Some exact forms for viscoelastic reflection coefficients

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ABSTRACT

An explicit analytical form of the scattering matrix for Homogeneous Isotropic Linear Viscoelastic (HILV) continuum is obtained. The reflection and transmission coefficients are the complex function related to the P- and S-velocities and corresponding quality factors Q_P and Q_S. The real part is the elastic scattering potential and the imaginary part is the term induced by the anelasticity of the medium. Linearized reflection coefficients can be used for viscoelastic AVO/AVA and full waveform inversion.

INTRODUCTION

Borcherdt (2009) has presented a complete theory for seismic waves propagating in layered anelastic media, assuming a viscoelastic model to hold. Borcherdt's formulation is particularly powerful in that it predicts a range of transverse, inhomogeneous wave types unique to viscoelastic media (Type I and II S waves), and develops rules for conversion of one type to another during interactions with planar boundaries. Problem of reflection and transmission coefficients at an interface between viscoelastic media in the context of inhomogeneous waves has studied by (Cooper, 1967; Schoenberg, 1971). (Krebes, 1983, 1984) has investigated the reflection coefficients of viscoelastic SII waves for some special cases. In this brief research we considered to the problem of reflection transmission in a viscoelastic layered media. We calculate the complete form of the viscoelastic scattering potential whose elements refers to the various types of reflection and transmission coefficients related to P- and SI-waves.

PRELIMINARIES

Inhomogeneous waves play an important role in viscoelastic wave theory. Such waves, which decrease in amplitude with propagation distance, are described with the use of complex vectors or *bivectors* (Morro, 1992). In the case of inhomogeneous waves, the attenuation and propagation vectors are not in the same direction. This makes the displacement vectors different from homogenous case. In what follows, we show that the particle motion for P waves is elliptical in the plane constructed by attenuation and propagation vectors. This elliptical motion reduces to a linear motion in the limit of homogenous case. Also, we have the two types of shear waves SI and SII. The first one, which is the generalization of SV wave, has an elliptical displacement vector in the propagation-attenuation plane. Finally SII type wave which is a generalization of SH wave types has a linear motion perpendicular to the propagation-attenuation plane. The wavenumber vector of inhomogeneous waves is represented by

$$\mathbf{K} = \mathbf{P} - i\mathbf{A}. \tag{1}$$

Here $\mathbf{P}$ is the propagation vector perpendicular to the constant phase plane $\mathbf{P}\cdot\mathbf{r} = constant$, and $\mathbf{A}$ is the attenuation vector perpendicular to the amplitude constant plane $\mathbf{A}\cdot\mathbf{r} = constant$. Attenuation vector $\mathbf{A}$ is in the direction of maximum decay of amplitude. In the case that attenuation and propagation vectors are in the same direction, wave is homogeneous. Elastic media is represented by $\mathbf{A} = 0$. If we represent the angle between $\mathbf{P}$ and $\mathbf{A}$ by δ, for inhomogeneous waves $\delta \neq 0$ we have

$$|\mathbf{P}| = 2^{-\frac{1}{2}} \left[\Re\mathbf{K}\cdot\mathbf{K} + \sqrt{(\Re\mathbf{K}\cdot\mathbf{K})^2 + (\Im\mathbf{K}\cdot\mathbf{K})^2 \sec^2\delta} \right]^{\frac{1}{2}}, \tag{2}$$

and

$$|\mathbf{A}| = 2^{-\frac{1}{2}} \left[-\Re\mathbf{K}\cdot\mathbf{K} + \sqrt{(\Re\mathbf{K}\cdot\mathbf{K})^2 + (\Im\mathbf{K}\cdot\mathbf{K})^2 \sec^2\delta} \right]^{\frac{1}{2}} \tag{3}$$

where

$$\mathbf{K}\cdot\mathbf{K} = |\mathbf{P}|^2 - |\mathbf{A}|^2 - 2i|\mathbf{P}||\mathbf{A}|\cos\delta = \left(\frac{\omega}{\alpha}\right)^2 = \frac{\omega^2\rho}{K + \frac{4}{3}M} \tag{4}$$

Here K and M are the viscoelastic Lamé parameters. Attenuation angle δ for isotropic viscoelastic medium always varies between zero and $\pi/2$. In anisotropic medium however it can exceed $\pi/2$. From this general framework we may now follow Borcherdt (2009) in analyzing three types of independently propagating wave. According to (4)

$$\Im(\mathbf{K}\cdot\mathbf{K}) = -\frac{1}{2}|\mathbf{P}||\mathbf{A}|\cos\delta \tag{5}$$

In a viscoelastic medium $\Im(\mathbf{K}\cdot\mathbf{K}) \neq 0$, this implies that the maximum attenuation $|\mathbf{A}|$ is not zero and also the direction of maximum attenuation can not be perpendicular to the direction of phase propagation. As a result the attenuation angle varies $0 \leq \delta < 90°$.

VISCOELASTIC WAVES

According to Helmholtz decomposition every vector field $\mathbf{U}$ can be written as a combination of an irrotational (curl-free) and a divergenceless vector as follows

$$\mathbf{U} = \boldsymbol{\nabla}\Phi + \boldsymbol{\nabla}\times\boldsymbol{\Psi} \tag{6}$$

with

$$\boldsymbol{\nabla}\cdot\boldsymbol{\Psi} = 0. \tag{7}$$

Constitutive equation that relates the stress and strain tensors imply that the wave equation for P- and S-wave are given by

$$\nabla^2\Phi - \alpha^{-2}\partial_t^2\Phi = 0 \tag{8}$$

$$\nabla^2\boldsymbol{\Psi} - \beta^{-2}\partial_t^2\boldsymbol{\Psi} = 0 \tag{9}$$

Where

$$\sqrt{\mathbf{K}^p\cdot\mathbf{K}^p} = K^p = \frac{\omega}{\alpha}, \tag{10}$$

$$\sqrt{\mathbf{K}^s\cdot\mathbf{K}^s} = K^s = \frac{\omega}{\beta}. \tag{11}$$

As we have seen, in presence of attenuation the wavenumber vector is complex which it's real part displays the propagation direction and imaginary part refers to the maximum direction of the wave attenuation. As a consequence P- and S-velocities in eqs.(10) and (11) are the complex velocities related to the elastic P- and S-velocities α_e and β_s as

$$\alpha = \alpha_e \left(1 + i\frac{Q_p^{-1}}{2}\right), \tag{12}$$

$$\beta = \beta_e \left(1 + i\frac{Q_s^{-1}}{2}\right), \tag{13}$$

where Q_p and Q_s are the quality factors for P- and S waves respectively. In above we used the low-loss viscoelastic medium approximation where $Q^{-1} \ll 1$. It is clear that even for complex velocity, the solutions of equations (8) and (9) have the plane wave form, namely

$$\Phi = \alpha\Phi_0 \exp\left[-i(\mathbf{K}^p \cdot \mathbf{r} - \omega t)\right], \tag{14}$$

$$\mathbf{\Psi} = \beta\mathbf{\Psi}_0 \exp\left[-i(\mathbf{K}^s \cdot \mathbf{r} - \omega t)\right], \tag{15}$$

where Φ_0 and $\mathbf{\Psi}_0$ are the complex scalar and vector constants. Now the displacement vectors for P- and S-waves are given by

$$\mathbf{U}^p = \mathbf{\nabla}\Phi = -i\alpha\mathbf{K}^p\Phi_0 \exp\left[-i(\mathbf{K}^p \cdot \mathbf{r} - \omega t)\right], \tag{16}$$

$$\mathbf{U}^s = \mathbf{\nabla} \times \mathbf{\Psi} = -i\beta\mathbf{K}^s \times \mathbf{\Psi}_0 \exp\left[-i(\mathbf{K}^s \cdot \mathbf{r} - \omega t)\right]. \tag{17}$$

In the case that $\mathbf{U}_S = U_S\mathbf{n}$, where $\mathbf{n}$ is a unit vector orthogonal to the plane of $\mathbf{P}_S - \mathbf{A}_S$, the corresponding wave is named S type-I(SI) wave. From study of complex vectors we know that they display the elliptical motion for a dynamic problem. Therefore we expect that displacement vectors for P- and S-waves with complex polarization vectors, describe an elliptical motion for particles. To understand the nature of the motion characterized by (16) and (17), let us consider to a general complex vector $V = V_R + iV_I$. Multiplication of V with the complex wavenumber vector $\mathbf{K}$ leads to

$$V\mathbf{K} = (V_R + iV_I)(\mathbf{P} - i\mathbf{A}) = (V_R\mathbf{P} + V_I\mathbf{A}) + i(V_I\mathbf{P} - V_R\mathbf{A}), \tag{18}$$

using the relationship between velocity and wavenumber vector namely $KV = \omega$, we can calculate the the real and imaginary parts of velocity V as

$$V_R = \frac{\omega K_R}{(K_R)^2 + (K_I)^2}, \tag{19}$$

$$V_I = -\frac{\omega K_I}{(K_R)^2 + (K_I)^2}. \tag{20}$$

Th polarization vector can be defined by a complex vector $\boldsymbol{\xi}$ as

$$\boldsymbol{\xi} = \boldsymbol{\xi}_R - i\boldsymbol{\xi}_I = \frac{V}{\omega}\mathbf{K}, \tag{21}$$

where the real and imaginary parts of the polarization vectors are

$$\boldsymbol{\xi}_R = \frac{K_R\mathbf{P} - K_I\mathbf{A}}{(K_R)^2 + (K_I)^2}, \tag{22}$$

$$\boldsymbol{\xi}_I = \frac{K_I\mathbf{P} + K_R\mathbf{A}}{(K_R)^2 + (K_I)^2}. \tag{23}$$

These two vectors are orthogonal, furthermore $|\boldsymbol{\xi}_R|^2 - |\boldsymbol{\xi}_I|^2 = 1$. A simple analysis indicates that particle motion related to the displacement for P-wave in equation (16) is an ellipse with major axes $\boldsymbol{\xi}_R$ and minor axes $\boldsymbol{\xi}_I$. In a similar manner, we can show that the polarization vector for SI wave can be written as

$$\boldsymbol{\zeta}_s = \boldsymbol{\zeta}_R - i\boldsymbol{\zeta}_I = (\boldsymbol{\xi}_{sR} - i\boldsymbol{\xi}_{sI}) \times \mathbf{n} \tag{24}$$

For low-loss viscoelastic media the elliptical polarization takes the following form

$$\boldsymbol{\xi} = \frac{\alpha_e}{\omega}\left\{\mathbf{K}_p + i\frac{Q_p^{-1}}{2}\mathbf{P}_p\right\} \tag{25}$$

$$\boldsymbol{\zeta}_s = \frac{\alpha_e}{\omega}\left\{\mathbf{K}_s + i\frac{Q_s^{-1}}{2}\mathbf{P}_s\right\} \times \mathbf{n} \tag{26}$$

Finally, we can redefine the the displacement vectors for P- and S-waves as

$$\mathbf{U}^p = \boldsymbol{\xi}_p\Phi_0 \exp\left[-i(\mathbf{K}_p\cdot\mathbf{r} - \omega t)\right] \tag{27}$$

$$\mathbf{U}^s = \boldsymbol{\zeta}_s\boldsymbol{\Psi}_0 \exp\left[-i(\mathbf{K}_s\cdot\mathbf{r} - \omega t)\right] \tag{28}$$

SCATTERING MATRIX

So far we considered to the viscoelastic plane wave propagate in an isotropic homogeneous medium. What happens if an inhomogeneous wave with a elliptical polarization hits the boundary of two half-space? To answer this question we need to define two half space medium with different physical properties separated by a boundary. To analyze the complete form of the reflection-transmission from boundary we used the same method by (Aki and Richards, 2002). Incident waves are defined in mediums V and V' by ${}^{\uparrow}\mathbf{U}^i$ and ${}^{\downarrow}\mathbf{U}^i$. The downarrow index refers to the wave that propagate in medium Vin $z < 0$ to the boundary and uparrow index refers to the wave that propagate in medium V' in $z > 0$ to the boundary. We identify the quantities releted to the V' by prim index and unprimed to the medium V.

With this definitions and basics, we are now ready to write the incident P- and S-waves in V and V' mediums

$${}^{\uparrow}\mathbf{U}_p^i = \Phi'^{\uparrow}_0(\xi'_{px}\mathbf{x} - \xi'_{pz}\mathbf{z}) \exp\left[i\omega\left(\mathbb{K}x - \frac{\xi'_{pz}}{\alpha'}z\right)\right], \tag{29}$$

$${}^{\downarrow}\mathbf{U}_p^i = \Phi_0{}^{\downarrow}(\xi_{px}\mathbf{x} + \xi_{pz}\mathbf{z}) \exp\left[i\omega\left(\mathbb{K}x + \frac{\xi_{pz}}{\alpha}z\right)\right], \tag{30}$$

$${}^{\uparrow}\mathbf{U}_s^i = \Psi'^{\uparrow}(\xi'_{sz}\mathbf{x} + \xi'_{sx}\mathbf{z}) \exp\left[i\omega\left(\mathbb{K}x - \frac{\xi'_{sz}}{\beta'}z\right)\right], \tag{31}$$

$${}^{\downarrow}\mathbf{U}_s^i = \Psi^{\downarrow}(\xi_{sz}\mathbf{x} - \xi_{sx}\mathbf{z}) \exp\left[i\omega\left(\mathbb{K}x + \frac{\xi_{sz}}{\beta}z\right)\right], \tag{32}$$

where we have defined the x- and z-components of polarization vectors as

$$\xi_x = \sin\theta + \frac{i}{2}Q^{-1}\tan\delta\cos\theta, \tag{33}$$

$$\xi_z = \cos\theta - \frac{i}{2}Q^{-1}\tan\delta\sin\theta. \tag{34}$$

In addition similar to the elastic case we define a complex ray parameter $\mathbb{K}$ that comes from the generalized Snell's law

$$\omega\mathbb{K} = |\mathbf{P}|\sin\theta - i|\mathbf{A}|\sin(\theta-\delta), \tag{35}$$

to having the complete form of reflection-transmission coefficients in addition to incident waves propagate in two half-space we also need to define the reflected and transmitted waves. With similar notations for incident waves, transmitted and reflected P- and S-waves are given by

$$^{\downarrow}\mathbf{U}_p^r = \Phi_0'^{\downarrow}(\xi'_{px}\mathbf{x} + \xi'_{pz}\mathbf{z})\exp\left[i\omega\left(\mathbb{K}x + \frac{\xi'_{pz}}{\alpha'}z\right)\right] \tag{36}$$

$$^{\uparrow}\mathbf{U}_p^t = \Phi_0{}^{\uparrow}(\xi_{px}\mathbf{x} - \xi_{pz}\mathbf{z})\exp\left[i\omega\left(\mathbb{K}x - \frac{\xi_{pz}}{\alpha}z\right)\right] \tag{37}$$

$$^{\downarrow}\mathbf{U}_s^r = \Psi_0'^{\downarrow}(\xi'_{sz}\mathbf{x} - \xi'_{px}\mathbf{z})\exp\left[i\omega\left(\mathbb{K}x + \frac{\xi'_{sz}}{\beta'}z\right)\right] \tag{38}$$

$$^{\uparrow}\mathbf{U}_s^t = \Psi_0{}^{\uparrow}(\xi_{sz}\mathbf{x} + \xi_{px}\mathbf{z})\exp\left[i\omega\left(\mathbb{K}x - \frac{\xi_{sz}}{\beta}z\right)\right] \tag{39}$$

where index r indicates the reflected wave and index t refers to the transmitted wave. The boundary condition implies that the displacements vectors in boundary should be continues. Also the components of the stress tensor are given by

$$P_{ij} = \delta_{ij}\left(K - \frac{2}{3}\right)E_{kk} + 2ME_{ij}, \tag{40}$$

with

$$E_{ij} = \frac{1}{2}(U_{i,j} + U_{j,i}). \tag{41}$$

Across the boundary should be continues. Namely the x-component and z-component of stress tensor are gives by

$$P_{31} = M(U_{3,1} + U_{1,3}), \tag{42}$$

$$P_{33} = \left(K - \frac{2}{3}M\right)U_{1,1} + \left(K + \frac{4}{3}M\right)U_{3,3} \tag{43}$$

have to be continued at $z = 0$. Implementation of boundary conditions as described to the displacements and stress tensor at $z = 0$ leads to

$$\hat{X}\begin{pmatrix}\Phi_0{}^{\uparrow}\\ \Psi^{\uparrow}\\ \Phi_0'^{\downarrow}\\ \Psi'^{\downarrow}\end{pmatrix} = \hat{Y}\begin{pmatrix}\Phi_0^{\downarrow}\\ \Psi^{\downarrow}\\ \Phi_0'^{\uparrow}\\ \Psi'^{\uparrow}\end{pmatrix}$$

where

$$\hat{X} = \begin{pmatrix} -\alpha\mathbb{K} & -\xi_{sz} & \alpha'\mathbb{K} & \xi'_{sz} \\ \xi_{pz} & -\beta\mathbb{K} & \xi'_{pz} & -\beta'\mathbb{K} \\ 2\rho\beta^2\mathbb{K}\xi_{pz} & \rho\beta(1-2\beta^2\mathbb{K}^2) & 2\rho'\beta'^2\mathbb{K}\xi'_{pz} & 2\rho'\beta'(1-2\beta'^2\mathbb{K}^2) \\ -\rho\alpha(1-2\beta^2\mathbb{K}^2) & 2\rho\beta^2\mathbb{K}\xi_{sz} & \rho'\alpha'(1-2\beta'^2\mathbb{K}^2) & -2\rho'\beta'^2\mathbb{K}\xi'_{sz} \end{pmatrix} \tag{44}$$

$$\hat{Y} = \begin{pmatrix} \alpha\mathbb{K} & \xi_{sz} & \alpha'\mathbb{K} & \xi'_{sz} \\ \xi_{pz} & -\beta\mathbb{K} & \xi'_{pz} & -\beta'\mathbb{K} \\ 2\rho\beta^2\mathbb{K}\xi_{pz} & \rho\beta(1-2\beta^2\xi^2) & 2\rho'\beta'^2\mathbb{K}\xi'_{pz} & 2\rho'\beta'(1-2\beta'^2\mathbb{K}^2) \\ \rho\alpha(1-2\beta^2\mathbb{K}^2) & -2\rho\beta^2\mathbb{K}\xi_{sz} & -\rho'\alpha'(1-2\beta'^2\mathbb{K}^2) & 2\rho'\beta'^2\mathbb{K}\xi'_{sz} \end{pmatrix}. \tag{45}$$

Accordingly the scattering matrix represents the all reflection-transmission coefficients is given by

$$X^{-1}Y = \begin{pmatrix} {}^{\downarrow}PP^{\uparrow} & {}^{\downarrow}SIP^{\uparrow} & {}^{\uparrow}PP^{\uparrow} & {}^{\downarrow}SIP^{\uparrow} \\ {}^{\downarrow}PSI^{\uparrow} & {}^{\downarrow}SISI^{\uparrow} & {}^{\downarrow}SISI^{\uparrow} & {}^{\uparrow}SISI^{\uparrow} \\ {}^{\downarrow}PP^{\downarrow} & {}^{\downarrow}SIP^{\downarrow} & {}^{\uparrow}PP^{\downarrow} & {}^{\uparrow}SIP^{\downarrow} \\ {}^{\downarrow}PSI^{\downarrow} & {}^{\downarrow}SISI^{\downarrow} & {}^{\uparrow}PSI^{\downarrow} & {}^{\uparrow}SISI^{\downarrow} \end{pmatrix}. \tag{46}$$

The diagonal elements of the scattering matrix represents the reflections that preserve the type of the waves. For example ${}^{\downarrow}PP^{\uparrow}$ refers to the reflected upgoing P-wave from downgoing incident P-wave and similar explanations for other diagonal elements. On the other hand some off-diagonal elements indicates the converted waves. For instance, ${}^{\downarrow}SIP^{\uparrow}$ displays a reflected upgoing P from incident downgoing SI wave. Other off-diagonal elements refers to the transmitted waves either converted modes or preserved modes. For example ${}^{\downarrow}SISI^{\downarrow}$ is related to the transmitted downgoing SI wave from a downgoing incident SI wave and ${}^{\downarrow}SIP^{\downarrow}$ is a downgoing transmitted P wave from a downgoing incident SI wave. It has been shown that (Moradi and Innanen, 2013) the waves with elliptical polarization can not converted to the waves with the linear polarizations. For example SII wave that has a linear polarization does not converted to the P or SI waves. Some elements of the scattering matrix (46) that might be interested are

$${}^{\downarrow}PP^{\uparrow} = \frac{E_- F_+ - G_+ H_- \mathbb{K}^2}{E_+ F_+ + G_- H_- \mathbb{K}^2}, \tag{47}$$

$${}^{\downarrow}SISI^{\downarrow} = -\frac{F_- E_+ - G_- H_+ \mathbb{K}^2}{E_+ F_+ + G_- H_- \mathbb{K}^2}, \tag{48}$$

where

$$A = \rho'(1-2\beta'^2\mathbb{K}^2) - \rho(1-2\beta^2\mathbb{K}^2), \tag{49}$$

$$B = A + \rho, \tag{50}$$

$$C = \rho' - A, \tag{51}$$

$$D = \rho' - \rho - A, \tag{52}$$

$$E_\pm = B\frac{\xi_{pz}}{\alpha} \pm C\frac{\xi'_{pz}}{\alpha'}, \tag{53}$$

$$F_\pm = B\frac{\xi_{sz}}{\beta} \pm C\frac{\xi'_{sz}}{\beta'}, \tag{54}$$

$$G_{\pm} = A \pm D\frac{\xi_{pz}}{\alpha}\frac{\xi'_{sz}}{\beta'}, \tag{55}$$

$$H_{\pm} = A \pm D\frac{\xi'_{pz}}{\alpha'}\frac{\xi_{sz}}{\beta}. \tag{56}$$

It is clear that all components of the scattering matrix are complex. To see what is the relation between the reflectivity functions we obtained and ones for elastic case, we divide the complex ray parameter $\mathbb{K}$ into the real and imaginary part

$$\mathbb{K} = p + i\mathbb{K}_I, \tag{57}$$

where p is the real ray parameter in the elastic medium and $\mathbb{K}_I$ is the term proportional to the reciprocal factor, it means in the low-loss viscoelastic medium $\mathbb{K}_I^2 \approx 0$. By this assumption the Taylor expansion series for (47) around $\mathbb{K}_I$

$$^{\downarrow}PP^{\uparrow} = \frac{E_-F_+ - G_+H_-p^2}{E_+F_+ + G_-H_-p^2} -$$

$$\left\{\frac{G_+H_-}{E_+F_+ + G_-H_-p^2} + \frac{E_-F_+ - G_+H_-p^2}{(E_+F_+ + G_-H_-p^2)^2}G_-H_-\right\} ip\mathbb{K}_I. \tag{58}$$

This equation is pseudo-approximation for $\mathbb{K}_I$ because in (58) still other coefficients are functions of $\mathbb{K}_I$. Consequently, for the low-loss viscoelastic medium we can expand (58) in terms of inverse quality factors Q_p^{-1} and $Q_p'^{-1}$ to reach the following expression

$$^{\downarrow}PP^{\uparrow} = (^{\downarrow}PP^{\uparrow})_{elastic} + Q_p^{-1}M + Q_p'^{-1}N + Q_s^{-1}R + Q_s'^{-1}S. \tag{59}$$

Although the approximation that we mentioned for PP mode is complicated to derive but we can examine that for a more simpler case. As we have mentioned earlier the SII-wave with a linear polarization can not reflected or transmitted in any type of waves with the elliptical polarization either inhomogeneous P-wave or SI waves. As a result the scattering matrix for SII-wave can be written separately by a 2×2 matrix as

$$\begin{pmatrix} ^{\downarrow}SIISII^{\uparrow} & ^{\uparrow}SIISII^{\uparrow} \\ ^{\downarrow}SIISII^{\downarrow} & ^{\uparrow}SIISII^{\downarrow} \end{pmatrix}, \tag{60}$$

with the elements

$$^{\downarrow}SIISII^{\uparrow} = -^{\uparrow}SIISII^{\downarrow} = \frac{\rho\beta\xi_{sz} - \rho'\beta'\xi'_{sz}}{\rho\beta\xi_{sz} + \rho'\beta'\xi'_{sz}}, \tag{61}$$

$$^{\downarrow}SIISII^{\downarrow} = \frac{2\rho\beta\xi_{sz}}{\rho\beta\xi_{sz} + \rho'\beta'\xi'_{sz}}, \tag{62}$$

$$^{\uparrow}SIISII^{\uparrow} = \frac{2\rho'\beta'\xi'_{sz}}{\rho\beta\xi_{sz} + \rho'\beta'\xi'_{sz}}. \tag{63}$$

The diagonal elements of this matrix represent the reflection coefficients and off-diagonal terms refer to the transmitted waves. Now we examine the relation between reflectivity function for SII to SII with that in the elastic case for SH to SH. Inserting the following quantities

$$\xi_{sz} = \cos\theta_s - \frac{i}{2} Q_s^{-1} \tan\delta_s \sin\theta_s, \tag{64}$$

$$\xi'_{sz} = \cos\theta'_s - \frac{i}{2} Q_s'^{-1} \tan\delta'_s \sin\theta'_s, \tag{65}$$

$$\beta = \beta_e \left(1 + \frac{iQ_s^{-1}}{2}\right), \tag{66}$$

$$\beta' = \beta'_e \left(1 + \frac{iQ_s'^{-1}}{2}\right), \tag{67}$$

in (61) we arrive at

$$^{\downarrow}SIISII^{\uparrow} = \frac{X_- + iQ_s^{-1}Y - iQ_s'^{-1}Y'}{X_+ + iQ_s^{-1}Y + iQ_s'^{-1}Y'}, \tag{68}$$

where we defined

$$X_\pm = \rho\beta_e \cos\theta_s \pm \rho'\beta'_h \cos\theta'_s, \tag{69}$$

$$2Y = \rho\beta_e(\cos\theta_s - \tan\delta_s \sin\theta_s), \tag{70}$$

$$2Y' = \rho'\beta'_e(\cos\theta'_s - \tan\delta'_s \sin\theta'_s). \tag{71}$$

Using the Taylor expansion with respect to the Q_s^{-1} and $Q_s'^{-1}$ and keep the first order terms (68) reduces to

$$^{\downarrow}SIISII^{\uparrow} \approx \frac{X_-}{X_+} + iQ_s^{-1}\frac{Y}{X_+}\left(1 - \frac{X_-}{X_+}\right) - iQ_s'^{-1}\frac{Y'}{X_+}\left(1 + \frac{X_-}{X_+}\right), \tag{72}$$

where the reflectivity function for SH to SH mode is given by

$$^{\downarrow}SHSH^{\uparrow} = \frac{X_-}{X_+} = \frac{\rho\beta_e \cos\theta_s - \rho'\beta'_e \cos\theta'_s}{\rho\beta_e \cos\theta_s + \rho'\beta'_e \cos\theta'_s}. \tag{73}$$

Let us now discuss in what conditions the reflected or transmitted waves can be homogeneous or inhomogeneous. In the case that wavenumber vector is complex the snell's law leads to two relation. The first one, which related to the real part of the complex ray parameter $\mathbb{K}$ is

$$\frac{\sin\theta_s}{\beta_e} = \frac{\sin\theta'_s}{\beta'_e}. \tag{74}$$

This is the snell's law for a elastic medium where we have real wavenumber vectors and real velocities. The second part which comes from the imaginary part of the $\mathbb{K}$ is

$$\frac{\sin(\theta_s - \delta_s)}{\beta_e \cos\delta_s} = \frac{\sin(\theta'_s - \delta'_s)}{\beta'_e \cos\delta'_s}. \tag{75}$$

The recent relation in the limit of elastic medium when attenuation angles goes to zero reduces to the (74). Expanding the $\sin$ functions in (75) and then inserting (74) in it we arrive at

$$\frac{\sin\theta_s}{\beta_e}\left(Q_s^{-1} - Q_s'^{-1}\right) = \frac{\tan\delta_s \cos\theta_s}{\beta_e Q_s} - \frac{\tan\delta'_s \cos\theta'_s}{\beta'_e Q'_s}. \tag{76}$$

For non-normal incident ($\theta_s \neq 0$), and the incident homogeneous wave $\delta_s = 0$ the transmitted wave is homogeneous if and only if $Q_s = Q'_s$.

LINEARIZED SCATTERING POTENTIAL

To perform the conventional AVO/AVA inversion and full waveform inversion method of seismic date for a viscoelastic medium we need the linearized scattering matrix. The linearized scattering potential for low-loss viscoelastic media has been obtained using the scattering integral method based on the born approximation (Moradi and Innanen, 2013). The scattering matrix we already obtained in this research directly from the continuity of the displacement vectors and stress tensor in the boundary should be linearized and compared with result obtained from green's function approach. In the elastic case this relationship is given by (Beylkin and Burridge, 1990)

$$R = -S\frac{\sqrt{\tan\theta_i \tan\theta_r}}{2\sin(\theta_i+\theta_r)}. \tag{77}$$

Here R denotes the linearized reflectivity functions and S is the scattering coefficients (Born approximation).

If the two half-space medium are very similar we can define the linearized reflectivity functions in previous section in terms of changes in density, velocities and quality factors as follows

$$A_\rho = \frac{\Delta\rho}{\bar{\rho}} = 2\frac{\rho'-\rho}{\rho'+\rho} \tag{78}$$

$$A_\alpha = \frac{\Delta\alpha}{\bar{\alpha}} = 2\frac{\alpha'-\alpha}{\alpha'+\alpha} \tag{79}$$

$$A_\beta = \frac{\Delta\beta}{\bar{\beta}} = 2\frac{\beta'-\beta}{\beta'+\beta} \tag{80}$$

$$A_{Q_S} = \frac{\Delta Q_S}{\bar{Q}_S} = 2\frac{Q_S'-Q_S}{Q_S'+Q_S} \tag{81}$$

$$A_{Q_P} = \frac{\Delta Q_P}{\bar{Q}_P} = 2\frac{Q_P'-Q_P}{Q_P'+Q_P} \tag{82}$$

where we defined

$$\Delta\rho = \rho'-\rho \tag{83}$$

and

$$\bar{\rho} = \frac{\rho'+\rho}{2} \tag{84}$$

One interesting results that come from the snell's law is that we can write the perturbation in quality factors in terms of velocities. Rewriting (75) as

$$\tan\bar{\theta}_s\left(1-\frac{Q_s}{Q_s'}\right) = \tan\bar{\delta}_s\left(1-\frac{Q_s}{Q_s'}\frac{\beta_e}{\beta_e'}\frac{\cos\theta'}{\cos\theta}\right) \tag{85}$$

and using the definitions of perturbations in(85) and keep the first order terms in perturbations we we arrive at

$$A_{Q_S} = \frac{\tan\bar{\delta}_s\cos^2\bar{\theta}_s}{\tan\bar{\theta}_s-\tan\bar{\delta}_s}A_\beta \tag{86}$$

in a similar way we have

$$A_{Q_P} = \frac{\tan\bar{\delta}_p \cos^2\bar{\theta}_p}{\tan\bar{\theta}_p - \tan\bar{\delta}_p} A_\alpha \tag{87}$$

It means that the perturbations in quality factors can be written in terms of perturbations in velocities.

SUMMARY AND CONCLUSION

In summary we obtained the complete form of the reflection transmission coefficients for the isotropic viscoelastic layered medium. The most important feature of the wave propagation in the viscoelastic medium is that the waves are inhomogeneous, in other words the direction of attenuation and propagation are not in the same direction. In this case the polarization vectors for P- and SI waves are the complex functions of quality factors. For inhomogeneous waves these polarizations display a elliptical motion in a plane of attenuation and propagation vectors. The scattering potential we obtained has the same form as the elastic case(Aki and Richards, 2002) but with the complex ray parameter and complex z-components of the polarizations. For SII wave that has the linear polarization we obtain the scattering potential, in this case we expand the reflection coefficients for SII reflected to SII in terms of inverse quality factors. As a future direction along these lines we can linearized the reflectivity functions to compare the result with those obtained using the Born approximation based on the scattering integral (Moradi and Innanen, 2013).

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APPENDIX: LINEARIZED SH-TO-SH SCATTERING POTENTIAL

Snell's law for incident and transmitted SH wave is given by

$$\frac{\beta}{\beta'} = \frac{\sin\theta_s}{\sin\theta'_s}. \tag{88}$$

Using the definition of perturbation parameters the right hand side reduces to

$$\frac{\sin\theta_s}{\sin\theta'_s} = \frac{\sin\left(\theta_s - \frac{\Delta\theta_s}{2}\right)}{\sin\left(\theta_s + \frac{\Delta\theta_s}{2}\right)} \approx 1 - 2\frac{\sin\frac{\Delta\theta_s}{2}}{\tan\bar{\theta}_s}, \tag{89}$$

also the left hand side reduces to

$$\frac{\beta}{\beta'} \approx 1 - \frac{\Delta\beta}{\bar{\beta}}. \tag{90}$$

equating (89) and (90) leads to

$$\sin\frac{\Delta\theta_s}{2} \approx \frac{\Delta\beta}{2\bar{\beta}}\tan\bar{\theta}_s = \frac{1}{2}A_\beta \tan\bar{\theta}_s. \tag{91}$$

In a similar manner

$$\frac{\cos\theta_s}{\cos\theta'_s} = \frac{\cos\left(\theta_s - \frac{\Delta\theta_s}{2}\right)}{\cos\left(\theta_s + \frac{\Delta\theta_s}{2}\right)} \approx 1 + 2\tan\bar{\theta}_s \sin\frac{\Delta\theta_s}{2} = 1 + A_\beta \tan^2\bar{\theta}_s \tag{92}$$

Now the reflectivity function is written as

$$^{\downarrow}SHSH^{\uparrow} = \frac{\frac{\rho}{\rho'}\frac{\beta}{\beta'}\frac{\cos\theta_s}{\cos\theta'_s} - 1}{\frac{\rho}{\rho'}\frac{\beta}{\beta'}\frac{\cos\theta_s}{\cos\theta'_s} + 1} = \frac{(1-A_\rho)(1-A_\beta)(1+A_\beta\tan^2\bar{\theta}_s) - 1}{(1-A_\rho)(1-A_\beta)(1+A_\beta\tan^2\bar{\theta}_s) + 1} \tag{93}$$

expanding and keep the first order of the perturbations we arrive at

$$^{\downarrow}SHSH^{\uparrow} \approx -\frac{1}{2}(A_\rho + A_\beta) + \frac{1}{2}A_\beta\tan^2\bar{\theta}_s \tag{94}$$

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