Scattering of homogeneous and inhomogeneous viscoelastic waves from arbitrary heterogeneities: update

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ABSTRACT

Motivated by the need to derive and characterize increasingly sophisticated seismic data analysis and inversion methods incorporating wave dissipation, we consider the problem of scattering of homogeneous and inhomogeneous waves from perturbations in five viscoelastic parameters (density, P- and S-wave velocities, and P- and S-wave quality factors), as formulated in the context of the Born approximation. Within this approximation the total wave field is the superposition of an incident plane wave and a scattered wave, the latter being a spherical wave weighted by a function of solid angle called the scattering potential. In elastic media the scattering potential is real, but if dissipation is included through a viscoelastic model, the potential becomes complex and thus impacts the amplitude and phase of the outgoing wave. The isotropic-elastic scattering framework of Stolt and Weglein, extended to admit viscoelastic media, exposes these amplitude and phase phenomena to study, and in particular allows certain well-known layered-medium viscoelastic results due to Borcherdt to be re-considered in an arbitrary heterogeneous Earth. The main theoretical challenge in doing this involves the choice of coordinate system over which to evaluate and analyze the waves, which in the viscoelastic case must be based on complex vector analysis. We present a candidate system within which several of Borcherdt's key results carry over; for instance, we show that elliptically polarized P- and SI- waves cannot to leading order be scattered into linearly polarized SI-waves. Furthermore, the elastic formulation is straightforwardly recovered in the limit as P- and S-wave quality factors tend to infinity.

INTRODUCTION

Accurate and tractable mathematical models of wave propagation are a key to reliable seismic data analysis. Our current ability to analytically describe seismic waves interacting with arbitrary viscoelastic heterogeneities, in support of the derivation and characterization of attenuation analysis methods and seismic full waveform inversion, is quite limited. Motivated by this, we present a particular approach to the problem of scattering of homogeneous and inhomogeneous viscoelastic waves from perturbations in density, P- and S-wave velocities, and P- and S-wave quality factors, as formulated in the context of the Born approximation.

The scattering of seismic waves by purely elastic heterogeneities under the Born approximation has been investigated by many authors (Wu and Aki, 1985; Beylkin and Burridge, 1990; Sato et al., 2012; Stolt and Weglein, 2012). Stolt and Weglein Stolt and Weglein (2012) introduced a formal theory for the description of the multidimensional scattering of seismic waves based on an isotropic-elastic model. We identify as a research priority the adaptation of this approach to incorporate other, more complete pictures of seismic wave propagation. Amongst these, the extension to include anelasticity and/or viscoelasticity (Flugge, 1967), which brings to the wave model the capacity to transform elastic energy into heat, ranks very high. Anelasticity is generally held to be a key contributor to seismic attenuation, or "seismic Q", which has received several decades worth of careful attention in the literature (e.g., Aki and Richards, 2002; Futterman, 1967). Development of methods for analysis (e.g., Tonn, 1991), processing (Bickel and Natarajan, 1985;

Hargreaves and Calvert, 1991; Wang, 2006; Zhang and Ulrych, 2007; Innanen and Lira, 2010), and inversion (Dahl and Ursin, 1992; Ribodetti and Virieux, 1998; Causse et al., 1999; Hicks and Pratt, 2001; Innanen and Weglein, 2007)of wave data exhibiting the attenuation and dispersion of seismic Q remains a very active research area.

Wave propagation in linear viscoelastic media has been extensively studied, both numerically (Carcione et al., 1988b; Carcione, 1993; Carcione et al., 1988a), and analytically for viscoelastic anisotropic media (Cervený and Psencík, 2005a,b,c). Borcherdt Borcherdt (2009) has presented a complete theory for seismic waves propagating in layered anelastic media, assuming a viscoelastic model to hold. Borcherdt in particular predicts a range of transverse inhomogeneous wave types unique to viscoelastic media (Type I and II S waves), and develops rules for conversion of one type to another during interactions with planar boundaries.

In the elastic-isotropic setting, beginning with a plane defined by the incoming wave vector and the outgoing wave vector, Stolt and Weglein Stolt and Weglein (2012) develop scattering quantities which in an intuitive manner generalize the layered-medium notions of reflections and conversions of P, Sv and Sh waves. The results are forms for the scattering operator whose diagonal elements describe the potential of a volume scattering element to scatter a P wave to a P wave, an Sv to an Sv wave, and an Sh to an Sh wave, and whose off-diagonal elements describe the potential to convert, from, say, a P wave to an Sv wave, etc. Having made a "good" choice of coordinate systems, canonical results, such as the lack of P-Sh and Sv-Sh mode conversions, are naturally reproduced in their formulation: the off-diagonal element corresponding to P-Sh scattering is seen to be identically zero.

Generalizing this approach to allow for viscoelastic waves of the type described by Borcherdt has several positive outcomes. First, and foremost, it provides an analytical framework for the examination of processes of scattering of viscoelastic waves from arbitrary three-dimensional heterogeneities, as opposed to layered media. Second, it provides a foundation for direct linear and nonlinear inversion methods for reflection seismic data, which go well beyond existing an-acoustic results (Innanen and Weglein, 2007; Innanen and Lira, 2010). And third, it provides a means to compute and analyze the gradient and Hessian quantities used in iterative seismic inversion (see the review by Virieux and Operto, 2009). The main challenge lies in the need to choose from a much wider range of possible systems of coordinates. Because the viscoelastic wave vectors are complex, and the real and imaginary components of these wave vectors are not necessarily parallel, the use of incident and scattered wave vectors as the starting point for coordinate system selection is a richer but more complicated idea. Nevertheless, judicious choices are possible, and we arrive at a complex, or bivector coordinate system which appears to naturally extend the concepts of Borcherdt (2009) to arbitrary 3D scattering.

We organize the paper as follows. Section 2 describes the mathematical framework used to evaluate the wave propagation in a viscoelastic medium. Section 3 provides an explanation of viscoelastic scattering potential in displacement space based on the perturbation theory. Section 4 develops the elements of scattering potential for various types of viscoelastic wave scattering. Finally, section 5 offers some concluding remarks.

MATHEMATICAL FORMULATION AND REVIEW

Homogeneous and inhomogeneous waves in viscoelastic wave theory

There are three types of waves in a viscoelastic medium: P, Type-I S, and Type-II S. For each wave type there is a corresponding seismic quality factor, Q_P , Q_{SI}, and Q_{SII}. For a low-loss viscoelastic medium, Q_P , Q_{SI}, and Q_{SII} reduce to Q_{hp} and Q_{hs} Borcherdt (2009). These quality factors have the standard definitions in terms of ratios of the real and imaginary parts of the complex moduli.

In this paper we will write quantities such as the viscoelastic wave vector and velocity for inhomogeneous waves in terms of the reciprocal quality factors for *homogenous waves*, i.e., Q_{hp} and Q_{hs}. Of the several mathematical possibilities this choice seems to be the most convenient, expressing our results in the language of standard exploration and monitoring seismology. As a consequence, the key result of this paper, the enumeration of the explicit elements of the multidimensional viscoelastic scattering operator, appears in terms of perturbations in Q_{hp} and Q_{hs}. These perturbations correspond to the relative-change quantities involved in anelastic amplitude-variation-with-angle (AVA) and amplitude-variation-with-frequency (AVF) expressions (Innanen, 2011).

In the case of inhomogeneous waves, the attenuation and propagation vectors are not in the same direction. This makes the displacement vectors different from those of the homogenous case. The particle motion for P waves is elliptical in the plane constructed by attenuation and propagation vectors. This elliptical motion reduces to a linear motion in the homogenous limit. Particle motion also distinguishes the two types of shear waves, SI and SII. The first, which is the generalization of the SV wave, has an elliptical displacement vector in the propagation-attenuation plane. This is the SI-type wave. The second one, which is the generalization of the SH wave, has linear particle motion perpendicular to the propagation-attenuation plane. This is the SII-type wave. The wavenumber vector of inhomogeneous waves is represented by

$$\mathbf{K} = \mathbf{P} - i\mathbf{A}. \tag{1}$$

Here $\mathbf{P}$ is the propagation vector, perpendicular to the constant phase plane $\mathbf{P} \cdot \mathbf{r} = constant$, and $\mathbf{A}$ is the attenuation vector perpendicular to the amplitude constant plane $\mathbf{A} \cdot \mathbf{r} = constant$. The attenuation vector $\mathbf{A}$ is in the direction of maximum amplitude decay. If the attenuation and propagation vectors are parallel, the wave is homogeneous (elastic behaviour is recovered in the limit as $\mathbf{A}$ vanishes). If we represent the angle between $\mathbf{P}$ and $\mathbf{A}$ by δ, for inhomogeneous waves $\delta \neq 0$ we have

$$|\mathbf{P}| = 2^{-\frac{1}{2}} \left[\Re\mathbf{K}\cdot\mathbf{K} + \sqrt{(\Re\mathbf{K}\cdot\mathbf{K})^2 + (\Im\mathbf{K}\cdot\mathbf{K})^2 \sec^2\delta}\right]^{\frac{1}{2}}, \tag{2}$$

and

$$|\mathbf{A}| = 2^{-\frac{1}{2}} \left[-\Re\mathbf{K}\cdot\mathbf{K} + \sqrt{(\Re\mathbf{K}\cdot\mathbf{K})^2 + (\Im\mathbf{K}\cdot\mathbf{K})^2 \sec^2\delta}\right]^{\frac{1}{2}} \tag{3}$$

where

$$\mathbf{K}\cdot\mathbf{K} = |\mathbf{P}|^2 - |\mathbf{A}|^2 - 2i|\mathbf{P}||\mathbf{A}|\cos\delta = \left(\frac{\omega}{\alpha}\right)^2 = \frac{\omega^2\rho}{K + \frac{4}{3}M}, \tag{4}$$

and where $\Re\mathbf{K}\cdot\mathbf{K}$ and $\Im\mathbf{K}\cdot\mathbf{K}$ represent the real and imaginary parts of $\mathbf{K}\cdot\mathbf{K}$ respectively, and K and M are the viscoelastic Lamé parameters. According to equation (4)

$$\Im(\mathbf{K}\cdot\mathbf{K}) = -2|\mathbf{P}||\mathbf{A}|\cos\delta, \tag{5}$$

so, because in a viscoelastic medium $\Im(\mathbf{K}\cdot\mathbf{K}) \neq 0$, this implies that the maximum attenuation $|\mathbf{A}|$ is not zero, and also that the direction of maximum attenuation cannot be perpendicular to the direction of phase propagation. As a result the attenuation angle varies between $0 \leq \delta < 90^\circ$. The attenuation angle δ for an isotropic viscoelastic medium thus always varies between zero and $\pi/2$ (in anisotropic media it can exceed $\pi/2$(Behura and Tsvankin, 2006)). From this general framework we may now follow Borcherdt (2009) in analyzing three types of independently propagating wave.

Viscoelastic waves

Wave equations for the P- and S-wave potentials Φ and $\mathbf{\Psi}$, respectively,

$$\nabla^2\Phi - \alpha^{-2}\partial_t^2\Phi = 0, \tag{6}$$

$$\nabla^2\mathbf{\Psi} - \beta^{-2}\partial_t^2\mathbf{\Psi} = 0, \tag{7}$$

have plane wave solutions

$$\Phi = \alpha\Phi_0 \exp\left[-i(\mathbf{K}_p\cdot\mathbf{r} - \omega t)\right], \tag{8}$$

$$\mathbf{\Psi} = \beta\mathbf{\Psi}_0 \exp\left[-i(\mathbf{K}_s\cdot\mathbf{r} - \omega t)\right], \tag{9}$$

where

$$\sqrt{\mathbf{K}_p\cdot\mathbf{K}_p} = K_p = \frac{\omega}{\alpha}, \tag{10}$$

and

$$\sqrt{\mathbf{K}_s\cdot\mathbf{K}_s} = K_s = \frac{\omega}{\beta}, \tag{11}$$

additionally, Φ_0 and $\mathbf{\Psi}_0$ are complex scalar and vector constants. In the presence of attenuation the wavenumber vector is complex, its real and imaginary parts corresponding to propagation direction and maximum attenuation direction respectively. As a consequence, the P- and S-wave velocities in equations (10) and (11) are also complex. These viscoelastic velocities are related to corresponding elastic P- and S-velocities α_e and β_e via

$$\alpha = \alpha_e \frac{\sqrt{2\chi_{hp}(1+Q_{hp}^{-2})}}{1 - iQ_{hp}^{-1}\chi_{hp}}, \tag{12}$$

$$\beta = \beta_e \frac{\sqrt{2\chi_{hs}(1+Q_{hs}^{-2})}}{1 - iQ_{hs}^{-1}\chi_{hs}}, \tag{13}$$

with

$$\chi = 1 + \sqrt{1 + Q^{-2}}, \tag{14}$$

where Q_{hp} and Q_{hs} are the quality factors for for homogeneous P- and S waves respectively. For the low-loss viscoelastic medium approximation where $Q^{-1} \ll 1$ the velocities reduce to

$$\alpha_L = \alpha_e\left(1 + \frac{i}{2}Q_{hp}^{-1}\right), \tag{15}$$

$$\beta_L = \beta_e \left(1 + \frac{i}{2} Q_{hs}^{-1}\right). \tag{16}$$

The displacement vectors for P- and S-waves are given by

$$\mathbf{U}_p = \boldsymbol{\nabla}\Phi = -i\alpha \mathbf{K}_p \Phi_0 \exp\left[-i(\mathbf{K}_p \cdot \mathbf{r} - \omega t)\right], \tag{17}$$

$$\mathbf{U}_s = \boldsymbol{\nabla} \times \boldsymbol{\Psi} = -i\beta \mathbf{K}_s \times \boldsymbol{\Psi}_0 \exp\left[-i(\mathbf{K}_s \cdot \mathbf{r} - \omega t)\right]. \tag{18}$$

If $\boldsymbol{\Psi}_0 = \Psi_0 \mathbf{n}$, where $\mathbf{n}$ is a unit vector orthogonal to the plane formed by $\mathbf{P}_S - \mathbf{A}_S$, the corresponding wave is considered to be of S type-I (SI) wave. Displacement vectors for P- and S-waves with complex polarization vectors describe elliptical particle motion. To understand the nature of the motion characterized by (17) and (18), consider a general complex vector $V = V_R + iV_I$. Multiplication of V with the complex wavenumber vector $\mathbf{K}$ leads to

$$V\mathbf{K} = (V_R + iV_I)(\mathbf{P} - i\mathbf{A}) = (V_R\mathbf{P} + V_I\mathbf{A}) + i(V_I\mathbf{P} - V_R\mathbf{A}). \tag{19}$$

If V is interpreted as a velocity, related to the wavenumber vector via $KV = \omega$, its real and imaginary parts are

$$V_R = \frac{\omega K_R}{(K_R)^2 + (K_I)^2}, \tag{20}$$

$$V_I = -\frac{\omega K_I}{(K_R)^2 + (K_I)^2}. \tag{21}$$

Because the polarization vector can be defined with the complex vector $\boldsymbol{\xi}$ as

$$\boldsymbol{\xi} = \boldsymbol{\xi}_R - i\boldsymbol{\xi}_I = \frac{V}{\omega}\mathbf{K}, \tag{22}$$

its real and imaginary parts of the polarization vectors are therefore

$$\boldsymbol{\xi}_R = \frac{K_R\mathbf{P} - K_I\mathbf{A}}{(K_R)^2 + (K_I)^2}, \tag{23}$$

$$\boldsymbol{\xi}_I = \frac{K_I\mathbf{P} + K_R\mathbf{A}}{(K_R)^2 + (K_I)^2}. \tag{24}$$

These two vectors are orthogonal, and $|\boldsymbol{\xi}_R|^2 - |\boldsymbol{\xi}_I|^2 = 1$. A simple analysis indicates that particle motion related to the displacement for P-wave in equation (17) is an ellipse with major axes $\boldsymbol{\xi}_R$ and minor axes $\boldsymbol{\xi}_I$. In a similar manner, we can show that the polarization vector for SI wave can be written as

$$\boldsymbol{\zeta}_s = \boldsymbol{\zeta}_R - i\boldsymbol{\zeta}_I = (\boldsymbol{\xi}_{sR} - i\boldsymbol{\xi}_{sI}) \times \mathbf{n} \tag{25}$$

For low-loss viscoelastic media the elliptical polarization is of the form

$$\boldsymbol{\xi}^L = \frac{\alpha_e}{\omega}\left\{\mathbf{K}_p^L + i\frac{Q_{hp}^{-1}}{2}\mathbf{P}_p^L\right\}, \tag{26}$$

$$\boldsymbol{\zeta}_s^L = \frac{\alpha_e}{\omega}\left\{\mathbf{K}_s^L + i\frac{Q_{hs}^{-1}}{2}\mathbf{P}_s^L\right\} \times \mathbf{n}. \tag{27}$$

Finally, we can redefine the the displacement vectors for P- and S-waves as

$$\mathbf{U}_p^L = \boldsymbol{\xi}_p^L \Phi_0' \exp\left[-i(\mathbf{K}_p^L \cdot \mathbf{r} - \omega t)\right], \tag{28}$$

$$\mathbf{U}_s^L = \boldsymbol{\zeta}_s^L \boldsymbol{\Psi}_0' \exp\left[-i(\mathbf{K}_s^L \cdot \mathbf{r} - \omega t)\right], \tag{29}$$

Where we have defined $\Phi_0' = -i\omega\Phi_0$ and $\boldsymbol{\Psi}_0' = -i\omega\boldsymbol{\Psi}_0$. We can rewrite the parameters that describe the elliptical motion for P- and SI-waves in terms of attenuation angles and quality factors. For low-loss media, the major axis of ellipse for P-wave particle motion, takes the form

$$|\boldsymbol{\xi}_{RP}^L|^2 \approx \frac{1}{2}\left(\sqrt{1 + Q_{hp}^{-2}\sec^2\delta_p} + 1\right), \tag{30}$$

and the minor axes

$$|\boldsymbol{\xi}_{IP}^L|^2 \approx \frac{1}{2}\left(\sqrt{1 + Q_{hp}^{-2}\sec^2\delta_p} - 1\right). \tag{31}$$

If wave is excited by a point source in a weakly attenuating medium, the attenuation angle is usually small (Yaping and Tsvankin, 2006). If the attenuation vector forms an angle with the propagation vector of $0^0 \leq \delta_p \leq 70^0$ always $Q_{hp}^{-2}\sec^2\delta_p \leq 0.1$, then the absolute values of the minor axes reduce to

$$|\boldsymbol{\xi}_{IP}^L| = \frac{1}{2} Q_{hp}^{-1}\sec\delta_p \tag{32}$$

In Figure 1 we plot the magnitude of the minor axis of the ellipse of the P-wave particle motion for the inhomogeneous P-wave vs. the attenuation angle δ (which varies from 0^0-70^0). We repeat the plot for three values of reciprocal Q_{hp}. The curves indicate that for fixed material parameter values, $|\boldsymbol{\xi}_{2P}^L|$ increases with increasing degree of inhomogeneity, but this increase is very small at small values of δ. For degrees of inhomogeneity of greater than 70 degrees the magnitude of the minor axis is significant for high absorption ($Q^{-1} > 0.1$). When $\boldsymbol{\xi}_{IP}^L = 0$, the wave is homogeneous and the elliptical motion degenerates into linear motion in the direction of propagation of the homogeneous wave. For high degrees of inhomogeneity ($\delta_P \to 90°$) equation (32) is not satisfied. In this limit, $Q_{hp}^{-1}\sec\delta_p \gg 1$, so the minor and major axes are equal:

$$|\boldsymbol{\xi}_{RP}^L| \approx |\boldsymbol{\xi}_{IP}^L| \approx \sqrt{\frac{1}{2} Q_{hp}^{-1}\sec\delta_p}. \tag{33}$$

In this case polarization for inhomogeneous waves is circular.

Next, we assume that $\boldsymbol{\Psi}_0$ is not simply a complex number multiplied by a real unit vector but is a complex vector in the xz plane. In this case, the corresponding wave is considered to be of SII-type (sometimes referred to as a linear S-wave). The particle motion for SII-wave is linear for both homogeneous and inhomogeneous waves and perpendicular to the wavenumber vector $\mathbf{K}_S$. In this case the displacement vector takes the form

$$\mathbf{U}_{SII} = \mathbf{y}\mathbf{K}_s \cdot (\Psi_{0z}\mathbf{x} - \Psi_{0x}\mathbf{z}) e^{i(\omega t - \mathbf{K}_s \cdot \mathbf{r})}. \tag{34}$$

Equation (34) indicates that the particle motion for SII wave is linear perpendicular to the $(\mathbf{P}_s - \mathbf{A}_s)$-plane.

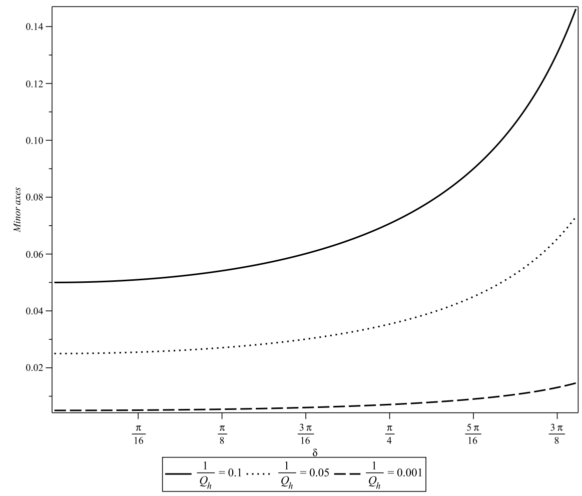

FIG. 1. Magnitude of the minor axes of the elliptical motion for an inhomogeneous wave in a low-loss viscoelastic media versus attenuation angle δ for three values of reciprocal quality factor Q_h^{-1}

THE VISCOELASTIC SCATTERING OPERATOR AND POTENTIALS

We now build from the above framework, formulating a description of scattering of homogeneous and inhomogeneous seismic waves from arbitrary viscoelastic heterogeneities in the Earth. Scattering theory is a framework within which various kinds of interactions of waves and particles can be analyzed. In the context of seismic exploration, scattering theory relates perturbations in the properties of the medium to the seismic waves that propagate through those perturbations (e.g., Weglein et al., 2003). The perturbations are assembled, along with reference medium properties, in a core quantity called the scattering operator, the construction of which for viscoelastic waves will be the subject of this section.

The Born approximation is a solution accurate to first order in the scattering operator, and is used as the basis for many types of migration and linearized inversion in seismic applications (Bleistein, 1979; Clayton and Stolt, 1981; Beylkin, 1985, etc.). Mapping between the scattering operator and the Born approximate model of seismic data usually involves integrating the product of the scattering operator with Green's functions, whose role is to model wave propagation through the smooth parts of the Earth model. In this framework the scattering operator studied in isolation provides insight into the physics of the interactions of the wave with rapidly varying parts of the Earth. It is akin to arriving at conclusions about waves in layered media by studying the mathematical structure of the P-P reflection coefficient (as in, e.g., Aki and Richards, 2002). In this section we arrive at interpretable forms of the viscoelastic scattering operator, including explicit expressions for the elements of the operator. Each element will represent the potential of a point in space to scatter a P to a P wave, a P to an SI wave, etc. Then we will be in a position to analyze the viscoelastic scattering problem for general insights.

The scattering operator in displacement space

In the scattering framework, the unperturbed medium is a reference medium and the perturbed medium is associated with the actual medium. The difference between the actual and reference medium wave operators is the perturbation operator or scattering operator. In the elastic-isotropic case, this operator can be expressed in terms of a 3×3 matrix, each element of which corresponds to the scattering of one wave type to another. The diagonal elements refer to scattering which conserves wave type, and off-diagonal elements refer to the those which convert wave type. To begin the process of form a viscoelastic scattering operator, we express the viscoelastic wave equation as

$$L_{ve}(\mathbf{r}, \omega)\mathbf{U}(\mathbf{r}, \omega) = 0. \tag{35}$$

Here the wave operator in Cartesian coordinates is

$$L_{ve} = \begin{pmatrix} L_{xx} & L_{xy} & L_{xz} \\ L_{yx} & L_{yy} & L_{yz} \\ L_{zx} & L_{zy} & L_{zz} \end{pmatrix}, \tag{36}$$

with the elements

$$(L_{ve})_{ij} = \rho\omega^2\delta_{ij} + \partial_i(\rho\alpha^2)\partial_j + \delta_{ij}\partial_k(\rho\beta^2)\partial_k - 2\partial_i(\rho\beta^2)\partial_j + \partial_j(\rho\beta^2)\partial_i, \tag{37}$$

for $i, j, k = x, y, z$. The P- and S-wave velocities are defined as

$$\alpha = \sqrt{\rho^{-1}\left(K + \frac{4M}{3}\right)}, \tag{38}$$

and

$$\beta = \sqrt{\rho^{-1}M}, \tag{39}$$

where ρ is the mass density, and M and K are viscoelastic moduli which are generally complex and frequency dependent. The scattering matrix is the difference between perturbed and unperturbed wave operators of the type in equation (36):

$$V_{ve}(\mathbf{r}, \omega) = L_{ve}(\mathbf{r}, \omega) - L_{ve_0}(\mathbf{r}, \omega) = \begin{pmatrix} V_{xx} & V_{xy} & V_{xz} \\ V_{yx} & V_{yy} & V_{yz} \\ V_{zx} & V_{zy} & V_{zz} \end{pmatrix}, \tag{40}$$

where subscript "0" denotes the reference medium. The scattering operator can be written as follows:

$$V_{ve}(\mathbf{r}, \omega) = \sum_k C_k^L(\mathbf{r}, \omega)A_k(\mathbf{r})C_k^R(\mathbf{r}, \omega), \tag{41}$$

where $A_i(\mathbf{r})$ are the perturbation parameters, and $C_i^L(\mathbf{r}, \omega)$ and $C_i^R(\mathbf{r}, \omega)$ are functions of first-order derivatives. Since we are interested in low-loss viscoelastic medium, in what follows we will ignore the superscript L for notational simplicity. We restrict ourself to scattering in locally isotropic viscoelastic medium. The fractional variation in density, velocities and quality factors for S and P-waves are

$$A_\tau = \frac{\delta\tau}{\bar{\tau}} = \frac{\tau - \tau_0}{2(\tau + \tau_0)} \ll 1, \tag{42}$$

where $\tau = \rho, \alpha_e, \beta_e, Q_{hp}, Q_{hs}$. Furthermore using equations (15) and (16), we can write

$$\rho\alpha^2 - \rho_0\alpha_0^2 = \rho_0\alpha_0^2(A_\rho + 2A_{\alpha_e} - iQ_{hp_0}^{-1}A_{Q_hp}) \tag{43}$$

and

$$\rho\beta^2 - \rho_0\beta_0^2 = \rho_0\beta_0^2(A_\rho + 2A_{\beta_e} - iQ_{hs_0}^{-1}A_{Q_hs}). \tag{44}$$

The elements of the scattering potential, in terms of these perturbations, are

$$\rho_0^{-1}(V_{ve}^{\rho})_{kl} = A_\rho \omega^2 \delta_{kl} + \alpha_0^2 \partial_k A_\rho \partial_l + \beta_0^2 \left(\delta_{kl}\partial_m A_\rho \partial_m - 2\partial_k A_\rho \partial_l + \partial_l A_\rho \partial_k\right), \tag{45}$$

$$\rho_0^{-1}(V_{ve}^{\alpha_e})_{kl} = 2\alpha_0^2 \partial_k A_{\alpha_e} \partial_l, \tag{46}$$

$$\rho_0^{-1}(V_{ve}^{\beta_e})_{kl} = 2\beta_0^2 \left(\delta_{kl}\partial_m A_{\beta_e} \partial_m - 2\partial_k A_{\beta_e} \partial_l + \partial_l A_{\beta_e} \partial_k\right), \tag{47}$$

$$\rho_0^{-1}(V_{ve}^{Q_{hp}})_{kl} = -i\alpha_0^2 Q_{hp0}^{-1} \partial_k A_{V_{QP}} \partial_l, \tag{48}$$

and

$$\rho_0^{-1}(V_{ve}^{Q_{hs}})_{kl} = -i\beta_0^2 \left(\delta_{kl}\partial_m A_{Q_{hs}} \partial_m - 2\partial_k A_{Q_{hs}} \partial_l + \partial_l A_{Q_{hs}} \partial_k\right), \tag{49}$$

where we have defined

$$V_{ve} = V_{ve}^{\rho} + V_{ve}^{\alpha_e} + V_{ve}^{\beta_e} + V_{ve}^{Q_{hp}} + V_{ve}^{Q_{hs}}. \tag{50}$$

The scattering operator in P, SI and SII space

The next task is to evaluate the scattering matrix in a system which naturally describes Borcherdt's viscoelastic modes $\mathrm{P}, \mathrm{SI}, \mathrm{SII}$, namely

$$\mathcal{V}_{ve} = \begin{pmatrix} {}^{P}_{P}\mathcal{V}_{ve} & {}^{P}_{SII}\mathcal{V}_{ve} & {}^{P}_{SI}\mathcal{V}_{ve} \\ {}^{SII}_{P}\mathcal{V}_{ve} & {}^{SII}_{SII}\mathcal{V}_{ve} & {}^{SII}_{SI}\mathcal{V}_{ve} \\ {}^{SI}_{P}\mathcal{V}_{ve} & {}^{SI}_{SII}\mathcal{V}_{ve} & {}^{SI}_{SI}\mathcal{V}_{ve} \end{pmatrix}. \tag{51}$$

Here, as in the original elastic-isotropic theory of Stolt and Weglein, the diagonal elements represent scattering which preserves the wave types and off-diagonal elements refer to scattering which converts the type of waves. For example, ${}^{P}_{SI}\mathcal{V}_{ve}$ represents the potential of a scattering point to convert a P-wave into an SI-type wave. Some elements are identically zero, for instance ${}^{P}_{SII}\mathcal{V}_{ve} = 0$, as we shall show presently. This means that a P-wave with elliptical polarization cannot convert into an SII wave with linear polarization. Since the particle motion for SII-type wave is in the $\mathbf{n}$ direction we can define the normal polarization vector in this direction.

Now, the scattering potential in displacement space can be written (Beylkin and Burridge, 1990)

$$^{R}_{I}\mathcal{V}_{ve} = \boldsymbol{\xi}_I^T V_{ve} \boldsymbol{\xi}_R. \tag{52}$$

Here $\boldsymbol{\xi}$ is the polarization vector, V_{ve} is the scattering operator and subscripts I and R respectively refer to the incident and reflected waves. Each type of displacement is determined by two angles, θ and δ, which indicate the directions of propagation and attenuation. The vectors $\mathbf{P}^i$ and $\mathbf{A}^i$ (propagation and attenuation vectors for the incident wave) point towards the scatterer while the vectors $\mathbf{P}^r$ and $\mathbf{A}^r$ for the reflected waves point away from the scatterer. Without loss of generality we assume that the incident propagation vector is in the z-direction, so the reflected and incident propagation and attenuation vectors can be written as

$$\mathbf{P}^r = \frac{\omega}{V_e}(\mathbf{x}\sin\theta + \mathbf{z}\cos\theta) \tag{53}$$

$$\mathbf{P}^i = \frac{\omega}{V_e}\mathbf{z} \tag{54}$$

$$\mathbf{A}^r = \frac{\omega}{2V_e} Q_h^{-1} (\mathbf{x}\sin\theta + \mathbf{z}\cos\theta + \tan\delta^r(\mathbf{x}\cos\theta - \mathbf{z}\sin\theta)) \tag{55}$$

$$\mathbf{A}^i = \frac{\omega}{2V_e} Q_h^{-1} (\mathbf{z} + \mathbf{x}\tan\delta^i) \tag{56}$$

Here θ, with $0 < \theta \leq \pi/2$, denotes the angle that the propagation vector makes with the z-axis and δ is the angle between propagation and attenuation vectors. In addition V refers to elastic velocities α_e and β_e. The angle that the attenuation vector makes with the z-axis is $\theta - \delta$. Regarding the displacements, for inhomogeneous P- and SI-waves, the polarization vectors display an elliptical motion in the x-z plane. In the homogeneous case P-wave particle motion is in the direction of the propagation vector $\mathbf{P}$ and for SI-waves is a unit vector in the direction of $\mathbf{P} \times \mathbf{y}$. For SII-waves for both inhomogeneous and homogeneous waves the polarization is linear in the y-direction. For P-waves involving elliptical particle motion, the polarization vector is

$$\boldsymbol{\xi}_p = \boldsymbol{\xi}_{RP} - i\boldsymbol{\xi}_{IP}. \tag{57}$$

For low-loss viscoelastic media the elliptical polarization takes the form

$$\boldsymbol{\xi}_p = \frac{\alpha_e}{\omega} \left\{ \mathbf{K}_p + i\frac{Q_{hp}^{-1}}{2}\mathbf{P}_p \right\}. \tag{58}$$

For the SI wave, since the displacement vector moves on an ellipse in the plane of attenuation-propagation, the polarization vector is defined as

$$\boldsymbol{\zeta}_s = \boldsymbol{\zeta}_{RS} - i\boldsymbol{\zeta}_{IS} = (\boldsymbol{\xi}_{RS} - i\boldsymbol{\xi}_{IS}) \times \mathbf{n}. \tag{59}$$

The polarizations vectors for incident and reflected P-waves using the propagation and attenuation vectors are given by

$$\boldsymbol{\xi}_p^i = \mathbf{z} - \mathbf{x}\frac{i}{2} Q_{hp}^{-1} \tan\delta_p^i \tag{60}$$

$$\boldsymbol{\xi}_p^r = [\mathbf{x}\sin\theta_p + \mathbf{z}\cos\theta_p] - \frac{i}{2} Q_{hp}^{-1} \tan\delta_p \, [\mathbf{x}\cos\theta_p - \mathbf{z}\sin\theta_p]\,. \tag{61}$$

The slowness vectors for incident and reflected waves are

$$\mathbf{k}_p^i = \frac{\mathbf{K}_p^i}{\omega} = \frac{1}{\alpha_e} \left(\mathbf{z} - \frac{i}{2} Q_{hp}^{-1} \left[\mathbf{x}\tan\delta_p^i + \mathbf{z} \right] \right) \tag{62}$$

$$\mathbf{k}_p^r = \frac{\mathbf{K}_p^r}{\omega} = \frac{1}{\alpha_e} [\mathbf{x}\sin\theta_p + \mathbf{z}\cos\theta_p] - \frac{i}{2\alpha_e} Q_{hp}^{-1} \left[\mathbf{x}\sin\theta_p + \mathbf{z}\cos\theta_p + \tan\delta_p^r(\mathbf{x}\cos\theta_p - \mathbf{z}\sin\theta_p) \right]. \tag{63}$$

In this case the polarizations for incident and reflected SI waves are given by

$$\boldsymbol{\zeta}_s^i = -\mathbf{x} - \frac{i}{2} Q_{hs}^{-1} \tan\delta_s^i \mathbf{z} \tag{64}$$

$$\boldsymbol{\zeta}_s^r = [\mathbf{z}\sin\theta_s - \mathbf{x}\cos\theta_s] - \frac{i}{2} Q_{hs}^{-1} \tan\delta_s^r \, [\mathbf{z}\cos\theta_s + \mathbf{x}\sin\theta_s] \tag{65}$$

The slowness vectors for incident and reflected waves are

$$\mathbf{k}_s^i = \frac{\mathbf{K}_s^i}{\omega} = \frac{1}{\beta_e} \left(\mathbf{z} - \frac{i}{2} Q_{hs}^{-1} \left[\mathbf{x}\tan\delta_s^i + \mathbf{z} \right] \right) \tag{66}$$

$$\mathbf{k}_s^r = \frac{\mathbf{K}_s^r}{\omega} = \frac{1}{\beta_e}\left[\mathbf{x}\sin\theta_s + \mathbf{z}\cos\theta_s\right] - \frac{i}{2\beta_e}Q_{hs}^{-1}\left[\mathbf{x}\sin\theta_s + \mathbf{z}\cos\theta_s + \tan\delta_s^r(\mathbf{x}\cos\theta_s - \mathbf{z}\sin\theta_s)\right] \tag{67}$$

Here, δ^r and δ^i are the angles between the attenuation and propagation vectors for reflected and incident waves.

ELEMENTS OF THE P-SI-SII SCATTERING MATRIX

We next write the scattering matrix element in frequency-independent form **?**. Since the differential operators are sandwiched between unperturbed Green's functions, we replace the left derivatives with i multiplied by the reflected wavenumber vector $\mathbf{K}^r$ and right derivative with i multiplied by the incident wavenumber vector $\mathbf{K}_s$. After replacing the left and right derivatives by the appropriate wavenumber vectors, the frequency independent parts of the scattering operator are

$$(\mathbb{V}_{ve}^{\rho})_{kl} = \delta_{kl} - \alpha_0^2 k_k^r k_l^i - \beta_0^2\left(\delta_{kl}k_m^r k_m^i - 2k_k^r k_l^i + k_l^r k_k^i\right), \tag{68}$$

$$(\mathbb{V}_{ve}^{\alpha_e})_{kl} = -2\alpha_0^2 k_k^r k_l^i, \tag{69}$$

$$(\mathbb{V}_{ve}^{\beta_e})_{kl} = -2\beta_0^2\left(\delta_{kl}k_m^r k_m^i - 2k_k^r k_l^i + k_l^r k_k^i\right), \tag{70}$$

$$(\mathbb{V}_{ve}^{Q_{hp}})_{kl} = iQ_{hp0}^{-1}\alpha_0^2 k_k^r k_l^i, \tag{71}$$

and

$$\rho_0^{-1}(\mathbb{V}_{ve}^{Q_{hs}})_{kl} = iQ_{hs0}^{-1}\beta_0^2\left(\delta_{kl}k_m^r k_m^i - 2k_k^r k_l^i + k_l^r k_k^i\right). \tag{72}$$

The frequency-independent components of the scattering potential are defined as

$$\frac{V_{ve}}{\rho_0\omega^2} = \mathbb{V}_{ve} = \mathbb{V}_{ve}^{\rho}A_\rho + \mathbb{V}_{ve}^{\alpha_e}A_{\alpha_e} + \mathbb{V}_{ve}^{\beta_e}A_{\beta_e} + \mathbb{V}_{ve}^{Q_{hp}}A_{Q_{hp}} + \mathbb{V}_{ve}^{Q_{hs}}A_{Q_{hs}}. \tag{73}$$

Now to obtain the scattering matrix we sandwich the above expressions with the proper polarization vectors. We use the vectors $\mathbf{R}$ and $\mathbf{I}$ to indicate the reflected and incident polarization vectors, respectively. For perturbation terms we will write

$${}_I^R\mathbb{V}_{ve}^{\rho} = \mathcal{F}_I^R - {}_I^R\mathcal{G}_\alpha - {}_I^R\mathcal{G}_\beta, \tag{74}$$

$${}_I^R\mathbb{V}_{ve}^{Q_{hp}} = -\frac{i}{2}Q_{hp0}^{-1}\left\{{}_I^R\mathbb{V}_{ve}^{\alpha_e}\right\} = iQ_{hp0}^{-1}\,{}_I^R\mathcal{G}_\alpha, \tag{75}$$

$${}_I^R\mathbb{V}_{ve}^{Q_{hs}} = -\frac{i}{2}Q_{hs0}^{-1}\left\{{}_I^R\mathbb{V}_{ve}^{\beta_e}\right\} = iQ_{hs0}^{-1}\,{}_I^R\mathcal{G}_\beta, \tag{76}$$

where we have defined

$$\mathcal{F}_I^R = \mathbf{R}\cdot\mathbf{I}, \tag{77}$$

$${}_I^R\mathcal{G}_\alpha = \alpha_0^2(\mathbf{R}\cdot\mathbf{k}^r)(\mathbf{I}\cdot\mathbf{k}^i), \tag{78}$$

$${}_I^R\mathcal{G}_\beta = \beta_0^2\left\{(\mathbf{R}\cdot\mathbf{I})(\mathbf{k}^r\cdot\mathbf{k}^i) - 2(\mathbf{R}\cdot\mathbf{k}^r)(\mathbf{I}\cdot\mathbf{k}^i) + (\mathbf{I}\cdot\mathbf{k}^r)(\mathbf{R}\cdot\mathbf{k}^i)\right\}. \tag{79}$$

To determine the explicit forms for each component of the scattering potential, we need to calculate $\mathcal{F}_I^R$, ${}_I^R\mathcal{G}_\alpha$ and ${}_I^R\mathcal{G}_\beta$.

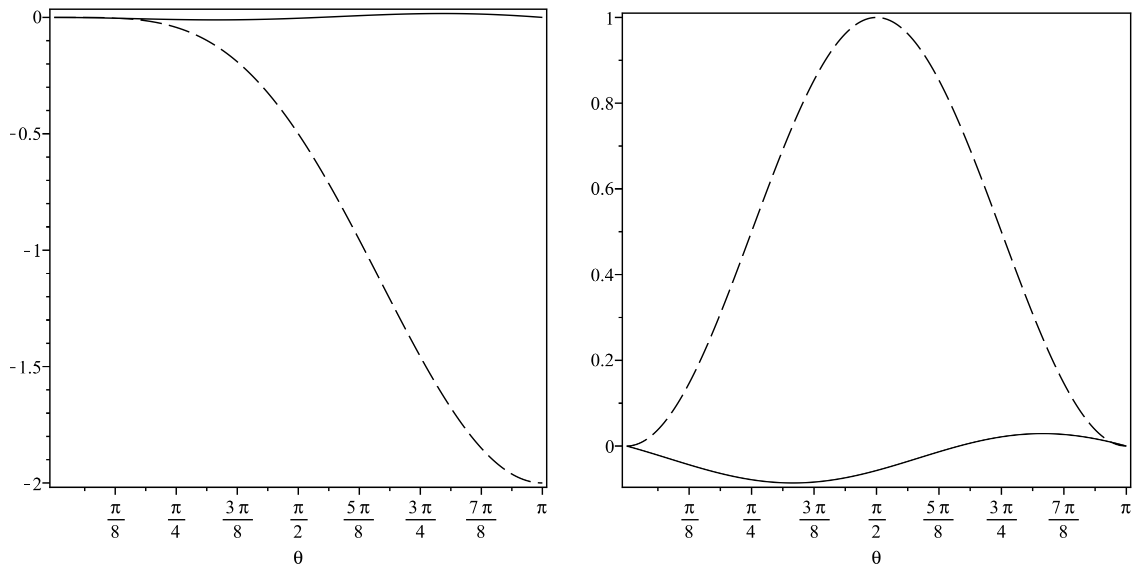

FIG. 2. Elastic and anelastic density(left) and S-velocity(right) components of the viscoelastic potential for scattering of incident homogeneous P-wave to inhomogeneous reflected P-wave versus of reflected wave angle θ, for $\delta_r = \frac{\pi}{3}$. Quality factor of P-wave for reference medium is to be 5 and for S-wave is 7. Also the S-to P-velocity ratio for reference medium is chosen to be $1/2$. Dash line is for elastic part and solid line for anelastic part.

Viscoelastic P-P scattering

This element quantifies the potential for a point in a viscoelastic medium to scatter a P-wave into a P-wave. The incident and reflected P-waves can be either inhomogeneous with elliptical motion or homogeneous with linear motion in the direction of propagation, depending on the angle between the propagation and attenuation vectors. In this case incident and reflected waves are P-waves, $\mathbf{R} = \boldsymbol{\xi}_p^r$ and $\mathbf{I} = \boldsymbol{\xi}_p^i$, and so we have

$$\mathcal{F}_p^p = \boldsymbol{\xi}_p^r \cdot \boldsymbol{\xi}_p^i \tag{80}$$

$${}^p_p\mathcal{G}_\alpha = \alpha_0^2 (\boldsymbol{\xi}_p^r \cdot \mathbf{k}_p^r)(\boldsymbol{\xi}_p^i \cdot \mathbf{k}_p^i) \tag{81}$$

$${}^p_p\mathcal{G}_\beta = \beta_0^2 \left\{ (\boldsymbol{\xi}_p^r \cdot \boldsymbol{\xi}_p^i)(\mathbf{k}_p^r \cdot \mathbf{k}_p^i) - 2(\boldsymbol{\xi}_p^r \cdot \mathbf{k}_p^r)(\boldsymbol{\xi}_p^i \cdot \mathbf{k}_p^i) + (\boldsymbol{\xi}_p^i \cdot \mathbf{k}_p^r)(\boldsymbol{\xi}_p^r \cdot \mathbf{k}_p^i) \right\}. \tag{82}$$

Using the dot products of various types of polarizations and slowness vectors we have

$$\boldsymbol{\xi}_p^r \cdot \boldsymbol{\xi}_p^i = \cos\theta + \frac{i}{2} Q_{hp_0}^{-1} \sin\theta \left(\tan\delta_p^r - \tan\delta_p^i\right) \tag{83}$$

$$\boldsymbol{\xi}_p^r \cdot \boldsymbol{k}_p^r = \boldsymbol{\xi}_p^i \cdot \boldsymbol{k}_p^i = \alpha_{H_0}^{-1} \left(1 - i\frac{Q_{hp_0}^{-1}}{2}\right) \tag{84}$$

$$\boldsymbol{\xi}_p^r \cdot \boldsymbol{k}_p^i = \boldsymbol{\xi}_p^i \cdot \boldsymbol{k}_p^r = \alpha_{H_0}^{-1} \left\{ \cos\theta \left(1 - i\frac{Q_{hp_0}^{-1}}{2}\right) + \frac{i}{2} Q_{hp_0}^{-1} \sin\theta \left(\tan\delta_p^r - \tan\delta_p^i\right) \right\} \tag{85}$$

$$\mathbf{k}_p^r \cdot \mathbf{k}_p^i = \frac{1}{\alpha_0^2} \left[\cos\theta \left(1 - iQ_{hp_0}^{-1}\right) + \frac{i}{2} Q_{hp_0}^{-1} \sin\theta \left(\tan\delta_p^r - \tan\delta_p^i\right) \right] \tag{86}$$

The scattering potential for PP mode is

$$ {}^{P}_{P}\mathbb{V}_{ve} = {}^{P}_{P}\mathbb{V}_{e} + i {}^{P}_{P}\mathbb{V}_{ane} \tag{87}$$

where elastic scattering potential $\mathbb{V}_e^{PP}$ is given by

$$ {}^{P}_{P}\mathbb{V}_{e} = \left(-1 + \cos\theta + \frac{2}{\gamma_0^2}\sin^2\theta\right) A_\rho + \left(\frac{4}{\gamma_0^2}\sin^2\theta\right) A_\beta - 2A_\alpha \tag{88}$$

and anelastic part of the scattering

$$ {}^{P}_{P}\mathbb{V}_{ane} = {}^{P}_{P}\mathbb{V}^{\rho}_{ane} A_\rho + {}^{P}_{P}\mathbb{V}^{\beta}_{ane} A_\beta + {}^{P}_{P}\mathbb{V}^{Q_{hs}}_{ane} A_{Q_{hs}} + {}^{P}_{P}\mathbb{V}^{Q_{hp}}_{ane} A_{Q_{hp}} \tag{89}$$

with

$$ {}^{P}_{P}\mathbb{V}^{\rho}_{ane} = \frac{1}{\gamma_0^2}\left\{-2\sin^2\theta(Q^{-1}_{hp_0} - Q^{-1}_{hs_0}) + \frac{Q^{-1}_{hp_0}}{2}(\sin\theta - \frac{2}{\gamma_0^2}\sin 2\theta)(\tan\delta^r_p - \tan\delta^i_p)\right\} \tag{90}$$

$$ {}^{P}_{P}\mathbb{V}^{\beta}_{ane} = -\frac{2}{\gamma_0^2}\left\{2\sin^2\theta(Q^{-1}_{hp_0} - Q^{-1}_{hs_0}) + Q^{-1}_{hp_0}\sin 2\theta(\tan\delta^r_p - \tan\delta^i_p)\right\} \tag{91}$$

$$ {}^{P}_{P}\mathbb{V}^{Q_{hs}}_{ane} = \frac{2Q^{-1}_{hp_0}}{\gamma_0^2}\sin^2\theta \tag{92}$$

$$ {}^{P}_{P}\mathbb{V}^{Q_{hp}}_{ane} A_{Q_{hp}} = Q^{-1}_{hp_0} \tag{93}$$

From equation (87) it is evident that the viscoelastic scattering potential is complex. The real part is the elastic scattering potential and the imaginary part is the term induced by the anelasticity of the medium. Let us consider as a special case the situation that the attenuation angles for incident and reflected waves are equal. In this case (90) and (91) reduce to

$$ {}^{P}_{P}\mathbb{V}^{\beta}_{ane} = 2 {}^{P}_{P}\mathbb{V}^{\rho}_{ane} = -\frac{4}{\gamma_0^2}(Q^{-1}_{hp_0} - Q^{-1}_{hs_0})\sin^2\theta. \tag{94}$$

Here the angle dependences of density, S-wave velocity and S-wave quality factor are the same. As a result the scattering patterns for these three parts are the same. Equations (90)-(92) also indicate that at normal incident ($\theta = \pi$), only changes in the P-wave quality factor affects the scattering potential.

In Figure 2 we plot the elastic and anelastic parts of the density and S-wave velocity components of the potential for scattering of a homogeneous P-wave to an inhomogeneous P-wave. We observe that the anelastic density component is comparatively small and the major contribution comes from the elastic part. In the limit of normal incidence, the absolute value of the density part of the elastic scattering potential goes to its maximum value, and the anelastic part approaches to zero. For S-wave velocity component of the scattering potential, similar to the density component, the major contribution to the reflectivity is from the elastic part. In this case both elastic and anelastic components approach zero for normal incidence as expected.

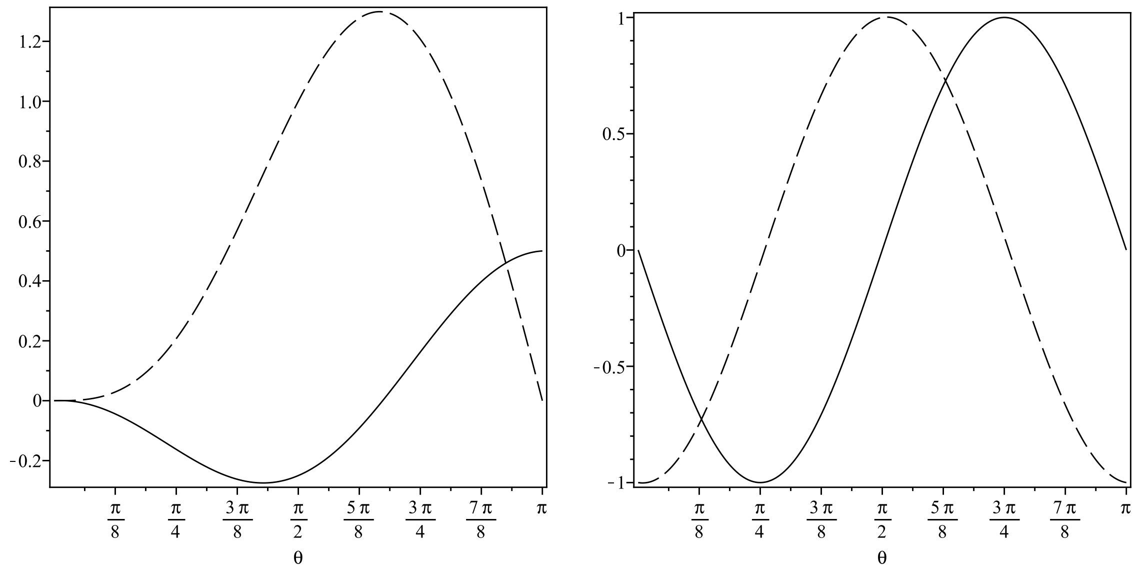

FIG. 3. Elastic and anelastic density(left) and S-velocity(right) components of the viscoelastic potential for scattering of incident homogeneous P-wave to inhomogeneous reflected SI-wave versus of reflected wave angle θ, for $\delta_r = \frac{\pi}{3}$. Quality factor of P-wave for reference medium is to be 5 and for S-wave is 7. Also the S-to P-velocity ratio for reference medium is chosen to be $1/2$. Dash line is for elastic part and solid line for anelastic part.

Viscoelastic $P - SI$ scattering

In this case the reflected wave is of type SI, $\mathbf{R} = \boldsymbol{\zeta}_s^r$, and the incident wave is a P-wave, $\mathbf{I} = \boldsymbol{\xi}_p^i$. We have

$$\mathcal{F}_{SI}^{P} = \boldsymbol{\zeta}_s^r \cdot \boldsymbol{\xi}_p^i \tag{95}$$

$${}^{P}_{SI}\mathcal{G}_{\alpha} = \alpha_0^2 (\boldsymbol{\zeta}_s^r \cdot \mathbf{k}_s^r)(\boldsymbol{\xi}_p^i \cdot \mathbf{k}_p^i) \tag{96}$$

$${}^{P}_{SI}\mathcal{G}_{\beta} = \beta_0^2 \left\{ (\boldsymbol{\zeta}_s^r \cdot \boldsymbol{\xi}_p^i)(\mathbf{k}_s^r \cdot \mathbf{k}_p^i) - 2(\boldsymbol{\zeta}_s^r \cdot \mathbf{k}_s^r)(\boldsymbol{\xi}_p^i \cdot \mathbf{k}_p^i) + (\boldsymbol{\xi}_p^i \cdot \mathbf{k}_s^r)(\boldsymbol{\zeta}_s^r \cdot \mathbf{k}_p^i) \right\}. \tag{97}$$

The dot products of the polarization and slowness vectors can be expressed in terms of the angle of the reflected propagation vector and the angles between the propagation and attenuation vectors:

$$\boldsymbol{\zeta}_s^r \cdot \boldsymbol{\xi}_p^i = \sin\theta - \frac{i}{2}\cos\theta (Q_{hs_0}^{-1}\tan\delta_s^r - Q_{hp_0}^{-1}\tan\delta_p^i) \tag{98}$$

$$\boldsymbol{\xi}_p^i \cdot \mathbf{k}_s^r = \beta_0^{-1}\left\{\cos\theta\left(1 - \frac{i}{2}Q_{hs_0}^{-1}\right) + \frac{i}{2}\sin\theta(Q_{hs_0}^{-1}\tan\delta_s^r - Q_{hp_0}^{-1}\tan\delta_p^i)\right\} \tag{99}$$

$$\boldsymbol{\zeta}_s^r \cdot \mathbf{k}_p^i = \alpha_0^{-1}\left\{\sin\theta\left(1 - \frac{i}{2}Q_{hp_0}^{-1}\right) - \frac{i}{2}\cos\theta(Q_{hs_0}^{-1}\tan\delta_s^r - Q_{hp_0}^{-1}\tan\delta_p^i)\right\} \tag{100}$$

$$\mathbf{k}_s^r \cdot \mathbf{k}_p^i = \alpha_0^{-1}\beta_0^{-1}\left\{\cos\theta\left(1 - \frac{i}{2}(Q_{hp_0}^{-1} + Q_{hs_0}^{-1})\right) + \frac{i}{2}\sin\theta(Q_{hs_0}^{-1}\tan\delta_s^r - Q_{hp_0}^{-1}\tan\delta_p^i)\right\} \tag{101}$$

The scattering potential for P to SI is, consequently,

$${}^{P}_{SI}\mathbb{V}_{ve} = {}^{P}_{SI}\mathbb{V}_{e} + i\,{}^{P}_{SI}\mathbb{V}_{ane} \tag{102}$$

where the elastic part of the scattering potential $\mathbb{V}_e^{PSI}$ is given by

$$ {}^{P}_{SI}\mathbb{V}_e = \left(\sin\theta - \frac{1}{\gamma_0}\sin 2\theta\right)A_\rho - \left(\frac{2}{\gamma_0}\sin 2\theta\right)A_\beta \tag{103} $$

and anelastic part is given by

$$ {}^{P}_{SI}\mathbb{V}_{ane} = {}^{P}_{SI}\mathbb{V}^{\rho}A_\rho + {}^{P}_{SI}\mathbb{V}^{\beta}A_\beta + {}^{P}_{SI}\mathbb{V}^{Q_{hs}}A_{Q_{hs}} \tag{104} $$

with

$$ {}^{P}_{SI}\mathbb{V}^{\rho} = \frac{1}{\gamma_0}\left\{\sin 2\theta(Q_{hp_0}^{-1} - Q_{hs_0}^{-1}) + (\cos 2\theta - \frac{\gamma_0}{2}\cos\theta)(Q_{hs_0}^{-1}\tan\delta_s^r - Q_{hp_0}^{-1}\tan\delta_p^i)\right\} \tag{105} $$

$$ {}^{P}_{SI}\mathbb{V}^{\beta} = \frac{2}{\gamma_0}\left\{\sin 2\theta(Q_{hp_0}^{-1} - Q_{hs_0}^{-1}) + \cos 2\theta(Q_{hs_0}^{-1}\tan\delta_s^r - Q_{hp_0}^{-1}\tan\delta_p^i)\right\} \tag{106} $$

$$ {}^{P}_{SI}\mathbb{V}^{Q_{hs}} = \frac{Q_{hs_0}^{-1}}{\gamma_0}\sin 2\theta. \tag{107} $$

Similarly to the P-P case, the P-SI scattering potential is a complex function whose real part corresponds to the potential for elastic P-SV scattering, and whose imaginary part corresponds to the part of the reflectivity due to the anelasticity of the medium. The correspondence is qualitatively as expected given standard AVO results. For instance, in P-SI scattering there is no contribution from the change in P-velocity to reflection response.

In the special case that both reflected and incident waves are homogeneous the density and S-wave velocity parts reduce to

$$ {}^{P}_{SI}\mathbb{V}^{\beta} = 2\,{}^{P}_{SI}\mathbb{V}^{\rho} = \frac{1}{\gamma_{H_0}}(Q_{hp_0}^{-1} - Q_{hs_0}^{-1})\sin 2\theta, \tag{108} $$

from which we discern that the contributions from density, S-wave velocity and its quality factor have the same angle dependencies. In contrast with the P-P scattering potential, for a P-SI element at small angles, the relative change in S-wave quality factor does make a contribution to the anelastic part. At normal incidence, the elastic part of the scattering potential is zero, however the anelastic part due to influenced by the relative change in density and S-wave velocity is not zero. It can be seen that in a special case that

$$ \frac{\tan\delta_s^r}{Q_{hs_0}} = \frac{\tan\delta_p^i}{Q_{hp_0}}, \tag{109} $$

the anelastic part of the scattering potential is zero. In Figure 3 we plot the elastic and anelastic parts of the potential of density and S-wave velocity perturbations to scatter a homogeneous P-wave into an inhomogeneous SI-wave. At normal incidence, the density component of the scattering potential is zero for any incident attenuation angle δ_p^i, but the anelastic part goes to its maximum value. In contrast to the case of P-P scattering, here the contribution of density to the anelastic part of the scattering is considerable. For the S-wave velocity component, the elastic and anelastic parts of the scattering potential have the same contribution to the change in reflectivity but the opposite phase. In other words, for weak scattering at $\theta = 0$ the anelastic part goes to its maximum value and the elastic part is zero. Comparing this to the case of P-P scattering, we conclude that the anelasticity of a scattering heterogeneity has a greater influence on the P-SI mode than on the P-P mode.

Viscoelastic SI-to-SI scattering

The third diagonal element of the scattering matrix refers to the scattering of an SI-wave to another SI-wave. In this case both incident and reflected waves are S-waves, respectively represented by wave number vectors $\boldsymbol{\zeta}_s^r$ and $\boldsymbol{\zeta}_s^i$. We have, to begin,

$$\mathcal{F}_{SI}^{SI} = \boldsymbol{\zeta}_s^r \cdot \boldsymbol{\zeta}_s^i \tag{110}$$

$${}_{SI}^{SI}\mathcal{G}_\beta = \beta_0^2 \left\{ (\boldsymbol{\zeta}_s^r \cdot \boldsymbol{\zeta}_s^i)(\mathbf{k}_s^r \cdot \mathbf{k}_s^i) + (\boldsymbol{\zeta}_s^i \cdot \mathbf{k}_s^r)(\boldsymbol{\zeta}_s^r \cdot \mathbf{k}_s^i) \right\}. \tag{111}$$

Using the dot product

$$\boldsymbol{\zeta}_s^r \cdot \boldsymbol{\zeta}_s^i = \cos\theta + \frac{i}{2} Q_{hs_0}^{-1} \sin\theta \left(\tan\delta_s^r - \tan\delta_s^i\right) \tag{112}$$

$$\boldsymbol{\zeta}_s^r \cdot \mathbf{k}_s^i = -\boldsymbol{\zeta}_s^i \cdot \mathbf{k}_s^r = \beta_0^{-1} \left\{ \sin\theta \left(1 - \frac{i}{2} Q_{hs_0}^{-1}\right) - \frac{i}{2} Q_{hs_0}^{-1} \cos\theta (\tan\delta_s^r - \tan\delta_s^i) \right\}, \tag{113}$$

the related scattering element is

$${}_{SI}^{SI}\mathbb{V}_{ve} = {}_{SI}^{SI}\mathbb{V}_e + i\,{}_{SI}^{SI}\mathbb{V}_{ane} \tag{114}$$

where the elastic part of the scattering potential V_e^{SISI} is given by

$${}_{SI}^{SI}\mathbb{V}_e = (\cos\theta - \cos 2\theta)\, A_\rho - (2\cos 2\theta) A_\beta \tag{115}$$

and anelastic part by

$${}_{SI}^{SI}\mathbb{V}_{ane} = {}_{SI}^{SI}\mathbb{V}^\rho A_\rho + {}_{SI}^{SI}\mathbb{V}^\beta A_\beta + {}_{SI}^{SI}\mathbb{V}^{Q_{hs}} A_{Q_{hs}}, \tag{116}$$

with

$${}_{SI}^{SI}\mathbb{V}^\rho = Q_{hs_0}^{-1} \left\{ \frac{1}{2}\sin\theta - \sin 2\theta \right\} \left(\tan\delta_s^r - \tan\delta_s^i\right) \tag{117}$$

$${}_{SI}^{SI}\mathbb{V}^\beta = -2 Q_{hs_0}^{-1} \sin 2\theta \left(\tan\delta_s^r - \tan\delta_s^i\right) \tag{118}$$

$${}_{SI}^{SI}\mathbb{V}^{Q_{hs}} = Q_{hs_0}^{-1} \cos 2\theta. \tag{119}$$

At normal incidence the anelastic part of the scattering potential is non-zero and influenced by the S-wave quality factor. Also, in the case of homogeneous incident and reflected waves, only the $A_{Q_{hs}}$ influences the scattering potential, with the contribution from density and the S-wave velocity vanishing. In Figure 4, we plot the elastic and anelastic parts of the potential of density and S-velocity heterogeneities to scatter a homogeneous SI-wave into an inhomogeneous SI-wave. We observe that the anelastic part of the density contribution is very small in comparison to the elastic part. Comparing the P-P, P-SI and SI-SI modes altogether, we conclude that the anelasticity of a scatterer has a more pronounced effect on the outgoing wave for the converted wave modes. We can see by comparing Figures 2, 3 and 4 that only for the P-SI mode are the anelastic part of the scattering potential and the elastic part comparable. We also note that for conserved modes (P-P and SI-SI), the density contribution to the anelastic part of the scattering is very small compared to the elastic part.

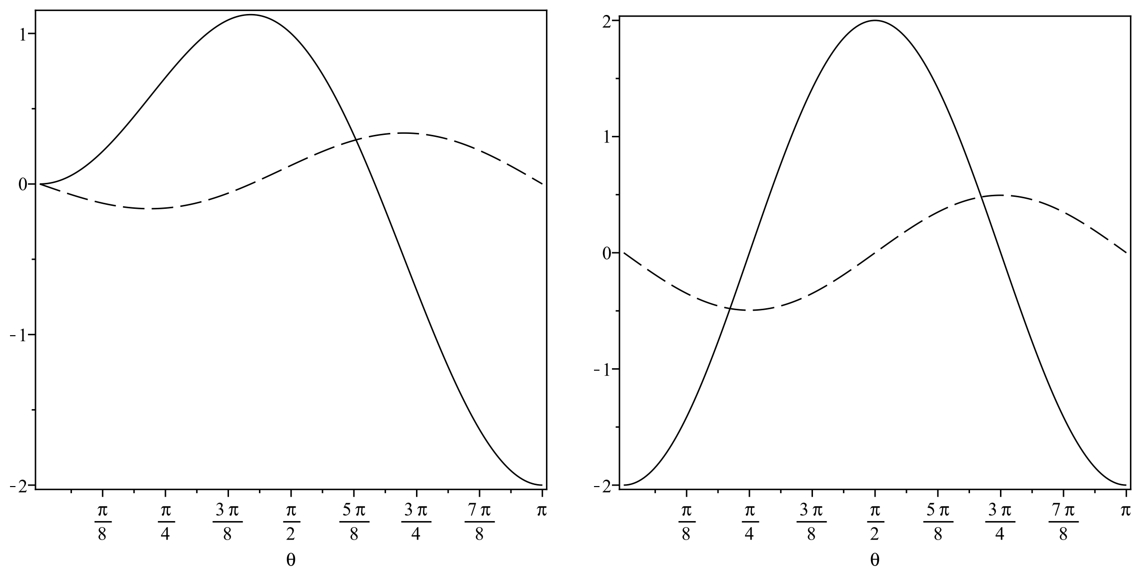

FIG. 4. Elastic and anelastic density(left) and S-velocity(right) components of the viscoelastic potential for scattering of incident homogeneous SI-wave to inhomogeneous reflected SI-wave versus of reflected wave angle θ, for $\delta_r = \frac{\pi}{3}$. Quality factor for S-wave is 7. Also the S-to P-velocity ratio for reference medium is chosen to be $1/2$. Dash line is for elastic part and solid line for anelastic part.

Viscoelastic scattering of SII-waves

Borcherdt (Borcherdt, 2009) proved that for an infinite planar boundary between to viscoelastic media in welded contact, neither an incident P- nor an SI-wave can convert through reflection or transmission into an SII-wave, and vice-versa. In the case of viscoelastic scattering from multidimensional heterogeneities we will show likewise that to first order (i.e., under the Born approximation) an incident SII-wave can only scatter into another SII-wave. For example let us consider to the scattering of P-wave to SII-wave, in this case we have

$$\mathcal{F}^{p}_{SII} = \mathbf{n}\cdot\boldsymbol{\xi}^i_p = 0 \tag{120}$$

$${}^{p}_{SII}\mathcal{G}_\alpha = \alpha_0^2(\mathbf{n}\cdot\mathbf{k}^r_s)(\boldsymbol{\xi}^i_p\cdot\mathbf{k}^i_p) = 0 \tag{121}$$

$${}^{p}_{SII}\mathcal{G}_\beta = \beta_0^2\left\{(\mathbf{n}\cdot\boldsymbol{\xi}^i_p)(\mathbf{k}^r_s\cdot\mathbf{k}^i_p) - 2(\mathbf{n}\cdot\mathbf{k}^r_s)(\boldsymbol{\xi}^i_p\cdot\mathbf{k}^i_p) + (\boldsymbol{\xi}^i_p\cdot\mathbf{k}^r_s)(\mathbf{n}\cdot\mathbf{k}^i_p)\right\} = 0, \tag{122}$$

where we have used the fact that polarization direction for SII wave is perpendicular to the plane constructed by the attenuation and propagation vectors. In a similar manner it is seen that the scattering elements for SI-SII, SII-P, P-SII, SII-SI and SI-SII are identically zero.

To analyse SII-SII scattering we calculate the second diagonal element of the scattering matrix, namely ${}^{SII}_{SII}\mathbb{V}_{ve}$, obtaining

$${}^{SII}_{SII}\mathbb{V}_{ve} = {}^{SII}_{SII}\mathbb{V}_e + i{}^{SII}_{SII}\mathbb{V}(2A_\beta + A_\rho) + i{}^{SII}_{SII}\mathbb{V}^{Q_{hs}}A_{Q_{hs}} \tag{123}$$

where the elastic part of the scattering potential is given by

$${}^{SII}_{SII}\mathbb{V}_e = (1-\cos\theta)A_\rho - (2\cos\theta)A_\beta \tag{124}$$

and the an elastic part by

$${}^{SII}_{SII}\mathbb{V} = -\frac{Q^{-1}_{hs_0}}{2}\sin\theta(\tan\delta^r_s - \tan\delta^i_s) \tag{125}$$

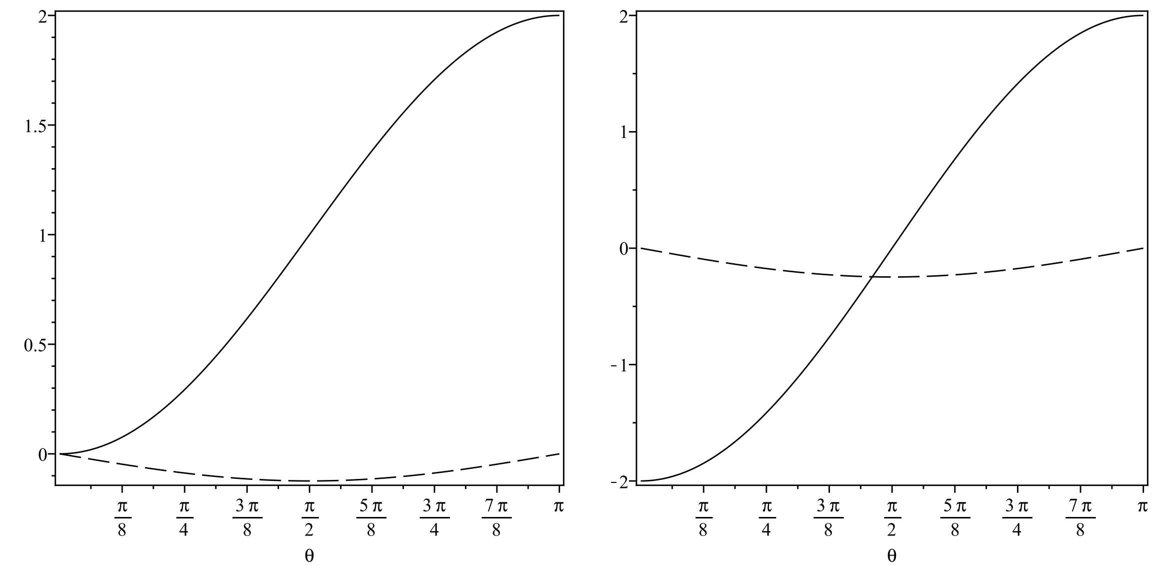

FIG. 5. Elastic and anelastic density(left) and S-velocity(right) components of the viscoelastic potential for scattering of incident homogeneous SII-wave to inhomogeneous reflected SII-wave versus of reflected wave angle θ, for $\delta_r = \frac{\pi}{3}$. Quality factor for S-wave is 5. Also the S-to P-velocity ratio for reference medium is chosen to be $1/2$. Dash line is for elastic part and solid line for anelastic part.

$$ {}^{SII}_{SII}\mathbb{V}^{Q_{hs}} = Q^{-1}_{hs_0} \cos\theta. \tag{126} $$

In Figure 5 we plot the the elastic and anelastic parts of the potential of density and S-wave velocity perturbations to scatter a homogeneous SII-wave into an inhomogeneous SII-wave.

SUMMARY AND CONCLUSION

The seismic response of the real earth deviates from the elastic-isotropic model often used to frame the seismic wave propagation problem. Here we investigate viscoelasticity in its capacity to reproduce the effect of dissipation on the propagation of a wave. Full formal theory for viscoelastic seismic waves exists, but the most powerful versions of it have largely been restricted to layered media. Exact, closed-form solutions for viscoelastic waves in arbitrary multidimensional media are not in general available, but, to first order, scattering formulations can provide interpretable approximate forms. These forms are important for obtaining physical insight into interactions of seismic waves with dissipative media, but also for posing and solving inverse scattering and full waveform inversion problems.

In a viscoelastic medium generally the directions of maximum attenuation and wave propagation are not aligned, in which case the wave is considered inhomogeneous. Allowing for inhomogeneity, there are three types of waves that propagate in a viscoelastic medium. P and SI waves with elliptical motion in the plane defined by propagation and attenuation directions, and SII with linear polarization perpendicular to that plane.

To make a scattering matrix whose elements refer to the scattering of viscoelastic wave modes, we first define the polarization vectors for P, SI and SII-waves. For P-wave the polarization vector is a complex vector in the plane of attenuation-propagation. This polarization involves the elliptical particle motion. For SI-waves, polarization vector is also a complex vector, this time perpendicular to the P-wave polarization. The particle motion for SI-wave is on an ellipse perpendicular to the

P-wave particle motion ellipse. Another S-wave is SII, which has a linear polarization and in the case of elastic reduces to the SH wave. We show that an SII-wave can only be scattered to an SII-wave. Also P- and SI-waves can not be converted to an SII-wave. So in the scattering matrix the only converted waves are PSI and SIP.

The scattering potential in displacement space is obtained by sandwiching the scattering operator between the incident and reflected polarization vectors. Since for the viscoelastic waves, polarizations are complex, the viscoelastic scattering potential we obtained is a complex function whose real part is elastic scattering potential and whose imaginary part is the related to the anelasticity of the medium. In contrast to the elastic scattering potential that only alters the amplitude of the outgoing field, the viscoelastic scattering potential alters both amplitude and phase of the outgoing field. Anelasticity appears to have more significant effect on converted waves than on conserved modes.

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