Minimized energy used to determine displacements at an internal boundary (finite-difference models)

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ABSTRACT

A method for calculation of internal boundary conditions, as opposed to edge boundary conditions, is explained. The method of minimizing energy within the rigid zone below a water bottom is developed, and the sequence of matrix equations required is presented in some detail.

INTRODUCTION

This paper developed from a CREWES report of last year (Manning and Wong, 2013). In that paper seismic traces were generated from a numerical model featuring water overlying an elastic medium, then compared to simulated seismic traces from a corresponding physical model. The models were quite consistent except that the physical model generated considerable energy propagating along the water bottom, which was not seen in the numerical model.

This mismatch prompted further study, which revealed that the physics of the water bottom is more complex than the numerical model could account for at the time. This paper is an attempt to quantify the physics in a way that is practical for model purposes.

THEORY

The model configuration used for the water bottom and its adjacent displacements is shown in Figure 1. The unique part of this depiction is the almost duplicate Ux displacements directly above and below the water bottom. With numerical models previously run by the author, calculations provided for only one set of displacements at this position. The upper set of displacements is assumed to be adequately calculated from pressure theory, as pressures must be continuous across the boundary. The lower set of displacements must be accounted for by the unique physics of the water bottom.

The indices which apply to the Ux and Uz displacements are shown at the top and left of the figure, and are centered on ix and iz. The index ix is a relative (arbitrary) index, but iz indicates the topmost grid-point where μ is greater than zero, and therefore within the solid water bottom.

A reduced size stress/strain matrix used to describe the displacements along the water bottom of Figure 1 is shown in Figure 2. Normally, a matrix used within a real model requires a vector of displacements equal to the width of the model (hundreds of points), but the 5 displacement miniature model here demonstrates the principles.

The upper half of the matrix is a pure diagonal matrix with a column of constants which defines the zero shear stress displacements. The lower half is a similar near-diagonal matrix where the column defines the zero compressional stress displacement differences.

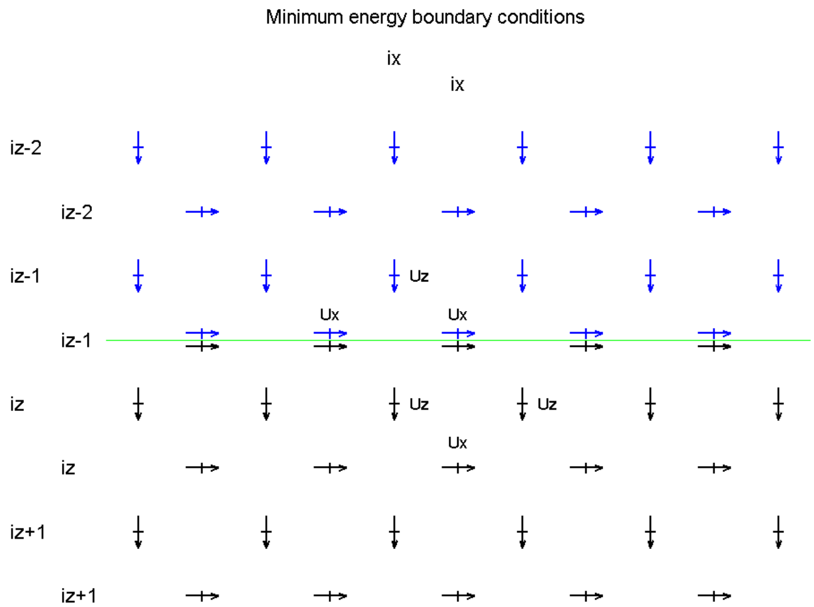

Figure 1: The displacements in the vicinity of the green line depicting the water bottom. It is assumed that at this point within a time step, all the Z-displacements have been calculated, and also all the X-displacements above and for one sample below the water bottom and lower. The X-displacements just below the water bottom must then be made consistent with the appropriate boundary conditions.

Combination shear and compressional stress/strain matrix

$$\begin{bmatrix} \mu_1 & 0 & 0 & 0 & 0 & \mu_1 Xs_1 \\ 0 & \mu_2 & 0 & 0 & 0 & \mu_2 Xs_2 \\ 0 & 0 & \mu_3 & 0 & 0 & \mu_3 Xs_3 \\ 0 & 0 & 0 & \mu_4 & 0 & \mu_4 Xs_4 \\ 0 & 0 & 0 & 0 & \mu_5 & \mu_5 Xs_5 \\ L_1 & 0 & 0 & 0 & 0 & L_1 Zc_1 \\ -L_2 & L_2 & 0 & 0 & 0 & L_2 Zc_2 \\ 0 & -L_3 & L_3 & 0 & 0 & L_3 Zc_3 \\ 0 & 0 & -L_4 & L_4 & 0 & L_4 Zc_4 \\ 0 & 0 & 0 & -L_5 & L_5 & L_5 Zc_5 \\ 0 & 0 & 0 & 0 & -L_6 & L_6 Zc_6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ 1 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \\ f_7 \\ f_8 \\ f_9 \\ f_{10} \\ f_{11} \end{bmatrix}$$

Figure 2: An abbreviated version of a matrix equation which shows how the Ux displacements immediately below the water bottom contribute to the shear and compressional stresses within the water bottom. The x vector specifies input displacement strains. Outputs f_1 through f_5 represent shear stresses, and f_6 through f_{11} represent compressional stresses.

The shear stresses across the water bottom are directly dependent on the displacements compared to the net shearing imposed from just below the water bottom (given by the Xs_i's). These stresses are dependent on the shear moduli (μ's) in the top half of the matrix. Each Xs_i is the zero shear-stress displacement determined by the box of displacements shown in Figure 1, where

$$Xs_{ix} = Ux(ix,iz+1) + Uz(iz+1,ix+1) - Uz(ix,iz+1), \qquad 1$$

and these displacements appear in part of the right column within the matrix.

The horizontal compressional stresses are dependent on the differences of the vertical displacements across the water bottom. The zero stress compressional displacement is then given by

$$Zc_{ix} = (Uz(ix,iz) - Uz(ix,iz-1))*\lambda/(\lambda+2\mu), \qquad 2$$

and these appear in the same right column, but in the lower positions.

The water bottom shear stress is directly dependent on the horizontal displacement from neutral, and therefore each has only one entry in the matrix. The water bottom compressional stress (in excess of the pure pressure wave stresses) depends on the difference of the horizontal displacements from neutral, and therefore has two adjacent entries in the matrix, with opposite polarities of equal amplitude.

The components of the pseudo-Toeplitz matrix

$$\begin{bmatrix} \mu_1 & 0 & 0 & 0 & 0 & L_1 & -L_2 & 0 & 0 & 0 & 0 \\ 0 & \mu_2 & 0 & 0 & 0 & 0 & L_2 & -L_3 & 0 & 0 & 0 \\ 0 & 0 & \mu_3 & 0 & 0 & 0 & 0 & L_3 & -L_4 & 0 & 0 \\ 0 & 0 & 0 & \mu_4 & 0 & 0 & 0 & 0 & L_4 & -L_5 & 0 \\ 0 & 0 & 0 & 0 & \mu_5 & 0 & 0 & 0 & 0 & L_5 & -L_6 \\ \mu_1 Xs_1 & \mu_2 Xs_2 & \mu_3 Xs_3 & \mu_4 Xs_4 & \mu_5 Xs_5 & L_1 Zc_1 & L_2 Zc_2 & L_3 Zc_3 & L_4 Zc_4 & L_5 Zc_5 & L_6 Zc_6 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & Xs_1 \\ 0 & 1 & 0 & 0 & 0 & Xs_2 \\ 0 & 0 & 1 & 0 & 0 & Xs_3 \\ 0 & 0 & 0 & 1 & 0 & Xs_4 \\ 0 & 0 & 0 & 0 & 1 & Xs_5 \\ 1 & 0 & 0 & 0 & 0 & Zc_1 \\ -1 & 1 & 0 & 0 & 0 & Zc_2 \\ 0 & -1 & 1 & 0 & 0 & Zc_3 \\ 0 & 0 & -1 & 1 & 0 & Zc_4 \\ 0 & 0 & 0 & -1 & 1 & Zc_5 \\ 0 & 0 & 0 & 0 & -1 & Zc_6 \end{bmatrix}$$

Figure 3: The components of the Toeplitz energy matrix. The matrix from Figure 2 is transposed and appears on the left of this Figure. The matrix on the right has the same form as the one in Figure 2, but the elastic constants are replaced by a +1 or -1, so that it may calculate displacements instead of stresses.

The energy content of a material depends on the stresses calculated in Figure 2 but must be multiplied by their corresponding relative displacements to obtain a measure of the energy stored. If the elastic constants in Figure 2 are replaced by 1's of the same polarity, the outputs are the relative displacements, and this is the matrix on the right in Figure 3.

The matrix on the left in Figure 3 shows the transpose of the matrix in Figure 2, and so the two matrices are in position to multiply together and produce the stress times distance measure of energy. The multiplication gives the matrix in Figure 4, and the displacement vectors are also included to give the total energy stored as a function of displacements.

$$\begin{bmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & 1 \end{bmatrix} \begin{bmatrix} \mu_1+L_{12} & -L_2 & 0 & 0 & 0 & C_1 \\ -L_2 & \mu_2+L_{23} & -L_3 & 0 & 0 & C_2 \\ 0 & -L_3 & \mu_3+L_{34} & -L_4 & 0 & C_3 \\ 0 & 0 & -L_4 & \mu_4+L_{45} & -L_5 & C_4 \\ 0 & 0 & 0 & -L_5 & \mu_5+L_{56} & C_5 \\ C_1 & C_2 & C_3 & C_4 & C_5 & \Sigma C_i^2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ 1 \end{bmatrix} = E$$

$$L_{ij} = L_i + L_j$$

$$C_i = \mu_i Xs_i + L_i Zc_i - L_{i+1} Zc_{i+1}$$

Figure 4: The matrix equation which sums the total elastic energy (E) in the water bottom. The square Toeplitz matrix here in the center results from multiplying the two matrices of Figure 3, and there is now room to display the two displacement vectors to complete the equation.

ZERO DERIVATIVES FROM A GENERAL SUM

The principle of finding maxima from the zeros of derivatives is well known, but it is difficult to do this for a sum laid out as in Figure 4. Consideration of a more general sum is easier to follow, and Figure 5 is a generalized version of Figure 4 after the multiplication with the vectors is carried out. For example $a_{21} = -L_2$.

$$\begin{bmatrix} +a_{11}x_1x_1 & +a_{21}x_2x_1 & +a_{31}x_3x_1 & +a_{41}x_4x_1 & +a_{51}x_5x_1 & +a_{61}x_1 \\ +a_{12}x_1x_2 & +a_{22}x_2x_2 & +a_{32}x_3x_2 & +a_{42}x_4x_2 & +a_{52}x_5x_2 & +a_{62}x_2 \\ +a_{13}x_1x_3 & +a_{23}x_2x_3 & +a_{33}x_3x_3 & +a_{43}x_4x_3 & +a_{53}x_5x_3 & +a_{63}x_3 \\ +a_{14}x_1x_4 & +a_{24}x_2x_4 & +a_{34}x_3x_4 & +a_{44}x_4x_4 & +a_{54}x_5x_4 & +a_{64}x_4 \\ +a_{15}x_1x_5 & +a_{25}x_2x_5 & +a_{35}x_3x_5 & +a_{45}x_4x_5 & +a_{55}x_5x_5 & +a_{65}x_5 \\ +a_{16}x_1 & +a_{26}x_2 & +a_{36}x_3 & +a_{46}x_4 & +a_{56}x_5 & +a_{66} \end{bmatrix}$$

Figure 5: A general sum of energy terms consisting of 5 variables and one constant. Note that the terms here do not make up a matrix, but are laid out in this form to show the pattern.

Extreme values are found from this sum by differentiating with respect to each variable and setting the resulting equation equal to zero. The principle is that a continuous function must be zero at a maximum or minimum. Figure 6 shows this as an equation for the extreme position along the X_i co-ordinate.

$$\partial/\partial x_i \left[\begin{array}{llllll} + a_{11}x_1x_1 & + a_{21}x_2x_1 & + a_{31}x_3x_1 & + a_{41}x_4x_1 & + a_{51}x_5x_1 & + a_{61}x_1 \\ + a_{12}x_1x_2 & + a_{22}x_2x_2 & + a_{32}x_3x_2 & + a_{42}x_4x_2 & + a_{52}x_5x_2 & + a_{62}x_2 \\ + a_{13}x_1x_3 & + a_{23}x_2x_3 & + a_{33}x_3x_3 & + a_{43}x_4x_3 & + a_{53}x_5x_3 & + a_{63}x_3 \\ + a_{14}x_1x_4 & + a_{24}x_2x_4 & + a_{34}x_3x_4 & + a_{44}x_4x_4 & + a_{54}x_5x_4 & + a_{64}x_4 \\ + a_{15}x_1x_5 & + a_{25}x_2x_5 & + a_{35}x_3x_5 & + a_{45}x_4x_5 & + a_{55}x_5x_5 & + a_{65}x_5 \\ + a_{16}x_1 & + a_{26}x_2 & + a_{36}x_3 & + a_{46}x_4 & + a_{56}x_5 & + a_{66} \end{array}\right] = 0$$

Figure 6: The equation which gives the position of the extreme value for variable X_i.

As an example, Figure 7 shows the derivative with respect to X_4, and highlights the terms which contain this variable. All other terms may be treated as constant with respect to X_4, and so do not contribute to the result. Of course the power of X_4 is reduced by one, etc.

$$\partial/\partial x_4 \left[\begin{array}{llllll} + a_{11}x_1x_1 & + a_{21}x_2x_1 & + a_{31}x_3x_1 & + a_{41}x_4x_1 & + a_{51}x_5x_1 & + a_{61}x_1 \\ + a_{12}x_1x_2 & + a_{22}x_2x_2 & + a_{32}x_3x_2 & + a_{42}x_4x_2 & + a_{52}x_5x_2 & + a_{62}x_2 \\ + a_{13}x_1x_3 & + a_{23}x_2x_3 & + a_{33}x_3x_3 & + a_{43}x_4x_3 & + a_{53}x_5x_3 & + a_{63}x_3 \\ + a_{14}x_1x_4 & + a_{24}x_2x_4 & + a_{34}x_3x_4 & + a_{44}x_4x_4 & + a_{54}x_5x_4 & + a_{64}x_4 \\ + a_{15}x_1x_5 & + a_{25}x_2x_5 & + a_{35}x_3x_5 & + a_{45}x_4x_5 & + a_{55}x_5x_5 & + a_{65}x_5 \\ + a_{16}x_1 & + a_{26}x_2 & + a_{36}x_3 & + a_{46}x_4 & + a_{56}x_5 & + a_{66} \end{array}\right] = 0$$

Figure 7: Only the highlighted terms contribute to the particular derivative equation in X_4.

$$\begin{array}{llllll} 2a_{11}x_1 + & (a_{12}+a_{21})x_2 + & (a_{13}+a_{31})x_3 + & (a_{14}+a_{41})x_4 + & (a_{15}+a_{51})x_5 + & (a_{16}+a_{61}) = 0 \\ (a_{21}+a_{12})x_1 + & 2a_{22}x_2 + & (a_{23}+a_{32})x_3 + & (a_{24}+a_{42})x_4 + & (a_{25}+a_{52})x_5 + & (a_{26}+a_{62}) = 0 \\ (a_{31}+a_{13})x_1 + & (a_{32}+a_{23})x_2 + & 2a_{33}x_3 + & (a_{34}+a_{43})x_4 + & (a_{35}+a_{53})x_5 + & (a_{36}+a_{63}) = 0 \\ (a_{41}+a_{14})x_1 + & (a_{42}+a_{24})x_2 + & (a_{43}+a_{34})x_3 + & 2a_{44}x_4 + & (a_{45}+a_{54})x_5 + & (a_{46}+a_{64}) = 0 \\ (a_{51}+a_{15})x_1 + & (a_{52}+a_{25})x_2 + & (a_{53}+a_{35})x_3 + & (a_{54}+a_{45})x_4 + & 2a_{55}x_5 + & (a_{56}+a_{65}) = 0 \end{array}$$

Figure 8: The five equations which define the maxima for each variable.

TRANSLATION TO REAL VALUES

The equations in Figure 8 may have the indexed constants replaced by the particular constants of Figure 4, and translated into matrix form in Figure 9. The basic matrices of the two figures are essentially identical because of the symmetry of the off diagonal terms.

$$\begin{bmatrix} 2\mu_1+2L_{12} & -2L_2 & 0 & 0 & 0 \\ -2L_2 & 2\mu_2+2L_{23} & -2L_3 & 0 & 0 \\ 0 & -2L_3 & 2\mu_3+2L_{34} & -2L_4 & 0 \\ 0 & 0 & -2L_4 & 2\mu_4+2L_{45} & -2L_5 \\ 0 & 0 & 0 & -2L_5 & 2\mu_5+2L_{56} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = - \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \\ C_5 \end{bmatrix}$$

$$L_{ij} = L_i + L_j$$

$$C_i = \mu_i Xs_i + L_i Zc_i - L_{i+1} Zc_{i+1}$$

Figure 9: The equation which results by substitution of the particular terms into the general terms of Figure 8 and by translation back into matrix format. The matrix is essentially identical to the original core matrix in Figure 4 because of the symmetry of the off diagonal terms.

The matrix equation of Figure 9 is an abbreviated version of the equation which applies to a real model, often with hundreds of terms. The X's, or the Ux's in the model are the unknowns, and so the equation must be inverted to find their values.

RESULTS

Use of the principles shown here requires a significant recoding of the finite-difference algorithms, and this has been only partially completed to date. Preliminary results on the tank model (Manning and Wong, 2013) have been similar to the results of the simple algorithm used there.

CONCLUSIONS

Internal boundary conditions are more complex than they at first appear. A better understanding should lead to better water bottom models, and the principles may help with other boundary condition problems.

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REFERENCES

Manning, P.M. and Wong, J., 2013. Finite-difference models with an internal boundary condition: CREWES Research Report, 25, 55.1-55.6.