Radiation patterns associated with the scattering from viscoelastic inclusions

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ABSTRACT

In this report we obtained the radiation patterns associated with the scattering of seismic waves from five viscoelastic inclusions; density, P- and S-wave velocities and quality factors for P- and S-waves. We show that the polarization and slowness of viscoelastic waves are complex. Basically the radiation patterns from elastic and anelastic inclusions are given by the scattering potentials which are the amplitude of the spherical scattered waves from scatter points. We show that the scattering potentials are complex functions of averages in phase and attenuation angles.

INTRODUCTION

Stolt and Weglein (2012) have introduced a formal theory for the description of the multidimensional scattering of seismic waves based on an isotropic-elastic model. We identify as a research priority the adaptation of this approach to incorporate other, more complete pictures of seismic wave propagation. Amongst these, the extension to include anelasticity and/or viscoelasticity , which brings to the wave model the capacity to transform elastic energy into heat, ranks very high. Anelasticity is generally held to be a key contributor to seismic attenuation, or "seismic Q", which has received several decades worth of careful attention in the literature (Aki and Richards, 2002; Futterman, 1967). Development of methods for analysis (Tonn, 1991), processing (Bickel and Natarajan, 1985; Hargreaves and Calvert, 1991; Wang, 2006; Zhang and Ulrych, 2007; Innanen and Lira, 2010), and inversion (Dahl and Ursin, 1992; Ribodetti and Virieux, 1998; Causse et al., 1999; Hicks and Pratt, 2001; Innanen and Weglein, 2007) of wave data exhibiting the attenuation and dispersion of seismic Q remains a very active research area.

Borcherdt (2009) has presented a complete theory for seismic waves propagating in layered anelastic media, assuming a viscoelastic model to hold. Borcherdt's formulation is particularly powerful in that it predicts a range of transverse, inhomogeneous wave types unique to viscoelastic media (Type I and II S waves), and develops rules for conversion of one type to another during interactions with planar boundaries.

Generalizing this approach to allow for viscoelastic waves of the type described by Borcherdt has several positive outcomes. First, and foremost, it provides an analytical framework for the examination of processes of scattering of viscoelastic waves from arbitrary three-dimensional heterogeneities, as opposed to layered media. Second, it provides a foundation for direct linear and nonlinear inversion methods for reflection seismic data, which go well beyond existing an-acoustic results (Innanen and Weglein, 2007; Innanen and Lira, 2010). And third, it provides a means to compute and analyze the gradient and Hessian quantities used in iterative seismic inversion (see the review by Virieux and Operto, 2009).

In this report we introduce the scattering potentials for of viscoelastic waves using the Born approximation based on the scattering theory. These scattering potentials represent the radiation patterns generated by elastic and anelastic inclusions in a viscoelastic background. The object of this report is the generalization of the viscoelastic scattering potentials obtained in low-loss medium to a medium with general attenuation properties.

EXACT FORM OF RAY PARAMETER AND SLOWNESS VECTORS

The most important feature of the waves in a viscoelastic medium is that the wavenumber vector is a complex number, which it's imaginary part refers to the amplitude damping. As a result, slowness and polarization vectors are complex numbers. The complex wavevector is given by

$$\mathbf{K} = \mathbf{P} + i\mathbf{A}, \tag{1}$$

where, propagation vector $\mathbf{P}$ is perpendicular to the wavefront and attenuation vector $\mathbf{A}$ is perpendicular to the plane of constant amplitudes and specified the direction of the maximum attenuation medium. The angle between these two vectors is called attenuation angle, δ, which is always less that $90°$. In the case that the attenuation and propagation vectors are parallel the wave is called homogenous. Otherwise it is inhomogeneous. Wave speed for a homogeneous P and S-wave may be written as

$$V_{\mathrm{H}} = V_E \sqrt{\frac{2\chi_{\mathrm{H}}^2}{1+\chi_{\mathrm{H}}}}, \tag{2}$$

where

$$\chi_H = \sqrt{1+Q^{-2}}, \tag{3}$$

with quality factor Q. Complex wave-number is defined as

$$K = \sqrt{\mathbf{K}\cdot\mathbf{K}} = \frac{\omega}{V}, \tag{4}$$

where complex velocity V and V are defined by

$$V = \frac{V_H}{1 - i\frac{Q^{-1}}{1+\chi_H}}, \tag{5}$$

In the case of low-loss viscoelastic media as $Q^{-1} \ll 1$, we have

$$V_H \approx V_E, \qquad V \approx V_E\left(1 + \frac{i}{2}Q_H^{-1}\right). \tag{6}$$

In above we used the low-loss viscoelastic medium approximation where $Q^{-1} \ll 1$. From study of complex vectors we know that they display the elliptical motion for a dynamic problem. Therefore we expect that displacement vectors for P- and S-waves with complex polarization vectors, describe an elliptical motion for particles. We assume that wavenumber vector is in the xz-plane, so the propagation and attenuation vectors are

$$\begin{aligned}\mathbf{P} &= \frac{\omega}{V_H}\sqrt{\frac{1+\chi}{1+\chi_H}}(\mathbf{x}\sin\theta + \mathbf{z}\cos\theta),\\ \mathbf{A} &= \frac{\omega}{V_H}\sqrt{\frac{-1+\chi}{1+\chi_H}}(\mathbf{x}\sin(\theta-\delta) + \mathbf{z}\cos(\theta-\delta)),\end{aligned} \tag{7}$$

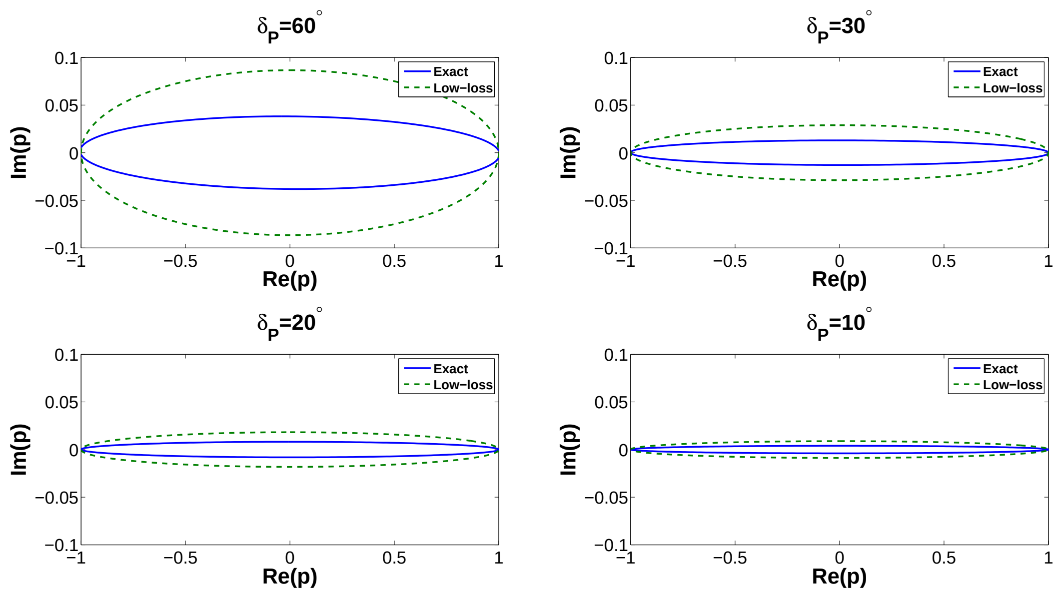

FIG. 1. Digram illustrating the ray parameter versus phase angle in complex plane for different values of attenuation angle δ. Solid line refers to the general attenuation and dash line refers to the low-loss attenuation.

with

$$\chi = \sqrt{1 + Q^{-2}\sec^2\delta}. \tag{8}$$

The slowness vector is given by

$$\mathbf{k} = \frac{\mathbf{K}}{\omega} = \mathrm{p}\mathbf{x} + \mathrm{q}\mathbf{z}, \tag{9}$$

where complex ray parameter and vertical slowness respectively are

$$\mathrm{p} = \frac{1}{\omega}(\mathrm{P}_x - i\mathrm{A}_x), \tag{10}$$

$$\mathrm{q} = \frac{1}{\omega}(\mathrm{P}_z - i\mathrm{A}_z). \tag{11}$$

Explicit forms of ray parameter and vertical slowness vectors are

$$\begin{aligned}
\mathrm{p} &= \frac{1}{V_H}\sqrt{\frac{1+\chi}{1+\chi_H}}\left\{\sin\theta\left(1 - i\frac{Q^{-1}}{1+\chi}\right) + i\frac{Q^{-1}}{1+\chi}\cos\theta\tan\delta\right\},\\
\mathrm{q} &= \frac{1}{V_H}\sqrt{\frac{1+\chi}{1+\chi_H}}\left\{\cos\theta\left(1 - i\frac{Q^{-1}}{1+\chi}\right) - i\frac{Q^{-1}}{1+\chi}\sin\theta\tan\delta\right\}.
\end{aligned} \tag{12}$$

In Fig. 1, we plot the ray parameter in complex plane versus phase angle for various values of attenuation angle. Diagram displays that the ray parameter is a an ellipse whose eccentricity decline as attenuation angle gets smaller. We also observe a higher deviation between general viscoelastic medium and low-loss attenuation medium for higher values of attenuation angle. The same interpretation is valid for polarization and slowness vectors. It is easy to show that ray parameter and vertical slowness vectors satisfy in

$$\mathrm{p}^2 + \mathrm{q}^2 = \frac{1}{V^2}. \tag{13}$$

To calculate the scattering potential we need the explicit form of the polarization and slowness vectors. Based on the complex wave number vector for viscoelastic waves, the polarization vectors for P- and S-waves are given by

$$\begin{aligned} \boldsymbol{\xi}_P &= V_P \mathbf{K}_P = V_P(\mathbf{P}_P - i\mathbf{A}_P), \\ \boldsymbol{\zeta}_P &= V_S \mathbf{K}_P \times \mathbf{y} = V_S(\mathbf{P}_S - i\mathbf{A}_S) \times \mathbf{y}, \end{aligned} \tag{14}$$

also the slowness vectors are defined as

$$\mathbf{k}_P = \frac{\mathbf{K}_P}{\omega}, \qquad \mathbf{k}_S = \frac{\mathbf{K}_S}{\omega}. \tag{15}$$

Consider to the case that polarization vectors for incident and reflected waves are in x-z plane

$$\begin{aligned} \mathbf{P}_{Pi} &= \frac{\omega}{V_{HP}}\sqrt{\frac{1+\chi_{Pi}}{1+\chi_{HP}}}(\mathbf{z}\cos\theta_{Pi} + \mathbf{x}\sin\theta_{Pi}), \\ \mathbf{A}_{Pi} &= \frac{\omega}{V_{HP}}\sqrt{\frac{-1+\chi_{Pi}}{1+\chi_{HP}}}(\mathbf{z}\cos(\theta_{Pi} - \delta_{Pi}) + \mathbf{x}\sin(\theta_{Pi} - \delta_{Pi})), \\ \mathbf{P}_{Pr} &= \frac{\omega}{V_{HP}}\sqrt{\frac{1+\chi_{Pr}}{1+\chi_{HP}}}(\mathbf{x}\sin\theta_{Pr} - \mathbf{z}\cos\theta_{Pr}), \\ \mathbf{A}_{Pr} &= \frac{\omega}{V_{HP}}\sqrt{\frac{-1+\chi_{Pr}}{1+\chi_{HP}}}(\mathbf{x}\sin(\theta_{Pr} - \delta_{Pr}) - \mathbf{z}\cos(\theta_{Pr} - \delta_{Pr})), \end{aligned} \tag{16}$$

whereδ_{P_i} and δ_{P_r} are the attenuation angles for incident and reflected P-waves. Homogeneous P-wave velocity, V_{PH} is the same for incident and reflected waves as they propagate in the reference medium.

SCATTERING POTENTIAL AND BORN APPROXIMATION

The elements of scattering theory can be found in many standard text book. The main idea in this theory is that, the actual medium that wave propagates in it can be considered as a reference medium with known physical parameters plus perturbations in properties which are unknown. The perturbations are proportional to the differences in elastic and anelastic properties between the reference and actual medium. Scattered wave can be regarded as a summation of the all possible single scattering from, one, two ....scatter points. Since the scattered wave from the first scatter point is weak comparing to the incident wave, practically scattering form two, three and more scatter points is negligible. This is called the Born approximation. The scattering potential in the first order Born approximation is given by (Beylkin and Burridge, 1990)

$$\mathbb{V} = \boldsymbol{\Psi}_0^I \cdot \mathbf{V} \cdot \boldsymbol{\Psi}_0^R. \tag{17}$$

In above equation $\mathbf{\Psi}_0$, is the Green's function or propagator in reference medium given by

$$\mathbf{\Psi}_0 \approx \omega^{-1}\boldsymbol{\xi} e^{i\omega(\mathbf{k}\cdot\mathbf{r})}, \tag{18}$$

where $\boldsymbol{\xi}$ is the polarization vector and $\mathbf{k}$ is the slowness vector. Scattering operator $\mathbf{V}$ is a kind of operator that contains the left and right partial derivatives respect to the space coordinate. It is a scattering tensor, which is the difference between the wave operator in actual and reference medium given as (Stolt and Weglein, 2012)

$$\mathbf{V} = \mathrm{L}_{\mathrm{VE}} - \mathrm{L}_{\mathrm{VE0}}, \tag{19}$$

where L, is the wave operator given by (Moradi and Innanen, 2015)

$$(L_{\mathrm{VE}})_{ij} = \rho\omega^2\delta_{ij} + \overleftarrow{\partial}_i(\rho V_P^2)\overrightarrow{\partial}_j + \delta_{ij}\overleftarrow{\partial}_k(\rho V_S^2)\overrightarrow{\partial}_k - 2\overleftarrow{\partial}_i(\rho V_S^2)\overrightarrow{\partial}_j + \overleftarrow{\partial}_j(\rho V_S^2)\overrightarrow{\partial}_i, \tag{20}$$

for $i, j, k = x, y, z$ with a sum rule notation on index k. The left(right) arrow on the partial derivatives indicates the operation on the left (right) hand side function. We define the fractional perturbations as a difference of the property in actual and reference media. For example the fractional perturbation in physical parameter x is given by

$$A_x = \frac{\Delta x}{x} = \frac{x - x_0}{x}, \tag{21}$$

where x_0 is the property in the reference medium. So that we can obtain the physical property in the actual medium in terms of it's value in reference medium and the fractional perturbation

$$x = x_0(1 + A_x). \tag{22}$$

Since the wave operator is complex in both actual and reference media, the scattering operator contains the perturbations in complex P- and S-wave velocities. We note that the perturbations are in the real quantities. Let us express the perturbation in complex number $z = x + iy$ in terms of perturbations in the real and imaginary parts. In this case we have

$$z = x_0(1 + A_x) + iy_0(1 + A_y), \tag{23}$$

So perturbation in complex number z can be expressed as

$$A_z = \frac{\Delta z}{z} = \frac{1}{x_0 + iy_0}(x_0 A_x + iy_0 A_y). \tag{24}$$

In the first attempt to calculate the scattering potential, we obtain the scattering operator. Fractional perturbations in density, P- and S-wave velocities and quality factors are easy to calculate. However the fractional perturbations in complex P- and S-wave velocities are more complicated. First consider to the first order approximation of factor χ_H

$$\chi_H \approx \chi_{H0}\left(1 - \frac{Q_0^{-2}}{\chi_{H0}^2}A_Q\right). \tag{25}$$

Inserting in the definitions of the velocities in the first order approximation we arrive at

$$A_V = 1 - \frac{V_0}{V} = A_{V_E} - \frac{i}{2}\mathbb{Q}^{-1}A_Q,$$

where we defined the analogous quality factor $\mathbb{Q}$ as

$$\mathbb{Q}^{-1} = \frac{2Q_0^{-1}}{1+\chi_{H0}} \left\{ \frac{V_0}{V_{H0}} \left(1 - \frac{Q_0^{-2}}{\chi_{H0}(1+\chi_{H0})} \right) - i \frac{Q_0^{-1}}{2\chi_{H0}^2} (2+\chi_{H0}) \right\}.$$

In the case of low-loss media $Q^{-1} \ll 1$, we have $\mathbb{Q} \rightarrow Q_0$. Next we write the scattering matrix element in frequency-independent form (Stolt and Weglein, 2012). Since the differential operators are sandwiched between unperturbed wave functions, we replace the left derivatives with i multiplied by the reflected wavenumber vector $\mathbf{K}_r$ and right derivative with i multiplied by the incident wavenumber vector $\mathbf{K}_i$. After replacing the left and right derivatives by the appropriate wavenumber vectors, the frequency independent parts of the scattering operator are

$$\begin{aligned}
(\mathbb{V}^{\rho}_{\mathrm{VE}})_{kl} &= \delta_{kl} - V_{\mathrm{P0}}^2 k_k^r k_l^i - V_{\mathrm{S0}}^2 \left(\delta_{kl} k_m^r k_m^i - 2k_k^r k_l^i + k_l^r k_k^i \right), \\
(\mathbb{V}^{V_{\mathrm{PE}}}_{\mathrm{VE}})_{kl} &= -2V_{\mathrm{P}}^2 k_k^r k_l^i, \\
(\mathbb{V}^{V_{\mathrm{SE}}}_{\mathrm{VE}})_{kl} &= -2V_{\mathrm{S0}}^2 \left(\delta_{kl} k_m^r k_m^i - 2k_k^r k_l^i + k_l^r k_k^i \right), \\
(\mathbb{V}^{Q_{\mathrm{P}}}_{\mathrm{VE}})_{kl} &= i\mathbb{Q}_P^{-1} V_{\mathrm{P0}}^2 k_k^r k_l^i, \\
(\mathbb{V}^{Q_{\mathrm{S}}}_{\mathrm{VE}})_{kl} &= i\mathbb{Q}_S^{-1} V_{\mathrm{S0}}^2 \left(\delta_{kl} k_m^r k_m^i - 2k_k^r k_l^i + k_l^r k_k^i \right).
\end{aligned} \tag{26}$$

The frequency-independent components of the scattering potential are defined as

$$\frac{V_{\mathrm{VE}}}{\rho_0 \omega^2} = \mathbb{V}_{\mathrm{VE}} = \mathbb{V}^{\rho}_{\mathrm{VE}} A_\rho + \mathbb{V}^{V_{\mathrm{PE}}}_{\mathrm{VE}} A_{V_{\mathrm{PE}}} + \mathbb{V}^{V_{\mathrm{SE}}}_{\mathrm{VE}} A_{V_{\mathrm{SE}}} + \mathbb{V}^{Q_{\mathrm{P}}}_{\mathrm{VE}} A_{Q_{\mathrm{P}}} + \mathbb{V}^{Q_{\mathrm{S}}}_{\mathrm{VE}} A_{Q_{\mathrm{S}}}. \tag{27}$$

Now to obtain the scattering matrix we sandwich the above expressions with the proper polarization vectors. We use the vectors $\mathbf{R}$ and $\mathbf{I}$ to indicate the reflected and incident polarization vectors, respectively. For perturbation terms we will write

$$\begin{aligned}
{}^{\mathrm{I}}_{\mathrm{R}}\mathbb{V}^{\rho}_{\mathrm{VE}} &= {}^{\mathrm{I}}_{\mathrm{R}}\mathcal{F} - {}^{\mathrm{I}}_{\mathrm{R}}\mathcal{G}_\alpha - {}^{\mathrm{I}}_{\mathrm{R}}\mathcal{G}_\beta, \\
{}^{\mathrm{I}}_{\mathrm{R}}\mathbb{V}^{Q_{\mathrm{P}}}_{\mathrm{VE}} &= -\frac{i}{2}\mathbb{Q}_P^{-1} \left\{ {}^{\mathrm{I}}_{\mathrm{R}}\mathbb{V}^{V_{\mathrm{PE}}}_{\mathrm{VE}} \right\} = \left(i\mathbb{Q}_P^{-1} \right) {}^{\mathrm{I}}_{\mathrm{R}}\mathcal{G}_{V_P}, \\
{}^{\mathrm{I}}_{\mathrm{R}}\mathbb{V}^{Q_{\mathrm{S}}}_{\mathrm{VE}} &= -\frac{i}{2}\mathbb{Q}_S^{-1} \left\{ {}^{\mathrm{I}}_{\mathrm{R}}\mathbb{V}^{\beta_{\mathrm{E}}}_{\mathrm{VE}} \right\} = \left(i\mathbb{Q}_S^{-1} \right) {}^{\mathrm{I}}_{\mathrm{R}}\mathcal{G}_\beta,
\end{aligned} \tag{28}$$

where we have defined

$$\begin{aligned}
{}^{\mathrm{I}}_{\mathrm{R}}\mathcal{F} &= \mathbf{R} \cdot \mathbf{I}, \\
{}^{\mathrm{I}}_{\mathrm{R}}\mathcal{G}_\alpha &= V_{\mathrm{P0}}^2 (\mathbf{R} \cdot \mathbf{k}_r)(\mathbf{I} \cdot \mathbf{k}_i), \\
{}^{\mathrm{I}}_{\mathrm{R}}\mathcal{G}_\beta &= V_{\mathrm{S0}}^2 \left\{ (\mathbf{R} \cdot \mathbf{I})(\mathbf{k}_r \cdot \mathbf{k}_i) - 2(\mathbf{R} \cdot \mathbf{k}_r)(\mathbf{I} \cdot \mathbf{k}_i) + (\mathbf{I} \cdot \mathbf{k}_r)(\mathbf{R} \cdot \mathbf{k}_i) \right\}.
\end{aligned} \tag{29}$$

To determine the explicit form of each component of the scattering potential, we need to calculate $\mathcal{F}_{\mathrm{I}}^{\mathrm{R}}$, ${}^{\mathrm{I}}_{\mathrm{R}}\mathcal{G}_\alpha$ and ${}^{\mathrm{I}}_{\mathrm{R}}\mathcal{G}_\beta$. To derive the scattering potential we use the polarizations and slowness vectors labeled by subscripts refers to the incident and scattered waves. The scattering potential should be expressed as a function of the parameters in reference medium and perturbations. Without loss of generality we assume that the change in the attenuation angle is very small $\Delta\delta_P = \delta_{Pr} - \delta_{Pi} \ll 1$, so we can write the fractional perturbation in attenuation angle as

$$A_{\delta_P} = 2\frac{\delta_{Pr} - \delta_{Pi}}{\delta_{Pi} + \delta_{Pr}}, \tag{30}$$

so that

$$\delta_{Pr} = \delta_P + \frac{\Delta\delta_P}{2},$$
$$\delta_{Pi} = \delta_P - \frac{\Delta\delta_P}{2},$$

where δ is the average in attenuation angle or reference medium attenuation angle. As a result

$$\sec\delta_{Pi} = \sec\delta_P\left(1 - \tan\delta_P \sin\frac{\Delta\delta_P}{2}\right),$$
$$\sec\delta_{Pr} = \sec\delta_P\left(1 + \tan\delta_P \sin\frac{\Delta\delta_P}{2}\right). \tag{31}$$

Since the above expressions multiplied by a fractional perturbation, in the first order in perturbation we have

$$\sqrt{1+\chi_{Pr}} \approx \sqrt{1+\chi_{Pi}} \approx \sqrt{1+\chi_P}.$$

In a similar manner

$$\tan\delta_{Pr} \approx \tan\delta_{Pi} \approx \tan\delta_P,$$

P-to-P scattering potential

Let us consider to the case that the inhomogeneous P-wave with attenuation angle δ_{Pi} scattered to an inhomogeneous P-wave with attenuation angle δ_{Pr}. In this case equations (29) reduce to

$$\begin{aligned}
{}^{\mathrm{P}}_{\mathrm{P}}\mathcal{F} &= \boldsymbol{\xi}_{\mathrm{P}r} \cdot \boldsymbol{\xi}_{\mathrm{P}i},\\
{}^{\mathrm{P}}_{\mathrm{P}}\mathcal{G}_{V_P} &= V_{P0}^2(\boldsymbol{\xi}_{\mathrm{P}r} \cdot \mathbf{k}_{\mathrm{P}r})(\boldsymbol{\xi}_{\mathrm{P}i} \cdot \mathbf{k}_{\mathrm{P}i}),\\
{}^{\mathrm{P}}_{\mathrm{P}}\mathcal{G}_{\beta} &= V_{S0}^2\left\{(\boldsymbol{\xi}_{\mathrm{P}r} \cdot \boldsymbol{\xi}_{\mathrm{P}i})(\mathbf{k}_{\mathrm{P}r} \cdot \mathbf{k}_{\mathrm{P}i}) - 2(\boldsymbol{\xi}_{\mathrm{P}r} \cdot \mathbf{k}_{\mathrm{P}r})(\boldsymbol{\xi}_{\mathrm{P}i} \cdot \mathbf{k}_{\mathrm{P}i}) + (\boldsymbol{\xi}_{\mathrm{P}i} \cdot \mathbf{k}_{\mathrm{P}r})(\boldsymbol{\xi}_{\mathrm{P}r} \cdot \mathbf{k}_{\mathrm{P}i})\right\}.
\end{aligned} \tag{32}$$

The detail of the dot product of various polarization and slowness vectors can be find in appendix A. The above expressions are the functions of the average phase and attenuation angles for incident and reflected P-waves. For low-loss media the analytical expressions can be found in (Moradi and Innanen, 2015). Below is the interpretation of the each component of the scattering potential for scattering of P-wave to P-wave

- ${}^{\mathrm{P}}_{\mathrm{P}}\mathbb{V}^{\rho}_{\mathrm{VE}} \neq 0$: component related to the scattering from density inclusion
- ${}^{\mathrm{P}}_{\mathrm{P}}\mathbb{V}^{V_P}_{\mathrm{VE}} \neq 0$ component related to the scattering from P-wave velocity inclusion
- ${}^{\mathrm{P}}_{\mathrm{P}}\mathbb{V}^{V_S}_{\mathrm{VE}} \neq 0$ component related to the scattering from S-wave velocity inclusion

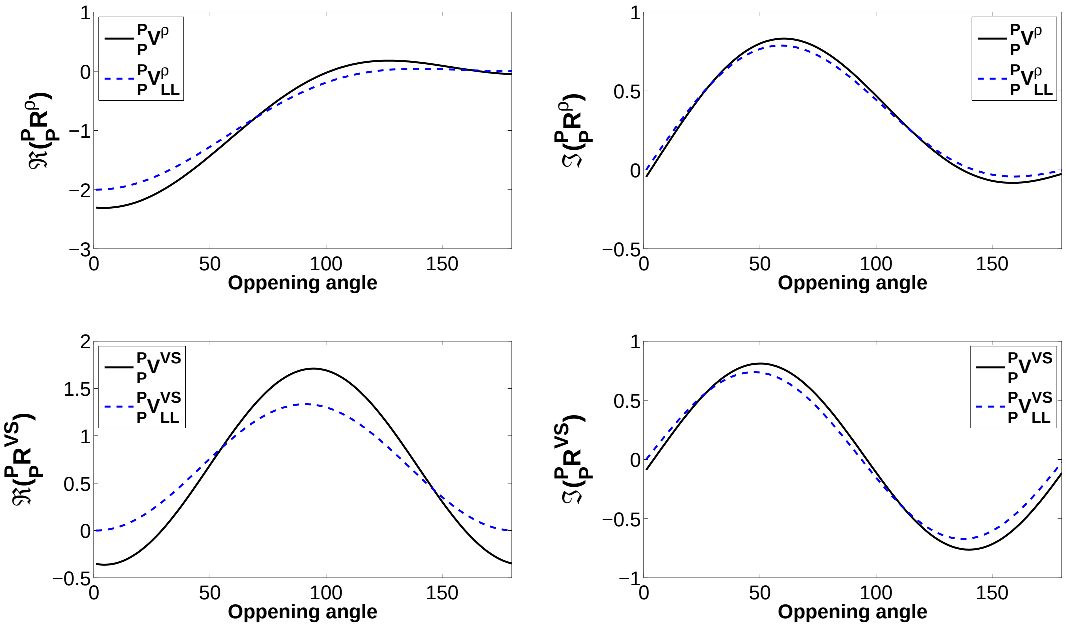

FIG. 2. Digram illustrating the real and imaginary of density and S-wave velocity part of P-to-P scattering potential for $Q_P = 5$ and $Q_S = 4$ and attenuation angles $\delta_P = \delta_S = 70°$. The solid line refers to the arbitrary attenuation and dash line refers to the low-loss case.

- ${}^{\mathrm{P}}_{\mathrm{P}}\mathbb{V}^{Q_P}_{\mathrm{VE}} \neq 0$ component related to the scattering from P-wave quality factor inclusion
- ${}^{\mathrm{P}}_{\mathrm{P}}\mathbb{V}^{Q_S}_{\mathrm{VE}} \neq 0$ component related to the scattering from S-wave quality factor inclusion

Consequently, any inclusion in elastic and anelastic properties can scatter the P-wave to P-wave. The first three components of the scttering potentials related to the density, P and S-wave velocity are complex functions with real and imaginary parts. In low-loss media the real part is equal to the elastic scattering potential, however for a general viscoelastic medium there is an extra term that is negligible in the low-loss limit. The components related to the inclusions in quality factors are pure imaginary terms. In Fig 2 we plot the real and imaginary parts of the P-to-P scattering potential for low-loss versus general viscoelastic medium. We can see the differences between the scattering potentials for medium with general anelastic properties and a medium with low-loss anelastic properties.

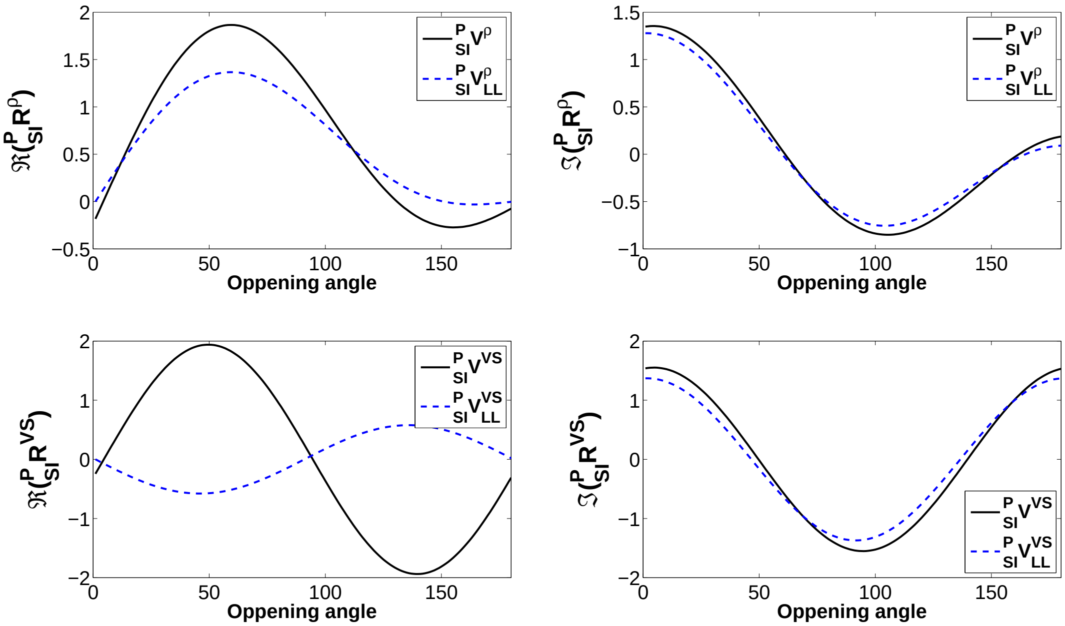

FIG. 3. Digram illustrating the real and imaginary of density and S-wave velocity part of P-to-SI scattering potential for $Q_P = 5$ and $Q_S = 4$ and attenuation angles $\delta_P = \delta_S = 70°$. The solid line refers to the arbitrary attenuation and dash line refers to the low-loss case.

P-SI scattering potential

In this case the reflected wave is of type SI, $\mathbf{R} = \boldsymbol{\zeta}_{\mathrm{S}r}$, and the incident wave is a P-wave, $\mathbf{I} = \boldsymbol{\xi}_{\mathrm{P}i}$. In this case equations (29) reduce to

$$\begin{aligned}
{}^{\mathrm{P}}_{\mathrm{SI}}\mathcal{F} &= \boldsymbol{\zeta}_{\mathrm{S}r} \cdot \boldsymbol{\xi}_{\mathrm{P}i} \\
{}^{\mathrm{P}}_{\mathrm{SI}}\mathcal{G}_{V_P} &= V_{P0}^2(\boldsymbol{\zeta}_{\mathrm{S}r} \cdot \mathbf{k}_{\mathrm{S}r})(\boldsymbol{\xi}_{\mathrm{P}i} \cdot \mathbf{k}_{\mathrm{P}i}) \\
{}^{\mathrm{P}}_{\mathrm{SI}}\mathcal{G}_{V_S} &= V_{S0}^2 \left\{(\boldsymbol{\zeta}_{\mathrm{S}r} \cdot \boldsymbol{\xi}_{\mathrm{P}i})(\mathbf{k}_{\mathrm{S}r} \cdot \mathbf{k}_{\mathrm{P}i}) - 2(\boldsymbol{\zeta}_{\mathrm{S}r} \cdot \mathbf{k}_{\mathrm{S}r})(\boldsymbol{\xi}_{\mathrm{P}i} \cdot \mathbf{k}_{\mathrm{P}i}) + (\boldsymbol{\xi}_{\mathrm{P}i} \cdot \mathbf{k}_{\mathrm{S}r})(\boldsymbol{\zeta}_{\mathrm{S}r} \cdot \mathbf{k}_{\mathrm{P}i})\right\} .
\end{aligned} \tag{33}$$

The detail of the the calculations can be found in the appendix. In this case the scattering potential component related to the change in P-wave velocity is zero. It means that inclusion in P-wave velocity can not convert the P-wave to SI-wave. The similar situation is satisfied for the scattering of elastic case, where the P-wave velocity inclusion can not convert the elastic P-wave to the SV-wave. Consequently, the P-wave quality factor component vanishes also means that P-wave can not be converted to SI-wave due to interaction with the P-wave quality factor inclusion. In Fig 3 we plot the real and imaginary parts of the P-to-SI scattering potential for low-loss versus general viscoelastic medium.

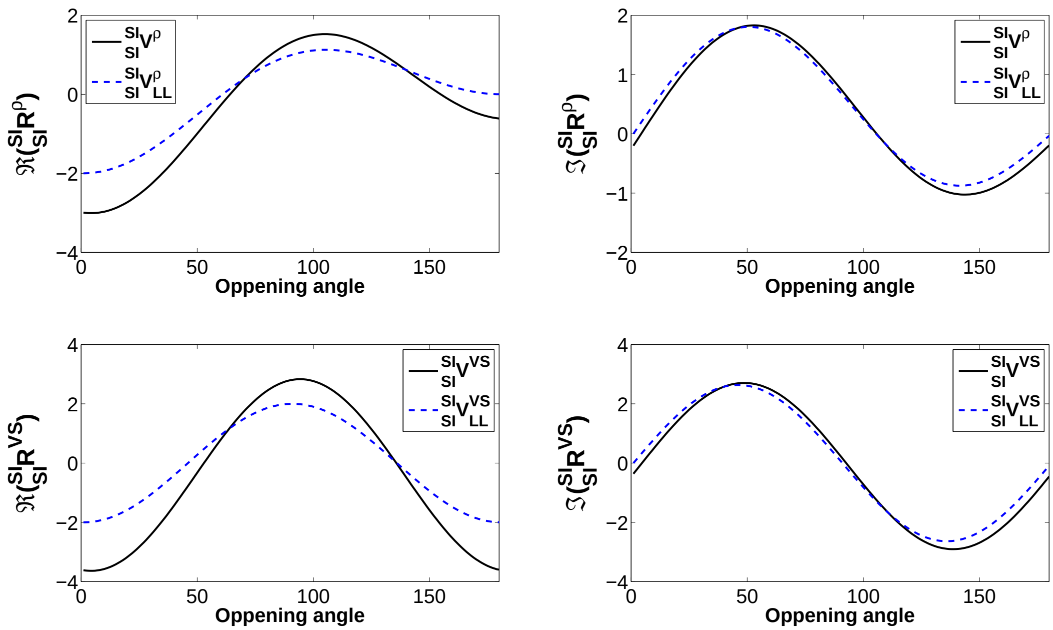

FIG. 4. Digram illustrating the real and imaginary of density and S-wave velocity part of SI-to-SI scattering potential for $Q_P = 5$ and $Q_S = 4$ and attenuation angles $\delta_P = \delta_S = 70°$. The solid line refers to the arbitrary attenuation and dash line refers to the low-loss case.

SI-SI scattering potential

In this case the reflected wave is of type SI, $\mathbf{R} = \boldsymbol{\zeta}_{\mathrm{S}r}$, and the incident wave is a SI-wave, $\mathbf{I} = \boldsymbol{\zeta}_{\mathrm{S}i}$. In this case equations (29) reduce to

$$\begin{aligned} {}^{\mathrm{SI}}_{\mathrm{SI}}\mathcal{F} &= \boldsymbol{\zeta}_{\mathrm{S}r} \cdot \boldsymbol{\zeta}_{\mathrm{S}i}, \\ {}^{\mathrm{SI}}_{\mathrm{SI}}\mathcal{G}_{V_S} &= V_{S0}^2 \left\{ (\boldsymbol{\zeta}_{\mathrm{S}r} \cdot \boldsymbol{\zeta}_{\mathrm{S}i})(\mathbf{k}_{\mathrm{S}r} \cdot \mathbf{k}_{\mathrm{S}i}) + (\boldsymbol{\zeta}_{\mathrm{S}i} \cdot \mathbf{k}_{\mathrm{S}r})(\boldsymbol{\zeta}_{\mathrm{S}r} \cdot \mathbf{k}_{\mathrm{S}i}) \right\}. \end{aligned} \tag{34}$$

Similar to the scattering potential for P-to-SI, the non zero components are, density, S-wave velocity and S-wave quality factor. No contributions from change in P-wave velocity and quality factor in scattering of SI-wave to SI-wave. In Fig 4 we plot the real and imaginary parts of the SI-to-SI scattering potential for low-loss versus general viscoelastic medium.

SUMMARY AND CONCLUSION

The scattering potential associated with the scattering in low-loss viscoelastic medium recently have been derived (Moradi and Innanen, 2015). In this paper we removed the low-loss assumption and derived the explicit forms of the scattering potential for a general viscoelastic medium. We used the Born approximation to obtain the scattering potentials.

In scattering theory approach, a low contrast medium can be simulated by couples of scatter points in a background medium. Mathematically scatter points are the perturbations that are added to an unperturbed medium to construct an actual perturbed medium. Compared to the low-contrast model, when waves travel through the medium including the scatter points, small portions of the wave interact with the scatter points. In this case only a small portion of the incidence wave is scattered from the scattering points and the majority of the incidence wave passes near through the scatter points without interacting with them. This non-interacting wave can be regarded as a transmitted wave compared to the case of a low-contrast medium. Scattered waves can themselves hit the scatter points again, but the resulted doubly-scattered wave would be very weak compared to the singly-scattered wave. Practically we can ignore all except the first order scattered waves. This is called the Born-approximation which deals only with the first order scattering.

In vescoelastic medium there are five parameters, density, P- and S-wave velocities and corresponding quality factors. To derive the scattering potential, first we calculate the scattering operator, which is the difference between the wave operator in actual and reference mediums. Second, the scattering operator is sandwiched between the polarization vectors. Polarization are complex vectors which displays a elliptical motions for P- and SI-waves. The scattering operator that we obtained is a complex function as a sum of perturbations in elastic and anelastic properties weighted by the the opening angle between the incident and reflected waves. In contrast to the exact form of the reflection coefficients, scattering potential is not a function of the incident reflected angles but it is function of the background material properties and opening angle.

ACKNOWLEDGMENTS

The authors thank the sponsors of CREWES for continued support. This work was funded by CREWES industrial sponsors and NSERC (Natural Science and Engineering Research Council of Canada) through the grant CRDPJ 461179-13.

APPENDIX A: COMPLEX POLARIZATION-SLOWNESS VECTORS ALGEBRA

Propagation and attenuation vectors for incident P-wave are

$$\begin{aligned}
\mathbf{P}_{\mathrm{Pi}} &= \frac{\omega}{V_{\mathrm{HP}}}\sqrt{\frac{1+\chi_{\mathrm{Pi}}}{1+\chi_{\mathrm{HP}}}}(\mathbf{z}\cos\theta_{\mathrm{Pi}} + \mathbf{x}\sin\theta_{\mathrm{Pi}}), \\
\mathbf{A}_{\mathrm{Pi}} &= \frac{\omega}{V_{\mathrm{HP}}}\sqrt{\frac{-1+\chi_{\mathrm{Pi}}}{1+\chi_{\mathrm{HP}}}}(\mathbf{z}\cos(\theta_{\mathrm{Pi}}-\delta_{\mathrm{Pi}}) + \mathbf{x}\sin(\theta_{\mathrm{Pi}}-\delta_{\mathrm{Pi}})),
\end{aligned} \tag{35}$$

reflected P-wave

$$\begin{aligned}\mathbf{P}_{\mathrm{Pr}} &= \frac{\omega}{V_{\mathrm{HP}}}\sqrt{\frac{1+\chi_{\mathrm{Pr}}}{1+\chi_{\mathrm{HP}}}}(\mathbf{x}\sin\theta_{\mathrm{Pr}} - \mathbf{z}\cos\theta_{\mathrm{Pr}}),\\ \mathbf{A}_{\mathrm{Pr}} &= \frac{\omega}{V_{\mathrm{HP}}}\sqrt{\frac{-1+\chi_{\mathrm{Pr}}}{1+\chi_{\mathrm{HP}}}}(\mathbf{x}\sin(\theta_{\mathrm{Pr}}-\delta_{\mathrm{Pr}}) - \mathbf{z}\cos(\theta_{\mathrm{Pr}}-\delta_{\mathrm{Pr}})),\end{aligned} \tag{36}$$

incident S-wave

$$\begin{aligned}\mathbf{P}_{\mathrm{Si}} &= \frac{\omega}{V_{\mathrm{HS}}}\sqrt{\frac{1+\chi_{\mathrm{Si}}}{1+\chi_{\mathrm{HS}}}}(\mathbf{z}\cos\theta_{\mathrm{Si}} + \mathbf{x}\sin\theta_{\mathrm{Si}}),\\ \mathbf{A}_{\mathrm{Si}} &= \frac{\omega}{V_{\mathrm{HS}}}\sqrt{\frac{-1+\chi_{\mathrm{Si}}}{1+\chi_{\mathrm{HS}}}}(\mathbf{z}\cos(\theta_{\mathrm{Si}}-\delta_{\mathrm{Si}}) + \mathbf{x}\sin(\theta_{\mathrm{Si}}-\delta_{\mathrm{Si}})),\end{aligned} \tag{37}$$

reflected S-wave

$$\begin{aligned}\mathbf{P}_{\mathrm{Sr}} &= \frac{\omega}{V_{\mathrm{HS}}}\sqrt{\frac{1+\chi_{\mathrm{Sr}}}{1+\chi_{\mathrm{HS}}}}(\mathbf{x}\sin\theta_{\mathrm{Sr}} - \mathbf{z}\cos\theta_{\mathrm{Sr}}),\\ \mathbf{A}_{\mathrm{Sr}} &= \frac{\omega}{V_{\mathrm{HS}}}\sqrt{\frac{-1+\chi_{\mathrm{Sr}}}{1+\chi_{\mathrm{HS}}}}(\mathbf{x}\sin(\theta_{\mathrm{Sr}}-\delta_{\mathrm{Sr}}) - \mathbf{z}\cos(\theta_{\mathrm{Sr}}-\delta_{\mathrm{Sr}})).\end{aligned} \tag{38}$$

Inner product of incident and reflected propagation and attenuation vectors for P-waves

$$\begin{aligned}\mathbf{P}_{\mathrm{Pi}}\cdot\mathbf{P}_{\mathrm{Pr}} &= -\frac{\omega^2}{V_{\mathrm{HP}}^2}\frac{\sqrt{(1+\chi_{\mathrm{Pi}})(1+\chi_{\mathrm{Pr}})}}{1+\chi_{\mathrm{HP}}}\cos(\theta_{\mathrm{Pi}}+\theta_{\mathrm{Pr}}),\\ \mathbf{P}_{\mathrm{Pi}}\cdot\mathbf{A}_{\mathrm{Pr}} &= -\frac{\omega^2}{V_{\mathrm{HP}}^2}\frac{\sqrt{(1+\chi_{\mathrm{Pi}})(-1+\chi_{\mathrm{Pr}})}}{1+\chi_{\mathrm{HP}}}\cos(\theta_{\mathrm{Pi}}+\theta_{\mathrm{Pr}}-\delta_{\mathrm{Pr}}),\\ \mathbf{P}_{\mathrm{Pr}}\cdot\mathbf{A}_{\mathrm{Pi}} &= -\frac{\omega^2}{V_{\mathrm{HP}}^2}\frac{\sqrt{(1+\chi_{\mathrm{Pr}})(-1+\chi_{\mathrm{Pi}})}}{1+\chi_{\mathrm{HP}}}\cos(\theta_{\mathrm{Pi}}+\theta_{\mathrm{Pr}}-\delta_{\mathrm{Pi}}).\\ \mathbf{A}_{\mathrm{Pi}}\cdot\mathbf{A}_{\mathrm{Pr}} &= -\frac{\omega^2}{V_{\mathrm{HP}}^2}\frac{\sqrt{(-1+\chi_{\mathrm{Pi}})(-1+\chi_{\mathrm{Pr}})}}{1+\chi_{\mathrm{HP}}}\cos(\theta_{\mathrm{Pi}}+\theta_{\mathrm{Pr}}-(\delta_{\mathrm{Pi}}+\delta_{\mathrm{Pr}})).\end{aligned} \tag{39}$$

Inner product of incident and reflected propagation and attenuation vectors for P-waves and S-waves

$$\begin{aligned}\mathbf{P}_{\mathrm{Pi}}\cdot\mathbf{P}_{\mathrm{Sr}} &= -\frac{\omega^2}{V_{\mathrm{HP}}V_{\mathrm{HS}}}\sqrt{\frac{(1+\chi_{\mathrm{Pi}})(1+\chi_{\mathrm{Sr}})}{(1+\chi_{\mathrm{HP}})(1+\chi_{\mathrm{HS}})}}\cos(\theta_{\mathrm{Pi}}+\theta_{\mathrm{Sr}}),\\ \mathbf{P}_{\mathrm{Pi}}\cdot\mathbf{A}_{\mathrm{Sr}} &= -\frac{\omega^2}{V_{\mathrm{HP}}V_{\mathrm{HS}}}\sqrt{\frac{(1+\chi_{\mathrm{Pi}})(-1+\chi_{\mathrm{Sr}})}{(1+\chi_{\mathrm{HP}})(1+\chi_{\mathrm{HS}})}}\cos(\theta_{\mathrm{Pi}}+\theta_{\mathrm{Sr}}-\delta_{\mathrm{Sr}}),\\ \mathbf{P}_{\mathrm{Pr}}\cdot\mathbf{A}_{\mathrm{Si}} &= -\frac{\omega^2}{V_{\mathrm{HP}}V_{\mathrm{HS}}}\sqrt{\frac{(1+\chi_{\mathrm{Pr}})(-1+\chi_{\mathrm{Si}})}{(1+\chi_{\mathrm{HP}})(1+\chi_{\mathrm{HS}})}}\cos(\theta_{\mathrm{Pr}}+\theta_{\mathrm{Si}}-\delta_{\mathrm{Si}}),\\ \mathbf{A}_{\mathrm{Pi}}\cdot\mathbf{A}_{\mathrm{Sr}} &= -\frac{\omega^2}{V_{\mathrm{HP}}V_{\mathrm{HS}}}\sqrt{\frac{(-1+\chi_{\mathrm{Pi}})(-1+\chi_{\mathrm{Sr}})}{(1+\chi_{\mathrm{HP}})(1+\chi_{\mathrm{HS}})}}\cos(\theta_{\mathrm{Pi}}+\theta_{\mathrm{Sr}}-(\delta_{\mathrm{Pi}}+\delta_{\mathrm{Sr}})),\\ \mathbf{A}_{\mathrm{Si}}\cdot\mathbf{A}_{\mathrm{Pr}} &= -\frac{\omega^2}{V_{\mathrm{HP}}V_{\mathrm{HS}}}\sqrt{\frac{(-1+\chi_{\mathrm{Si}})(-1+\chi_{\mathrm{Pr}})}{(1+\chi_{\mathrm{HP}})(1+\chi_{\mathrm{HS}})}}\cos(\theta_{\mathrm{Si}}+\theta_{\mathrm{Pr}}-(\delta_{\mathrm{Si}}+\delta_{\mathrm{Pr}})).\end{aligned} \tag{40}$$

Using the above expressions we obtain the inner product of wave number vector for incident and reflected P-waves

$$\begin{aligned}
\mathbf{K}_{\mathrm{Pi}}\cdot\mathbf{K}_{\mathrm{Pr}} &= \mathbf{P}_{\mathrm{Pi}}\cdot\mathbf{P}_{\mathrm{Pr}} - \mathbf{A}_{\mathrm{Pi}}\cdot\mathbf{A}_{\mathrm{Pr}} - i\mathbf{P}_{\mathrm{Pi}}\cdot\mathbf{A}_{\mathrm{Pr}} - i\mathbf{A}_{\mathrm{Pi}}\cdot\mathbf{P}_{\mathrm{Pr}} \\
&= -\frac{\omega^2}{V_{\mathrm{HP}}^2}\frac{\sqrt{(1+\chi_{\mathrm{Pi}})(1+\chi_{\mathrm{Pr}})}}{1+\chi_{\mathrm{HP}}}\times \\
&\Bigg[\cos(\theta_{\mathrm{Pi}}+\theta_{\mathrm{Pr}}) - \frac{Q_{\mathrm{P}}^{-2}\sec\delta_{\mathrm{Pi}}\sec\delta_{\mathrm{Pr}}}{(1+\chi_{\mathrm{Pi}})(1+\chi_{\mathrm{Pr}})}\cos(\theta_{\mathrm{Pi}}+\theta_{\mathrm{Pr}}-(\delta_{\mathrm{Pi}}+\delta_{\mathrm{Pr}})) \\
&-i\frac{Q_{\mathrm{P}}^{-1}\sec\delta_{\mathrm{Pr}}}{1+\chi_{\mathrm{Pr}}}\cos(\theta_{\mathrm{Pi}}+\theta_{\mathrm{Pr}}-\delta_{\mathrm{Pr}}) - i\frac{Q_{\mathrm{P}}^{-1}\sec\delta_{\mathrm{Pi}}}{1+\chi_{\mathrm{Pi}}}\cos(\theta_{\mathrm{Pi}}+\theta_{\mathrm{Pr}}-\delta_{\mathrm{Pi}})\Bigg] \\
&= -\frac{\omega^2}{V_{\mathrm{HP}}^2}\frac{\sqrt{(1+\chi_{\mathrm{Pi}})(1+\chi_{\mathrm{Pr}})}}{1+\chi_{\mathrm{HP}}}\times \\
&\Bigg[\cos\sigma_{\mathrm{PP}} - \frac{Q_{\mathrm{P}}^{-2}\sec\delta_{\mathrm{Pi}}\sec\delta_{\mathrm{Pr}}}{(1+\chi_{\mathrm{Pi}})(1+\chi_{\mathrm{Pr}})}\cos(\sigma_{\mathrm{PP}}-(\delta_{\mathrm{Pi}}+\delta_{\mathrm{Pr}})) \\
&-i\frac{Q_{\mathrm{P}}^{-1}\sec\delta_{\mathrm{Pr}}}{1+\chi_{\mathrm{Pr}}}\cos(\sigma_{\mathrm{PP}}-\delta_{\mathrm{Pr}}) - i\frac{Q_{\mathrm{P}}^{-1}\sec\delta_{\mathrm{Pi}}}{1+\chi_{\mathrm{Pi}}}\cos(\sigma_{\mathrm{PP}}-\delta_{\mathrm{Pi}})\Bigg] \\
&= -\frac{\omega^2}{V_{\mathrm{HP}}^2}\frac{\sqrt{(1+\chi_{\mathrm{Pi}})(1+\chi_{\mathrm{Pr}})}}{1+\chi_{\mathrm{HP}}}\times \\
&\Bigg[\cos\sigma_{\mathrm{PP}} - \frac{Q_{\mathrm{P}}^{-2}}{(1+\chi_{\mathrm{Pi}})(1+\chi_{\mathrm{Pr}})}[\cos\sigma_{\mathrm{PP}}(1-\tan\delta_{\mathrm{Pr}}\tan\delta_{\mathrm{Pi}}) + \sin\sigma_{\mathrm{PP}}(\tan\delta_{\mathrm{Pr}}+\tan\delta_{\mathrm{Pi}})] \\
&-i\frac{Q_{\mathrm{P}}^{-1}}{1+\chi_{\mathrm{Pr}}}[\cos\sigma_{\mathrm{PP}} + \sin\sigma_{\mathrm{PP}}\tan\delta_{\mathrm{Pr}}] - i\frac{Q_{\mathrm{P}}^{-1}}{1+\chi_{\mathrm{Pi}}}[\cos\sigma_{\mathrm{PP}} + \sin\sigma_{\mathrm{PP}}\tan\delta_{\mathrm{Pi}}]\Bigg].
\end{aligned} \tag{41}$$

Finally

$$\begin{aligned}
\mathbf{K}_{\mathrm{Pi}}\cdot\mathbf{K}_{\mathrm{Pr}} = -\frac{\omega^2}{V_{\mathrm{HP}}^2}&\frac{\sqrt{(1+\chi_{\mathrm{Pi}})(1+\chi_{\mathrm{Pr}})}}{1+\chi_{\mathrm{HP}}}\times \\
&\Bigg[\cos\sigma_{\mathrm{PP}}\left(1 - i\frac{Q_{\mathrm{P}}^{-1}}{1+\chi_{\mathrm{Pr}}} - i\frac{Q_{\mathrm{P}}^{-1}}{1+\chi_{\mathrm{Pi}}} - \frac{Q_{\mathrm{P}}^{-2}}{(1+\chi_{\mathrm{Pi}})(1+\chi_{\mathrm{Pr}})}\right) \\
&-\frac{Q_{\mathrm{P}}^{-2}}{(1+\chi_{\mathrm{Pi}})(1+\chi_{\mathrm{Pr}})}[\sin\sigma_{\mathrm{PP}}(\tan\delta_{\mathrm{Pr}}+\tan\delta_{\mathrm{Pi}}) - \cos\sigma_{\mathrm{PP}}(\tan\delta_{\mathrm{Pr}}\tan\delta_{\mathrm{Pi}})] \\
&-i\sin\sigma_{\mathrm{PP}}\left[\frac{Q_{\mathrm{P}}^{-1}\tan\delta_{\mathrm{Pr}}}{1+\chi_{\mathrm{Pr}}} + \frac{Q_{\mathrm{P}}^{-1}\tan\delta_{\mathrm{Pi}}}{1+\chi_{\mathrm{Pi}}}\right]\Bigg].
\end{aligned} \tag{42}$$

Also

$$\begin{aligned}
\mathbf{K}_{\mathrm{Pi}}\cdot\mathbf{K}_{\mathrm{Pi}} &= \mathbf{P}_{\mathrm{Pi}}\cdot\mathbf{P}_{\mathrm{Pi}} - \mathbf{A}_{\mathrm{Pi}}\cdot\mathbf{A}_{\mathrm{Pi}} - 2i\mathbf{P}_{\mathrm{Pi}}\cdot\mathbf{A}_{\mathrm{Pi}} \\
&= \frac{\omega^2}{V_{\mathrm{HP}}^2}\left[\frac{1+\chi_{\mathrm{Pi}}}{1+\chi_{\mathrm{HP}}} - \frac{-1+\chi_{\mathrm{Pi}}}{1+\chi_{\mathrm{HP}}} - 2i\frac{1+\chi_{\mathrm{Pi}}}{1+\chi_{\mathrm{HP}}}\frac{-1+\chi_{\mathrm{Pi}}}{1+\chi_{\mathrm{HP}}}\cos\delta_{\mathrm{Pi}}\right] \\
&= 2\frac{\omega^2}{V_{\mathrm{HP}}^2}\frac{1-iQ_{\mathrm{P}}^{-1}}{1+\chi_{\mathrm{HP}}} = \frac{\omega^2}{V_{\mathrm{P}}^2}.
\end{aligned} \tag{43}$$

The inner product of wave number vector for incident P-wave and reflected S-waves

$$
\begin{aligned}
\mathbf{K}_{\mathrm{Pi}}\cdot\mathbf{K}_{\mathrm{Sr}} = &-\frac{\omega^2}{V_{\mathrm{HP}}V_{\mathrm{HS}}}\sqrt{\frac{(1+\chi_{\mathrm{Pi}})(1+\chi_{\mathrm{Sr}})}{(1+\chi_{\mathrm{HP}})(1+\chi_{\mathrm{HS}})}}\times\\
&\left[\cos\sigma_{\mathrm{PS}}\left(1-i\frac{Q_{\mathrm{S}}^{-1}}{1+\chi_{\mathrm{Sr}}}-i\frac{Q_{\mathrm{P}}^{-1}}{1+\chi_{\mathrm{Pi}}}-\frac{Q_{\mathrm{P}}^{-1}Q_{\mathrm{S}}^{-1}}{(1+\chi_{\mathrm{Pi}})(1+\chi_{\mathrm{Sr}})}\right)\right.\\
&-\frac{Q_{\mathrm{P}}^{-1}Q_{\mathrm{S}}^{-1}}{(1+\chi_{\mathrm{Pi}})(1+\chi_{\mathrm{Sr}})}[\sin\sigma_{\mathrm{PS}}(\tan\delta_{\mathrm{Sr}}+\tan\delta_{\mathrm{Pi}})-\cos\sigma_{\mathrm{PS}}(\tan\delta_{\mathrm{Sr}}\tan\delta_{\mathrm{Pi}})]\\
&\left.-i\sin\sigma_{\mathrm{PS}}\left(\frac{Q_{\mathrm{S}}^{-1}\tan\delta_{\mathrm{Sr}}}{1+\chi_{\mathrm{Sr}}}+\frac{Q_{\mathrm{P}}^{-1}\tan\delta_{\mathrm{Pi}}}{1+\chi_{\mathrm{Pi}}}\right)\right].
\end{aligned}
\tag{44}
$$

$$
\begin{aligned}
\mathbf{K}_{\mathrm{Si}}\cdot\mathbf{K}_{\mathrm{Sr}} = &-\frac{\omega^2}{V_{\mathrm{HS}}^2}\frac{\sqrt{(1+\chi_{\mathrm{Si}})(1+\chi_{\mathrm{Sr}})}}{1+\chi_{\mathrm{HS}}}\times\\
&\left[\cos\sigma_{\mathrm{SS}}\left(1-i\frac{Q_{\mathrm{S}}^{-1}}{1+\chi_{\mathrm{Sr}}}-i\frac{Q_{\mathrm{S}}^{-1}}{1+\chi_{\mathrm{Si}}}-\frac{Q_{\mathrm{S}}^{-2}}{(1+\chi_{\mathrm{Si}})(1+\chi_{\mathrm{Sr}})}\right)\right.\\
&-\frac{Q_{\mathrm{S}}^{-2}}{(1+\chi_{\mathrm{Si}})(1+\chi_{\mathrm{Sr}})}[\sin\sigma_{\mathrm{SS}}(\tan\delta_{\mathrm{Sr}}+\tan\delta_{\mathrm{Si}})-\cos\sigma_{\mathrm{SS}}(\tan\delta_{\mathrm{Sr}}\tan\delta_{\mathrm{Si}})]\\
&\left.-i\sin\sigma_{\mathrm{SS}}\left[\frac{Q_{\mathrm{S}}^{-1}\tan\delta_{\mathrm{Sr}}}{1+\chi_{\mathrm{Sr}}}+\frac{Q_{\mathrm{S}}^{-1}\tan\delta_{\mathrm{Si}}}{1+\chi_{\mathrm{Si}}}\right]\right].
\end{aligned}
\tag{45}
$$

In low-loss case we have

$$
\begin{aligned}
\mathbf{K}_{\mathrm{Pi}}\cdot\mathbf{K}_{\mathrm{Pr}} &= -\frac{\omega^2}{V_{\mathrm{EP0}}^2}\left[\cos\sigma_{\mathrm{PP}}(1-iQ_{\mathrm{P}}^{-1})-\frac{i}{2}Q_{\mathrm{P}}^{-1}\sin\sigma_{\mathrm{PP}}\left[\tan\delta_{\mathrm{Pr}}+\tan\delta_{\mathrm{Pi}}\right]\right]\\
\mathbf{K}_{\mathrm{Si}}\cdot\mathbf{K}_{\mathrm{Sr}} &= -\frac{\omega^2}{V_{\mathrm{ES0}}^2}\left[\cos\sigma_{\mathrm{SS}}(1-iQ_{\mathrm{S}}^{-1})-\frac{i}{2}Q_{\mathrm{S}}^{-1}\sin\sigma_{\mathrm{SS}}\left[\tan\delta_{\mathrm{Sr}}+\tan\delta_{\mathrm{Si}}\right]\right]\\
\mathbf{K}_{\mathrm{Pi}}\cdot\mathbf{K}_{\mathrm{Sr}} &= \frac{\omega^2}{V_{\mathrm{EP0}}V_{\mathrm{ES0}}}\left[\cos\sigma_{\mathrm{PS}}\left(-1+i\frac{Q_{\mathrm{S}}^{-1}}{2}+i\frac{Q_{\mathrm{P}}^{-1}}{2}\right)+\frac{i}{2}\sin\sigma_{\mathrm{PS}}\left(Q_{\mathrm{S}}^{-1}\tan\delta_{\mathrm{Sr}}+Q_{\mathrm{P}}^{-1}\tan\delta_{\mathrm{Pi}}\right)\right]
\end{aligned}
\tag{46}
$$

Now consider to the cross product of the vectors

$$
\begin{aligned}
\mathbf{P}_{\mathrm{Sr}} \times \mathbf{P}_{\mathrm{Pi}} &= \frac{\omega^2}{V_{\mathrm{HP}} V_{\mathrm{HS}}} \sqrt{\frac{(1+\chi_{\mathrm{Pi}})(1+\chi_{\mathrm{Sr}})}{(1+\chi_{\mathrm{HP}})(1+\chi_{\mathrm{HS}})}} \times \\
&\quad (\mathbf{x}\sin\theta_{\mathrm{Sr}} - \mathbf{z}\cos\theta_{\mathrm{Sr}}) \times (\mathbf{z}\cos\theta_{\mathrm{Pi}} + \mathbf{x}\sin\theta_{\mathrm{Pi}}) \\
\mathbf{A}_{\mathrm{Sr}} \times \mathbf{A}_{\mathrm{Pi}} &= \frac{\omega^2}{V_{\mathrm{HS}} V_{\mathrm{HP}}} \sqrt{\frac{(-1+\chi_{\mathrm{Sr}})(-1+\chi_{\mathrm{Pi}})}{(1+\chi_{\mathrm{HS}})(1+\chi_{\mathrm{HP}})}} \times \\
&\quad (\mathbf{x}\sin(\theta_{\mathrm{Sr}} - \delta_{\mathrm{Sr}}) - \mathbf{z}\cos(\theta_{\mathrm{Sr}} - \delta_{\mathrm{Sr}})) \times (\mathbf{z}\cos(\theta_{\mathrm{Pi}} - \delta_{\mathrm{Pi}}) + \mathbf{x}\sin(\theta_{\mathrm{Pi}} - \delta_{\mathrm{Pi}})) \\
\mathbf{A}_{\mathrm{Sr}} \times \mathbf{P}_{\mathrm{Pi}} &= \frac{\omega^2}{V_{\mathrm{HS}} V_{\mathrm{HP}}} \sqrt{\frac{(-1+\chi_{\mathrm{Sr}})(1+\chi_{\mathrm{Pi}})}{(1+\chi_{\mathrm{HS}})(1+\chi_{\mathrm{HP}})}} \times \\
&\quad (\mathbf{x}\sin(\theta_{\mathrm{Sr}} - \delta_{\mathrm{Sr}}) - \mathbf{z}\cos(\theta_{\mathrm{Sr}} - \delta_{\mathrm{Sr}})) \times (\mathbf{z}\cos\theta_{\mathrm{Pi}} + \mathbf{x}\sin\theta_{\mathrm{Pi}})
\end{aligned}
\tag{47}
$$

$$
\begin{aligned}
\mathbf{A}_{\mathrm{Sr}} \times \mathbf{A}_{\mathrm{Pi}} &= -\mathbf{y}\frac{\omega^2}{V_{\mathrm{HS}} V_{\mathrm{HP}}} \sqrt{\frac{(-1+\chi_{\mathrm{Sr}})(-1+\chi_{\mathrm{Pi}})}{(1+\chi_{\mathrm{HS}})(1+\chi_{\mathrm{HP}})}} \sin(\theta_{\mathrm{Sr}} + \theta_{\mathrm{Pi}} - (\delta_{\mathrm{Sr}} + \delta_{\mathrm{Pi}})) \\
\mathbf{P}_{\mathrm{Sr}} \times \mathbf{P}_{\mathrm{Pi}} &= -\mathbf{y}\frac{\omega^2}{V_{\mathrm{HP}} V_{\mathrm{HS}}} \sqrt{\frac{(1+\chi_{\mathrm{Pi}})(1+\chi_{\mathrm{Sr}})}{(1+\chi_{\mathrm{HP}})(1+\chi_{\mathrm{HS}})}} \sin(\theta_{\mathrm{Sr}} + \theta_{\mathrm{Pi}}) \\
\mathbf{A}_{\mathrm{Sr}} \times \mathbf{P}_{\mathrm{Pi}} &= -\mathbf{y}\frac{\omega^2}{V_{\mathrm{HS}} V_{\mathrm{HP}}} \sqrt{\frac{(-1+\chi_{\mathrm{Sr}})(1+\chi_{\mathrm{Pi}})}{(1+\chi_{\mathrm{HS}})(1+\chi_{\mathrm{HP}})}} \sin(\theta_{\mathrm{Sr}} + \theta_{\mathrm{Pi}} - \delta_{\mathrm{Sr}}) \\
\mathbf{P}_{\mathrm{Sr}} \times \mathbf{A}_{\mathrm{Pi}} &= -\mathbf{y}\frac{\omega^2}{V_{\mathrm{HS}} V_{\mathrm{HP}}} \sqrt{\frac{(1+\chi_{\mathrm{Sr}})(-1+\chi_{\mathrm{Pi}})}{(1+\chi_{\mathrm{HS}})(1+\chi_{\mathrm{HP}})}} \sin(\theta_{\mathrm{Sr}} + \theta_{\mathrm{Pi}} - \delta_{\mathrm{Pi}})
\end{aligned}
\tag{48}
$$

so we have

$$
\begin{aligned}
\mathbf{K}_{\mathrm{Sr}} \times \mathbf{K}_{\mathrm{Pi}} &= \mathbf{P}_{\mathrm{Sr}} \times \mathbf{P}_{\mathrm{Pi}} - \mathbf{A}_{\mathrm{Sr}} \times \mathbf{A}_{\mathrm{Pi}} - i\mathbf{A}_{\mathrm{Sr}} \times \mathbf{P}_{\mathrm{Pi}} - i\mathbf{P}_{\mathrm{Sr}} \times \mathbf{A}_{\mathrm{Pi}} \\
&= -\mathbf{y}\frac{\omega^2}{V_{\mathrm{HP}} V_{\mathrm{HS}}} \sqrt{\frac{(1+\chi_{\mathrm{Pi}})(1+\chi_{\mathrm{Sr}})}{(1+\chi_{\mathrm{HP}})(1+\chi_{\mathrm{HS}})}} \sin(\theta_{\mathrm{Sr}} + \theta_{\mathrm{Pi}}) \\
&\quad + \mathbf{y}\frac{\omega^2}{V_{\mathrm{HS}} V_{\mathrm{HP}}} \sqrt{\frac{(-1+\chi_{\mathrm{Sr}})(-1+\chi_{\mathrm{Pi}})}{(1+\chi_{\mathrm{HS}})(1+\chi_{\mathrm{HP}})}} \sin(\theta_{\mathrm{Sr}} + \theta_{\mathrm{Pi}} - (\delta_{\mathrm{Sr}} + \delta_{\mathrm{Pi}})) \\
&\quad + i\mathbf{y}\frac{\omega^2}{V_{\mathrm{HS}} V_{\mathrm{HP}}} \sqrt{\frac{(-1+\chi_{\mathrm{Sr}})(1+\chi_{\mathrm{Pi}})}{(1+\chi_{\mathrm{HS}})(1+\chi_{\mathrm{HP}})}} \sin(\theta_{\mathrm{Sr}} + \theta_{\mathrm{Pi}} - \delta_{\mathrm{Sr}}) \\
&\quad + i\mathbf{y}\frac{\omega^2}{V_{\mathrm{HS}} V_{\mathrm{HP}}} \sqrt{\frac{(1+\chi_{\mathrm{Sr}})(-1+\chi_{\mathrm{Pi}})}{(1+\chi_{\mathrm{HS}})(1+\chi_{\mathrm{HP}})}} \sin(\theta_{\mathrm{Sr}} + \theta_{\mathrm{Pi}} - \delta_{\mathrm{Pi}})
\end{aligned}
\tag{49}
$$

$$
\begin{aligned}
&= \mathbf{y}\frac{\omega^2}{V_{\mathrm{HP}}V_{\mathrm{HS}}}\sqrt{\frac{(1+\chi_{\mathrm{Pi}})(1+\chi_{\mathrm{Sr}})}{(1+\chi_{\mathrm{HP}})(1+\chi_{\mathrm{HS}})}}\times \\
&-\sin\sigma_{\mathrm{PS}} + \frac{Q_{\mathrm{P}}^{-1}Q_{\mathrm{S}}^{-1}\sec\delta_{\mathrm{Pi}}\sec\delta_{\mathrm{Sr}}}{(1+\chi_{\mathrm{Sr}})(1+\chi_{\mathrm{Pi}})}\sin(\sigma_{\mathrm{PS}}-(\delta_{\mathrm{Sr}}+\delta_{\mathrm{Pi}})) \\
&+i\frac{Q_{\mathrm{S}}^{-1}\sec\delta_{\mathrm{Sr}}}{(1+\chi_{\mathrm{Sr}})}\sin(\sigma_{\mathrm{PS}}-\delta_{\mathrm{Sr}}) + i\frac{Q_{\mathrm{P}}^{-1}\sec\delta_{\mathrm{Pi}}}{(1+\chi_{\mathrm{Pi}})}\sin(\sigma_{\mathrm{PS}}-\delta_{\mathrm{Pi}})
\end{aligned} \tag{50}
$$

$$
\begin{aligned}
&= \mathbf{y}\frac{\omega^2}{V_{\mathrm{HP}}V_{\mathrm{HS}}}\sqrt{\frac{(1+\chi_{\mathrm{Pi}})(1+\chi_{\mathrm{Sr}})}{(1+\chi_{\mathrm{HP}})(1+\chi_{\mathrm{HS}})}}\times \\
&-\sin\sigma_{\mathrm{PS}} + \frac{Q_{\mathrm{P}}^{-1}Q_{\mathrm{S}}^{-1}}{(1+\chi_{\mathrm{Sr}})(1+\chi_{\mathrm{Pi}})}[\sin\sigma_{PS}(1-\tan\delta_{\mathrm{Sr}}\tan\delta_{\mathrm{Pi}}) - \cos\sigma_{PS}(\tan\delta_{\mathrm{Sr}}+\tan\delta_{\mathrm{Pi}})] \\
&+i\frac{Q_{\mathrm{S}}^{-1}}{1+\chi_{\mathrm{Sr}}}[\sin\sigma_{\mathrm{PS}}-\cos\sigma_{\mathrm{PS}}\tan\delta_{\mathrm{Sr}}] + i\frac{Q_{\mathrm{P}}^{-1}}{1+\chi_{\mathrm{Pi}}}[\sin\sigma_{\mathrm{PS}}-\cos\sigma_{\mathrm{PS}}\tan\delta_{\mathrm{Pi}}]
\end{aligned} \tag{51}
$$

$$
\begin{aligned}
\mathbf{K}_{\mathrm{Sr}}\times\mathbf{K}_{\mathrm{Pi}} &= \mathbf{y}\frac{\omega^2}{V_{\mathrm{HP}}V_{\mathrm{HS}}}\sqrt{\frac{(1+\chi_{\mathrm{Pi}})(1+\chi_{\mathrm{Sr}})}{(1+\chi_{\mathrm{HP}})(1+\chi_{\mathrm{HS}})}}\times \\
&\sin\sigma_{\mathrm{PS}}\left(-1+i\frac{Q_{\mathrm{S}}^{-1}}{1+\chi_{\mathrm{Sr}}}+i\frac{Q_{\mathrm{P}}^{-1}}{1+\chi_{\mathrm{Pi}}}+\frac{Q_{\mathrm{P}}^{-1}Q_{\mathrm{S}}^{-1}}{(1+\chi_{\mathrm{Sr}})(1+\chi_{\mathrm{Pi}})}\right) \\
&-\frac{Q_{\mathrm{P}}^{-1}Q_{\mathrm{S}}^{-1}}{(1+\chi_{\mathrm{Sr}})(1+\chi_{\mathrm{Pi}})}[\sin\sigma_{PS}\tan\delta_{\mathrm{Sr}}\tan\delta_{\mathrm{Pi}}+\cos\sigma_{PS}(\tan\delta_{\mathrm{Sr}}+\tan\delta_{\mathrm{Pi}})]
\end{aligned} \tag{52}
$$

$$
\begin{aligned}
&-i\cos\sigma_{\mathrm{PS}}\left[\frac{Q_{\mathrm{S}}^{-1}}{1+\chi_{\mathrm{Sr}}}\tan\delta_{\mathrm{Sr}}+\frac{Q_{\mathrm{P}}^{-1}}{1+\chi_{\mathrm{Pi}}}\tan\delta_{\mathrm{Pi}}\right] \\
&\mathbf{K}_{\mathrm{Si}}\times\mathbf{K}_{\mathrm{Sr}} = -\mathbf{y}\frac{\omega^2}{V_{\mathrm{HS}}^2}\frac{\sqrt{(1+\chi_{\mathrm{Si}})(1+\chi_{\mathrm{Sr}})}}{(1+\chi_{\mathrm{HS}})}\times \\
&\sin\sigma_{\mathrm{SS}}\left(-1+i\frac{Q_{\mathrm{S}}^{-1}}{1+\chi_{\mathrm{Sr}}}+i\frac{Q_{\mathrm{S}}^{-1}}{1+\chi_{\mathrm{Si}}}+\frac{Q_{\mathrm{S}}^{-2}}{(1+\chi_{\mathrm{Sr}})(1+\chi_{\mathrm{Si}})}\right) \\
&-\frac{Q_{\mathrm{S}}^{-2}}{(1+\chi_{\mathrm{Sr}})(1+\chi_{\mathrm{Si}})}[\sin\sigma_{SS}\sin\delta_{\mathrm{Sr}}\sin\delta_{\mathrm{Si}}+\cos\sigma_{SS}(\sin\delta_{\mathrm{Sr}}+\sin\delta_{\mathrm{Si}})] \\
&-i\cos\sigma_{\mathrm{SS}}\left[\frac{Q_{\mathrm{S}}^{-1}}{1+\chi_{\mathrm{Sr}}}\tan\delta_{\mathrm{Sr}}+\frac{Q_{\mathrm{S}}^{-1}}{1+\chi_{\mathrm{Si}}}\tan\delta_{\mathrm{Si}}\right]
\end{aligned} \tag{53}
$$

In low-loss case

$$
\begin{aligned}
\mathbf{K}_{\mathrm{Sr}}\times\mathbf{K}_{\mathrm{Pi}} &= \mathbf{y}\frac{\omega^2}{V_{\mathrm{EP0}}V_{\mathrm{ES0}}}\left[\sin\sigma_{\mathrm{PS}}\left(-1+i\frac{Q_{\mathrm{S}}^{-1}}{2}+i\frac{Q_{\mathrm{P}}^{-1}}{2}\right)-\frac{i}{2}\cos\sigma_{\mathrm{PS}}\left[Q_{\mathrm{S}}^{-1}\tan\delta_{\mathrm{Sr}}+Q_{\mathrm{P}}^{-1}\tan\delta_{\mathrm{Pi}}\right]\right] \\
\mathbf{K}_{\mathrm{Si}}\times\mathbf{K}_{\mathrm{Sr}} &= \mathbf{y}\frac{\omega^2}{V_{\mathrm{ES0}}^2}\left[\sin\sigma_{\mathrm{SS}}\left(1-iQ_{\mathrm{S}}^{-1}\right)+\frac{i}{2}\cos\sigma_{\mathrm{SS}}Q_{\mathrm{S}}^{-1}\left[\tan\delta_{\mathrm{Sr}}+\tan\delta_{\mathrm{Si}}\right]\right]
\end{aligned} \tag{54}
$$

Consider to the P-to-P scattering potential

$$
\begin{aligned}
{}^{\mathrm{P}}_{\mathrm{P}}\mathcal{F} &= \boldsymbol{\xi}_{\mathrm{P}r}\cdot\boldsymbol{\xi}_{\mathrm{P}i} = \frac{V_{\mathrm{P0}}^2}{\omega^2}\mathbf{K}_{\mathrm{Pi}}\cdot\mathbf{K}_{\mathrm{Pr}},\\
{}^{\mathrm{P}}_{\mathrm{P}}\mathcal{G}_{V_{\mathrm{P}}} &= V_{\mathrm{P0}}^2(\boldsymbol{\xi}_{\mathrm{P}r}\cdot\mathbf{k}_{\mathrm{P}r})(\boldsymbol{\xi}_{\mathrm{P}i}\cdot\mathbf{k}_{\mathrm{P}i}) = \frac{V_{\mathrm{P0}}^4}{\omega^4}(\mathbf{K}_{\mathrm{Pi}}\cdot\mathbf{K}_{\mathrm{Pi}})(\mathbf{K}_{\mathrm{Pr}}\cdot\mathbf{K}_{\mathrm{Pr}}),\\
{}^{\mathrm{P}}_{\mathrm{P}}\mathcal{G}_{V_{\mathrm{S}}} &= V_{\mathrm{S0}}^2\left\{(\boldsymbol{\xi}_{\mathrm{P}r}\cdot\boldsymbol{\xi}_{\mathrm{P}i})(\mathbf{k}_{\mathrm{P}r}\cdot\mathbf{k}_{\mathrm{P}i}) - 2(\boldsymbol{\xi}_{\mathrm{P}r}\cdot\mathbf{k}_{\mathrm{P}r})(\boldsymbol{\xi}_{\mathrm{P}i}\cdot\mathbf{k}_{\mathrm{P}i}) + (\boldsymbol{\xi}_{\mathrm{P}i}\cdot\mathbf{k}_{\mathrm{P}r})(\boldsymbol{\xi}_{\mathrm{P}r}\cdot\mathbf{k}_{\mathrm{P}i})\right\}\\
&= 2V_{\mathrm{S0}}^2\frac{V_{\mathrm{P0}}^2}{\omega^4}\left\{(\mathbf{K}_{\mathrm{Pi}}\cdot\mathbf{K}_{\mathrm{Pr}})^2 - (\mathbf{K}_{\mathrm{Pi}}\cdot\mathbf{K}_{\mathrm{Pi}})(\mathbf{K}_{\mathrm{Pr}}\cdot\mathbf{K}_{\mathrm{Pr}})\right\}\\
&= 2\frac{V_{\mathrm{S0}}^2}{V_{\mathrm{P0}}^2}\left\{\frac{V_{\mathrm{P0}}^4}{\omega^4}(\mathbf{K}_{\mathrm{Pi}}\cdot\mathbf{K}_{\mathrm{Pr}})^2 - 1)\right\}
\end{aligned}
\tag{55}
$$

Let us consider to the low-loss media

$$
(\mathbf{K}_{\mathrm{Pi}}\cdot\mathbf{K}_{\mathrm{Pr}})^2 = \frac{\omega^4}{V_{\mathrm{EP0}}^4}\left[\cos^2\sigma_{\mathrm{PP}}(1-2iQ_{\mathrm{P}}^{-1}) - iQ_{\mathrm{P}}^{-1}\sin\sigma_{\mathrm{PP}}\cos\sigma_{\mathrm{PP}}\left[\tan\delta_{\mathrm{Pr}}+\tan\delta_{\mathrm{Pi}}\right]\right]
$$

so we have

$$
\begin{aligned}
{}^{\mathrm{P}}_{\mathrm{P}}\mathcal{F} &= \boldsymbol{\xi}_{\mathrm{P}r}\cdot\boldsymbol{\xi}_{\mathrm{P}i} = -\cos\sigma_{\mathrm{PP}} + \frac{i}{2}Q_{\mathrm{P}}^{-1}\sin\sigma_{\mathrm{PP}}\left[\tan\delta_{\mathrm{Pr}}+\tan\delta_{\mathrm{Pi}}\right]\\
{}^{\mathrm{P}}_{\mathrm{P}}\mathcal{G}_{V_{\mathrm{P}}} &= 1,\\
{}^{\mathrm{P}}_{\mathrm{P}}\mathcal{G}_{V_{\mathrm{S}}} &= -\frac{V_{\mathrm{ES0}}^2}{V_{\mathrm{EP0}}^2}\left\{2\sin^2\sigma_{\mathrm{PP}}(1+iQ_{\mathrm{S}}^{-1}-iQ_{\mathrm{P}}^{-1}) + iQ_{\mathrm{P}}^{-1}\sin 2\sigma_{\mathrm{PP}}\left[\tan\delta_{\mathrm{Pr}}+\tan\delta_{\mathrm{Pi}}\right]\right\}
\end{aligned}
\tag{56}
$$

we have

$$
\begin{aligned}
{}^{\mathrm{I}}_{\mathrm{R}}\mathbb{V}^{\rho}_{\mathrm{VE}} &= {}^{\mathrm{I}}_{\mathrm{R}}\mathcal{F} - {}^{\mathrm{I}}_{\mathrm{R}}\mathcal{G}_{V_{\mathrm{EP}}} - {}^{\mathrm{I}}_{\mathrm{R}}\mathcal{G}_{V_{\mathrm{ES}}},\\
{}^{\mathrm{I}}_{\mathrm{R}}\mathbb{V}^{Q_{\mathrm{P}}}_{\mathrm{VE}} &= -\frac{i}{2}Q_P^{-1}\left\{{}^{\mathrm{I}}_{\mathrm{R}}\mathbb{V}^{V_{\mathrm{PE}}}_{\mathrm{VE}}\right\} = \left(iQ_P^{-1}\right){}^{\mathrm{I}}_{\mathrm{R}}\mathcal{G}_{V_{\mathrm{EP}}},\\
{}^{\mathrm{I}}_{\mathrm{R}}\mathbb{V}^{Q_{\mathrm{S}}}_{\mathrm{VE}} &= -\frac{i}{2}Q_S^{-1}\left\{{}^{\mathrm{I}}_{\mathrm{R}}\mathbb{V}^{V_{\mathrm{ES}}}_{\mathrm{VE}}\right\} = \left(iQ_S^{-1}\right){}^{\mathrm{I}}_{\mathrm{R}}\mathcal{G}_{V_{\mathrm{ES}}},
\end{aligned}
\tag{57}
$$

so

$$
\begin{aligned}
{}^{\mathrm{I}}_{\mathrm{R}}\mathbb{V}^{\rho}_{\mathrm{VE}} = -1 - \cos\sigma_{\mathrm{PP}} + 2\left(\frac{V_{\mathrm{ES0}}}{V_{\mathrm{EP0}}}\right)^2\sin^2\sigma_{\mathrm{PP}} + 2i\left(\frac{V_{\mathrm{ES0}}}{V_{\mathrm{EP0}}}\right)^2\times\\
\left\{\sin^2\sigma_{\mathrm{PP}}(Q_{\mathrm{S}}^{-1}-Q_{\mathrm{P}}^{-1}) + Q_{\mathrm{P}}^{-1}\left(\sin 2\sigma_{\mathrm{PP}} + \frac{1}{2}\left(\frac{V_{\mathrm{EP0}}}{V_{\mathrm{ES0}}}\right)^2\sin\sigma_{\mathrm{PP}}\right)\tan\delta_{\mathrm{P}}\right\}
\end{aligned}
\tag{58}
$$

$$
\begin{aligned}
{}^{\mathrm{P}}_{\mathrm{P}}\mathbb{V}^{V_{\mathrm{PE}}}_{\mathrm{VE}} &= -2({}^{\mathrm{P}}_{\mathrm{P}}\mathcal{G}_{V_{\mathrm{EP}}}) = -2 \\
{}^{\mathrm{P}}_{\mathrm{P}}\mathbb{V}^{V_{\mathrm{SE}}}_{\mathrm{VE}} &= -2({}^{\mathrm{P}}_{\mathrm{P}}\mathcal{G}_{V_{\mathrm{ES}}}) = 4\left(\frac{V_{\mathrm{ES0}}}{V_{\mathrm{EP0}}}\right)^2 \sin^2\sigma_{\mathrm{PP}} \\
&\quad + 4i\left(\frac{V_{\mathrm{ES0}}}{V_{\mathrm{EP0}}}\right)^2 \left\{\sin^2\sigma_{\mathrm{PP}}(Q_{\mathrm{S}}^{-1} - Q_{\mathrm{P}}^{-1}) + Q_{\mathrm{P}}^{-1}\sin 2\sigma_{\mathrm{PP}}\tan\delta_{\mathrm{P}}\right\} \qquad (59) \\
{}^{\mathrm{P}}_{\mathrm{P}}\mathbb{V}^{Q_{\mathrm{P}}}_{\mathrm{VE}} &= \left(iQ_P^{-1}\right){}^{\mathrm{I}}_{\mathrm{R}}\mathcal{G}_{V_{\mathrm{EP}}} = iQ_P^{-1}, \\
{}^{\mathrm{P}}_{\mathrm{P}}\mathbb{V}^{Q_{\mathrm{S}}}_{\mathrm{VE}} &= \left(iQ_S^{-1}\right){}^{\mathrm{I}}_{\mathrm{R}}\mathcal{G}_{V_{\mathrm{ES}}} = -2iQ_{\mathrm{S}}^{-1}\left(\frac{V_{\mathrm{ES0}}}{V_{\mathrm{EP0}}}\right)^2 \sin^2\sigma_{\mathrm{PP}}.
\end{aligned}
$$

Let us consider to the scattering of P-to-SI wave. We have

$$
\begin{aligned}
{}^{\mathrm{P}}_{\mathrm{SI}}\mathcal{F} &= \boldsymbol{\zeta}_{\mathrm{S}r}\cdot\boldsymbol{\xi}_{\mathrm{P}i} \\
{}^{\mathrm{P}}_{\mathrm{SI}}\mathcal{G}_{V_P} &= V_{P0}^2(\boldsymbol{\zeta}_{\mathrm{S}r}\cdot\mathbf{k}_{\mathrm{S}r})(\boldsymbol{\xi}_{\mathrm{P}i}\cdot\mathbf{k}_{\mathrm{P}i}) \\
{}^{\mathrm{P}}_{\mathrm{SI}}\mathcal{G}_{V_S} &= V_{S0}^2\left\{(\boldsymbol{\zeta}_{\mathrm{S}r}\cdot\boldsymbol{\xi}_{\mathrm{P}i})(\mathbf{k}_{\mathrm{S}r}\cdot\mathbf{k}_{\mathrm{P}i}) - 2(\boldsymbol{\zeta}_{\mathrm{S}r}\cdot\mathbf{k}_{\mathrm{S}r})(\boldsymbol{\xi}_{\mathrm{P}i}\cdot\mathbf{k}_{\mathrm{P}i}) + (\boldsymbol{\xi}_{\mathrm{P}i}\cdot\mathbf{k}_{\mathrm{S}r})(\boldsymbol{\zeta}_{\mathrm{S}r}\cdot\mathbf{k}_{\mathrm{P}i})\right\}.
\end{aligned} \qquad (60)
$$

First we have

$$
\begin{aligned}
\boldsymbol{\zeta}_{\mathrm{S}r}\cdot\boldsymbol{\xi}_{\mathrm{P}i} &= \boldsymbol{\xi}_{\mathrm{P}i}\cdot(\mathbf{y}\times\boldsymbol{\xi}_{\mathrm{S}r}) = \mathbf{y}\cdot(\boldsymbol{\xi}_{\mathrm{S}r}\times\boldsymbol{\xi}_{\mathrm{P}i}) = \frac{V_{\mathrm{P}}V_{\mathrm{S}}}{\omega^2}\mathbf{y}\cdot(\mathbf{K}_{\mathrm{S}r}\times\mathbf{K}_{\mathrm{P}i}) \\
\boldsymbol{\zeta}_{\mathrm{S}r}\cdot\mathbf{k}_{\mathrm{S}r} &= \mathbf{k}_{\mathrm{S}r}\cdot(\mathbf{y}\times\boldsymbol{\xi}_{\mathrm{S}r}) = \mathbf{y}\cdot(\boldsymbol{\xi}_{\mathrm{S}r}\times\mathbf{k}_{\mathrm{S}r}) = 0 \\
\boldsymbol{\zeta}_{\mathrm{S}r}\cdot\mathbf{k}_{\mathrm{P}i} &= \mathbf{k}_{\mathrm{P}i}\cdot(\mathbf{y}\times\boldsymbol{\xi}_{\mathrm{S}r}) = \mathbf{y}\cdot(\boldsymbol{\xi}_{\mathrm{S}r}\times\mathbf{k}_{\mathrm{P}i}) = \mathbf{y}\frac{V_{\mathrm{S}}}{\omega^2}\cdot(\mathbf{K}_{\mathrm{S}r}\times\mathbf{K}_{\mathrm{P}i}) \\
\boldsymbol{\xi}_{\mathrm{P}i}\cdot\mathbf{k}_{\mathrm{S}r} &= \frac{V_{\mathrm{P}}}{\omega^2}\mathbf{K}_{\mathrm{P}i}\cdot\mathbf{K}_{\mathrm{S}r}
\end{aligned} \qquad (61)
$$

finally

$$
\begin{aligned}
{}^{\mathrm{P}}_{\mathrm{SI}}\mathcal{F} &= \frac{V_{\mathrm{P}}V_{\mathrm{S}}}{\omega^2}\mathbf{y}\cdot(\mathbf{K}_{\mathrm{S}r}\times\mathbf{K}_{\mathrm{P}i}) \\
{}^{\mathrm{P}}_{\mathrm{SI}}\mathcal{G}_{V_P} &= 0 \\
{}^{\mathrm{P}}_{\mathrm{SI}}\mathcal{G}_{V_S} &= 2V_{S0}^2\frac{V_{\mathrm{P}}V_{\mathrm{S}}}{\omega^4}\left\{\mathbf{y}\cdot(\mathbf{K}_{\mathrm{S}r}\times\mathbf{K}_{\mathrm{P}i})(\mathbf{K}_{\mathrm{P}i}\cdot\mathbf{K}_{\mathrm{S}r})\right\}.
\end{aligned} \qquad (62)
$$

so we have

$$
\begin{aligned}
{}^{\mathrm{P}}_{\mathrm{SI}}\mathcal{F} &= -\sin\sigma_{\mathrm{PS}} - \frac{i}{2}\cos\sigma_{\mathrm{PS}}\left[Q_{\mathrm{S}}^{-1}\tan\delta_{\mathrm{Sr}} + Q_{\mathrm{P}}^{-1}\tan\delta_{\mathrm{Pi}}\right] \\
{}^{\mathrm{P}}_{\mathrm{SI}}\mathcal{G}_{V_S} &= \frac{V_{\mathrm{ES0}}}{V_{\mathrm{EP0}}}\left[\sin 2\sigma_{\mathrm{PS}}\left(1 + \frac{i}{2}(Q_{\mathrm{S}}^{-1} - Q_{\mathrm{P}}^{-1})\right)\right. \\
&\qquad \left. + i\cos 2\sigma_{\mathrm{PS}}\left(Q_{\mathrm{S}}^{-1}\tan\delta_{\mathrm{Sr}} + Q_{\mathrm{P}}^{-1}\tan\delta_{\mathrm{Pi}}\right)\right]
\end{aligned} \qquad (63)
$$

finally

$$
\begin{aligned}
{}^{\mathrm{P}}_{\mathrm{SI}}\mathbb{V}_{\rho} &= -\sin\sigma_{\mathrm{PS}} - \frac{V_{\mathrm{ES0}}}{V_{\mathrm{EP0}}}\sin 2\sigma_{\mathrm{PS}} \\
&- i\frac{V_{\mathrm{ES0}}}{2V_{\mathrm{EP0}}}\Big[\sin 2\sigma_{\mathrm{PS}}(Q_{\mathrm{S}}^{-1} - Q_{\mathrm{P}}^{-1}) \\
&+ \left(2\cos 2\sigma_{\mathrm{PS}} + \frac{V_{\mathrm{EP0}}}{V_{\mathrm{ES0}}}\cos\sigma_{\mathrm{PS}}\right)\left(Q_{\mathrm{S}}^{-1}\tan\delta_{\mathrm{Sr}} + Q_{\mathrm{P}}^{-1}\tan\delta_{\mathrm{Pi}}\right)\Big] \\
{}^{\mathrm{P}}_{\mathrm{SI}}\mathbb{V}_{V_{\mathrm{S}}} &= -2\frac{V_{\mathrm{ES0}}}{V_{\mathrm{EP0}}}\sin 2\sigma_{\mathrm{PS}} \\
&- i\frac{V_{\mathrm{ES0}}}{V_{\mathrm{EP0}}}\left[\sin 2\sigma_{\mathrm{PS}}\left(Q_{\mathrm{S}}^{-1} - Q_{\mathrm{P}}^{-1}\right) + 2\cos 2\sigma_{\mathrm{PS}}\left(Q_{\mathrm{S}}^{-1}\tan\delta_{\mathrm{Sr}} + Q_{\mathrm{P}}^{-1}\tan\delta_{\mathrm{Pi}}\right)\right] \\
{}^{\mathrm{P}}_{\mathrm{SI}}\mathbb{V}_{Q_{\mathrm{S}}^{-1}} &= iQ_{\mathrm{S}}^{-1}\frac{V_{\mathrm{ES0}}}{V_{\mathrm{EP0}}}\sin 2\sigma_{\mathrm{PS}}
\end{aligned}
\tag{64}
$$

Consider to the scattering SI-to-SI

$$
\begin{aligned}
{}^{\mathrm{SI}}_{\mathrm{SI}}\mathcal{F} &= \boldsymbol{\zeta}_{\mathrm{S}r}\cdot\boldsymbol{\zeta}_{\mathrm{S}i} = \boldsymbol{\xi}_{\mathrm{S}r}\cdot\boldsymbol{\xi}_{\mathrm{S}i} = \frac{V_{\mathrm{S0}}^2}{\omega^2}\mathbf{K}_{\mathrm{Si}}\cdot\mathbf{K}_{\mathrm{Sr}}, \\
{}^{\mathrm{SI}}_{\mathrm{SI}}\mathcal{G}_{V_S} &= V_{\mathrm{S0}}^2\left\{(\boldsymbol{\zeta}_{\mathrm{S}r}\cdot\boldsymbol{\zeta}_{\mathrm{S}i})(\mathbf{k}_{\mathrm{S}r}\cdot\mathbf{k}_{\mathrm{S}i}) + (\boldsymbol{\zeta}_{\mathrm{S}i}\cdot\mathbf{k}_{\mathrm{S}r})(\boldsymbol{\zeta}_{\mathrm{S}r}\cdot\mathbf{k}_{\mathrm{S}i})\right\},
\end{aligned}
\tag{65}
$$

where

$$
\begin{aligned}
\boldsymbol{\zeta}_{\mathrm{S}i}\cdot\mathbf{k}_{\mathrm{S}r} &= \mathbf{k}_{\mathrm{S}r}\cdot(\mathbf{y}\times\boldsymbol{\xi}_{\mathrm{S}i}) = \mathbf{y}\cdot(\boldsymbol{\xi}_{\mathrm{S}i}\times\mathbf{k}_{\mathrm{S}r}) = \mathbf{y}\frac{V_{\mathrm{S}}}{\omega^2}\cdot(\mathbf{K}_{\mathrm{S}i}\times\mathbf{K}_{\mathrm{S}r}) \\
\boldsymbol{\zeta}_{\mathrm{S}r}\cdot\mathbf{k}_{\mathrm{S}i} &= \mathbf{k}_{\mathrm{S}i}\cdot(\mathbf{y}\times\boldsymbol{\xi}_{\mathrm{S}r}) = \mathbf{y}\cdot(\boldsymbol{\xi}_{\mathrm{S}r}\times\mathbf{k}_{\mathrm{S}i}) = -\mathbf{y}\frac{V_{\mathrm{S}}}{\omega^2}\cdot(\mathbf{K}_{\mathrm{S}i}\times\mathbf{K}_{\mathrm{S}r})
\end{aligned}
\tag{66}
$$

as a result

$$
{}^{\mathrm{SI}}_{\mathrm{SI}}\mathcal{G}_{V_{\mathrm{S}}} = \frac{V_{\mathrm{S0}}^4}{\omega^4}\left\{(\mathbf{K}_{\mathrm{Si}}\cdot\mathbf{K}_{\mathrm{Sr}})^2 - |\mathbf{K}_{\mathrm{Si}}\times\mathbf{K}_{\mathrm{Sr}}|^2\right\} = (1 + 2iQ_{\mathrm{S}}^{-1})\times \tag{67}
$$

$$
\left[\cos^2\sigma_{\mathrm{SS}}\left(1 - 2iQ_{\mathrm{S}}^{-1}\right) - i\sin\sigma_{\mathrm{SS}}\cos\sigma_{\mathrm{SS}}Q_{\mathrm{S}}^{-1}\left(\tan\delta_{\mathrm{Sr}} + \tan\delta_{\mathrm{Si}}\right)\right] \tag{68}
$$

$$
-\left[\sin^2\sigma_{\mathrm{SS}}\left(1 - 2iQ_{\mathrm{S}}^{-1}\right) + i\sin\sigma_{\mathrm{SS}}\cos\sigma_{\mathrm{SS}}Q_{\mathrm{S}}^{-1}\left[\tan\delta_{\mathrm{Sr}} + \tan\delta_{\mathrm{Si}}\right]\right], \tag{69}
$$

finally

$$
{}^{\mathrm{SI}}_{\mathrm{SI}}\mathcal{G}_{V_{\mathrm{S}}} = \cos 2\sigma_{\mathrm{SS}} - iQ_{\mathrm{S}}^{-1}\sin 2\sigma_{\mathrm{SS}}\left[\tan\delta_{\mathrm{Sr}} + \tan\delta_{\mathrm{Si}}\right]. \tag{70}
$$

In low-loss media

$$
{}^{\mathrm{SI}}_{\mathrm{SI}}\mathcal{F} = -\cos\sigma_{\mathrm{S}} + \frac{i}{2}Q_{\mathrm{S}}^{-1}\sin\sigma_{\mathrm{SS}}\left(\tan\delta_{\mathrm{Sr}} + \tan\delta_{\mathrm{Si}}\right), \tag{71}
$$

as a result

$$
\begin{aligned}
{}^{\mathrm{SI}}_{\mathrm{SI}}\mathbb{V}^{\rho} &= -\cos\sigma_{\mathrm{S}} - \cos 2\sigma_{\mathrm{SS}} + iQ_{\mathrm{S}}^{-1}\left(\sin\sigma_{\mathrm{SS}} + 2\sin 2\sigma_{\mathrm{SS}}\right), \tan\delta_{\mathrm{S}} \\
{}^{\mathrm{SI}}_{\mathrm{SI}}\mathbb{V}^{V_{\mathrm{S}}} &= -2\cos 2\sigma_{\mathrm{SS}} + 4iQ_{\mathrm{S}}^{-1}\sin 2\sigma_{\mathrm{SS}}\tan\delta_{\mathrm{S}}, \\
{}^{\mathrm{SI}}_{\mathrm{SI}}\mathbb{V}^{Q_{\mathrm{S}}} &= iQ_{\mathrm{S}}^{-1}\cos 2\sigma_{\mathrm{SS}}.
\end{aligned}
\tag{72}
$$

ACKNOWLEDGMENTS

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