\beta_2^2 - \rho_1\beta_1^2\right) = a' \tag{B.29}$$

$$b'' = -4\left(\rho_2\beta_2^2 - \rho_1\beta_1^2\right) = a'' \tag{B.30}$$

$$c = \rho_1\left(1-2\beta_1^2p^2\right) + 2\rho_2\beta_2^2p^2 \tag{B.31}$$

$$c' = -4p\left(\rho_1\beta_1^2 - \rho_2\beta_2^2\right) = -a' \tag{B.32}$$

$$c'' = -4\left(\rho_1\beta_1^2 - \rho_2\beta_2^2\right) = -a'' \tag{B.33}$$

$$d = 2\left(\rho_2\beta_2^2 - \rho_1\beta_1^2\right) \tag{B.34}$$

$$d' = 0 \tag{B.35}$$

$$d'' = 0 \tag{B.36}$$

$$a' = b' = -c' \tag{B.37}$$

$$a'' = b'' = -c'' \tag{B.38}$$