Exact solutions for reflection coefficients, in 1D and 2D

Heather K. Hardeman* and Michael P. Lamoureux*

ABSTRACT

In this paper, we solve the 1D and 2D elastic wave equation using two different velocity fields: a velocity jump and a velocity ramp. We require the density and modulus satisfy the relation established in (Lamoureux et al., 2012) and (Lamoureux et al., 2013) using some parameter α. We find the reflection coefficient for the 1D case of a velocity jump given general α. Extending these velocities to the two dimensions, we compute the analytic solutions to the 2D elastic wave equation and find the reflection coefficients for a plane wave hitting the jump and ramp at normal incidence. Finally, we conclude with discussion of the case where the plane wave hits the transition zone of the 2D velocity ramp at non-normal incidence given varying density. The motivation for this paper is to extend the work of the authors in (Lamoureux et al., 2012) and (Lamoureux et al., 2013) and to demonstrate explicit reflection coefficients in a continuously varying velocity field.

INTRODUCTION

In (Lamoureux et al., 2012), the authors found reflection coefficients of solutions to the 1D elastic wave equation

$$\rho(x)\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}\left(K(x)\frac{\partial u}{\partial x}\right) \tag{1}$$

where they established a relationship between the density $\rho(x)$ and the bulk modulus $K(x)$ using some parameter α. Specifically, they defined this relation as $\rho(x) = c^{\alpha-2}(x)$ and $K(x) = c^{\alpha}(x)$ where $c(x)$ is the velocity. Notice the relation preserves the ratio

$$\frac{K(x)}{\rho(x)} = c^2(x). \tag{2}$$

The authors considered two different cases: varying density and varying modulus. Respectively, these are the cases when $\alpha = 0$ and $\alpha = 2$. They did these cases for two different velocity ramps in the 1D case and found the reflection coefficients for each case. In particular, they were interested in the velocity with a jump discontinuity

$$c(x) = \begin{cases} 1 & x < 0 \\ 2 & 0 < x. \end{cases} \tag{3}$$

and the velocity ramp

$$c(x) = \begin{cases} 1 & x < 1 \\ x & 1 < x < 2 \\ 2 & 2 < x. \end{cases} \tag{4}$$

*University of Calgary

After studying the case when $\alpha = 0, 2$ for the velocity ramp (Eqn.4), the authors solved for an exact solution for the reflection coefficient with regards to the 1D velocity ramp for general α:

$$R(\omega) = \frac{e^{2i\omega}(2^{n_1} - 2^{n_2})(n_1 + n_2)}{2^{n_2}(2i\omega + 2\sqrt{1/4 - \omega^2}) + 2^{n_1}(-2i\omega + 2\sqrt{1/4 - \omega^2})} \tag{5}$$

where $n_{1,2} = (1 - \alpha) \pm \sqrt{(1-\alpha)^2/4 - \omega^2}$.

The following year, the authors extended their work to find exact solutions for reflection coefficients given more general velocities (Lamoureux et al., 2013). In particular, they considered a velocity jump

$$c(x) = \begin{cases} c_1 & x < 0 \\ c_2 & 0 < x \end{cases} \tag{6}$$

and a velocity ramp

$$c(x) = \begin{cases} c_1 & x < 0 \\ c_1 + mx & 0 \leq x \leq L \\ c_2 & L < x \end{cases} \tag{7}$$

where $m = \frac{c_2 - c_1}{L}$.

For the velocity ramp (Eqn. 7) when only the modulus varies ($\alpha = 0$), they found the reflection coefficient

$$R(\omega) = \frac{(r_2 - r_1)(n_1 + n_2)}{2(r_1 - r_2)(i\omega/m) + 2(r_1 + r_2)\sqrt{1/4 - (\omega/m)^2}} \tag{8}$$

where $r_1 = (c_2/c_1)^{n_1}$, $r_2 = (c_2/c_1)^{n_2}$. The parameters n_1 and n_2 are defined as before but with $\alpha = 0$, i.e. $n_1 = -1/4 + \sqrt{1/4 - (\omega/m)^2}$ and $n_2 = -1/4 - \sqrt{1/4 - (\omega/m)^2}$.

In this paper, we extend their work to the 2D case; however, we will consider the velocity jump

$$c(x) = \begin{cases} 1000 & x < 0 \\ 2000 & 0 < x \end{cases} \tag{9}$$

where c(x) is in meters per second, and the velocity ramp

$$c(x) = \begin{cases} 1000 & x < 0 \\ 1000 + 100x & 0 \leq x \leq 10 \\ 2000 & 10 < x. \end{cases} \tag{10}$$

The velocity jump (Eqn. 9) models a quick jump from a low velocity to a high velocity with values more commonly found in seismic. Similarly, the velocity ramp (Eqn. 10) gives a better model of the speed of a seismic wave through a transition zone of 10 m.

In our study, we begin by finding the reflection coefficient equation for the 1D velocity jump (Eqn. 9). With the 1D case resolved, we extend the velocities (Eqn. 9) and (Eqn. 10)

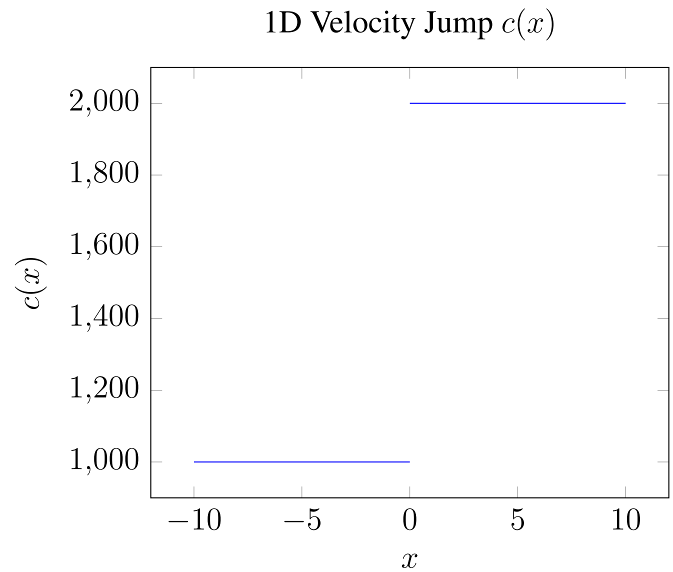

FIG. 1: A velocity field with a jump discontinuity at $x = 0$. The velocity jumps from $c_1 = 1000\ m/s$ to $c_2 = 2000\ m/s$ at $x = 0$.

to two dimensions and consider the 2D case; at which point, we will begin by studying the 2D velocity jump case and finding an exact solution for the reflection coefficient. Then, we will move on to the 2D velocity ramp case and solve for the exact solution for the reflection coefficient. In both 2D cases, we consider a plane wave incident to the line of jump discontinuities for the velocity jump and the transition zone for the velocity ramp, i.e. the case when the solution is not dependent on x. Finally, we will consider the case when the wave is not incident to the transition zone for a velocity jump in 2D.

REFLECTION AND TRANSMISSION COEFFICIENTS FOR THE 1D VELOCITY JUMP - VARYING α

For the velocity jump (Eqn. 9), the general solution to the (Eqn. 1) is

$$u(x,t) = \begin{cases} e^{i\omega(\pm x/1000-t)} & x < 0, t \geq 0 \\ e^{i\omega(\pm x/2000-t)} & x > 0, t \geq 0. \end{cases} \tag{11}$$

For this case, the wave moves from left to right across a jump discontinuity at $x = 0$. Therefore, the regional solutions are

$$u_{\text{left}}(x,t) = e^{i\omega(x/1000-t)} + Re^{i\omega(-x/1000-t)}, \tag{12}$$
$$u_{\text{right}}(x,t) = Te^{i\omega(x/2000-t)}, \tag{13}$$

where R is the portion of the wave which is reflected by the discontinuity and T represents the part of the wave which is transmitted through the discontinuity.

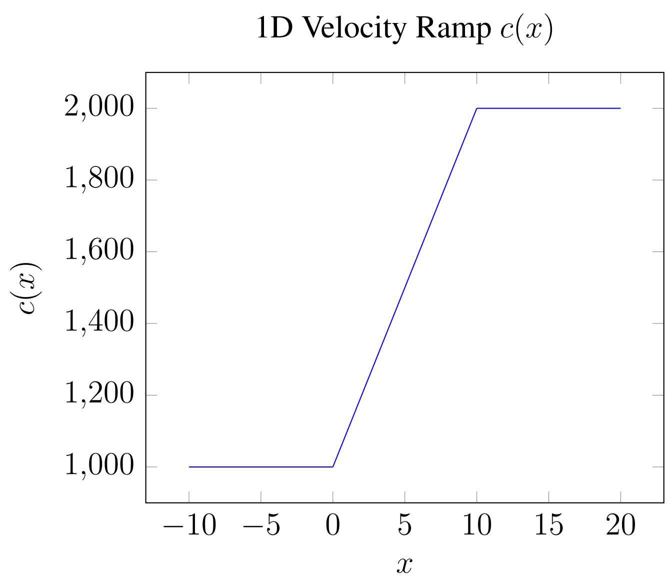

FIG. 2: A velocity field with a linear ramp between $c_1 = 1000$ m/s and $c_2 = 2000$ m/s.

Now, we set the following continuity conditions for the problem:

$$u_{\text{left}} = u_{\text{right}} \quad \text{at } x = 0, \text{ and} \tag{14}$$
$$K_{\text{left}}\partial_x(u_{\text{left}}) = K_{\text{right}}\partial_x(u_{\text{right}}) \quad \text{at } x = 0. \tag{15}$$

The first equation preserves continuity across $x = 0$, and the second equation enforces continuity of force. These continuity conditions give the following system of equations:

$$1 + R = T \tag{16}$$
$$K_{\text{left}}\left(\frac{i\omega}{1000} - \frac{i\omega}{1000}R\right) = \frac{K_{\text{right}}}{2000}i\omega T \tag{17}$$

Recall $K(x) = c^{\alpha}(x)$. Therefore, $K_{\text{left}} = 1000^{\alpha}$ and $K_{\text{right}} = 2000^{\alpha}$. Assuming $\omega \neq 0$, then we get

$$1 + R = T \tag{18}$$
$$1000^{\alpha-1} - 1000^{\alpha-1}R = 2000^{\alpha-1}T \tag{19}$$

Using MATLAB to solve (Eqn. 18) for R and T, we get

$$R(\alpha) = \frac{2 \cdot 1000^{\alpha} - 2000^{\alpha}}{2 \cdot 1000^{\alpha} + 2000^{\alpha}} \tag{20}$$
$$T(\alpha) = \frac{4 \cdot 1000^{\alpha}}{2 \cdot 1000^{\alpha} + 2000^{\alpha}}. \tag{21}$$

Impedance at $\alpha = 1$

Given the relation defined for the modulus $K(x)$ and the density $\rho(x)$, Eqn. (20) provides the opportunity to compare the effect of varying the modulus and density on the

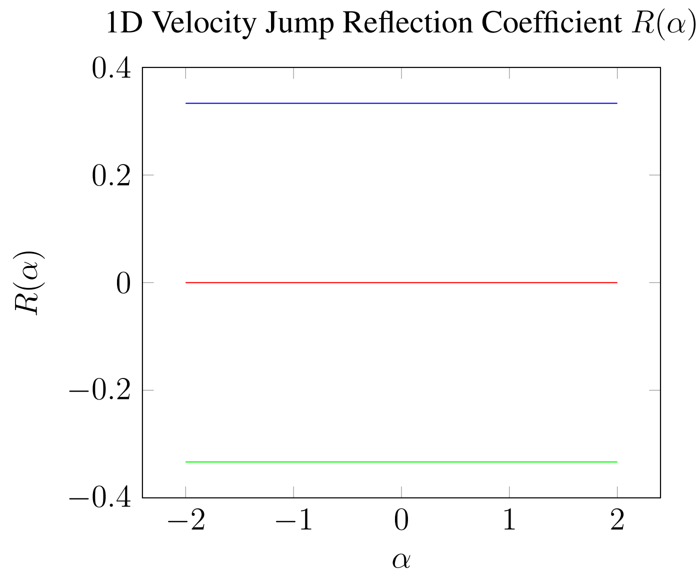

FIG. 3: Reflection coefficient for 1D velocity jump when $\alpha = 0$ (top), $\alpha = 1$ (middle), and $\alpha = 2$ (bottom)

reflection which occurs when the wave crosses a jump discontinuity. In (Lamoureux et al., 2012), the authors took particular interest in the cases when $\alpha = 0$ and $\alpha = 2$. They noted that a consequence in the different choice of α shows a flip in polarity once the wave is reflected. We can see this represented in Fig. 3.

Consider the case when $\alpha = 1$. Using Eqn. (20),

$$R(1) = \frac{2 \cdot 1000^1 - 2000^1}{2 \cdot 1000^1 + 2000^1} = 0. \tag{22}$$

The fact that impedance remains constant in the case when $\alpha = 1$ provides a good explanation for why the reflection coefficient $R(1) = 0$. Recall that the impedance $I = \rho(x)c(x)$. For $\alpha = 1$,

$$I = \rho(x)c(x) = c^{1-2}(x)c(x) = \frac{1}{c(x)}c(x) = 1. \tag{23}$$

Reflections are dependent on a change in impedance between layers in the Earth. If the impedance is constant, then there should not be a reflection. The case $\alpha = 1$ shows this fact.

REFLECTION AND TRANSMISSION COEFFICIENTS FOR 2D VELOCITY JUMP - VARYING α

Now, we extend the velocity jump (Eqn. 9) to two dimensions by defining it as

$$c(x,z) = \begin{cases} 1000 & z < 0, x \in \mathbb{R} \\ 2000 & z > 0, x \in \mathbb{R}. \end{cases} \tag{24}$$

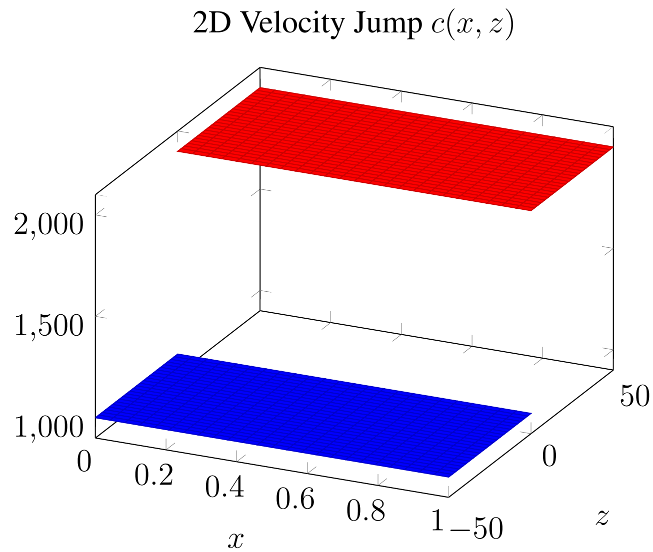

FIG. 4: A velocity field with jump from $1000m/s$ to $2000m/s$ at $z = 0$.

We preserve the relation between the density and modulus established in (Lamoureux et al., 2012) by setting $\rho(x, z) = c^{\alpha-2}(x, z)$ and $K(x, z) = c^{\alpha}(x, z)$.

Given that we are working in 2D now, we consider the case when the wave hits the jump interface at normal incidence, i.e. when the incident angle measure from the vertical is at $0°$. As such, solving the 2D elastic wave equation

$$\rho(x, z)\frac{\partial^2 u}{\partial t^2} = \nabla \cdot (K(x, z)\nabla u) \tag{25}$$

for the velocity (Eqn. 24), we get the general solution

$$u(x, z, t) = \begin{cases} e^{i\omega(\pm z/1000-t)} & z < 0, t \geq 0 \\ e^{i\omega(\pm z/2000-t)} & z > 0, t \geq 0 \end{cases} \tag{26}$$

when it is incident to the velocity jump.

For this case, we want to consider a plane wave moving down across a velocity jump. Thus, we represent the wave in the top half by

$$u_{\text{top}}(x, z, t) = e^{i\omega(z/1000-t)} + Re^{i\omega(-z/1000-t)} \tag{27}$$

and the wave which is transmitted across the velocity jump as

$$u_{\text{bottom}}(x, z, t) = Te^{i\omega(z/2000-t)} \tag{28}$$

where R is the reflection coefficient and T is the transmission coefficient.

Again, we want to enforce displacement continuity across the jump as well as continuity of force. As such, we consider the continuity conditions

$$u_{\text{top}} = u_{\text{bottom}} \quad \text{at } z = 0\text{, and} \tag{29}$$

$$K_{\text{top}}\nabla(u_{\text{top}}) = K_{\text{bottom}}\nabla(u_{\text{bottom}}) \quad \text{at } z = 0 \tag{30}$$

resulting in

$$e^{-i\omega t} + Re^{-i\omega t} = Te^{-i\omega t} \tag{31}$$

$$K_{\text{top}}\left(0 + \frac{i\omega}{1000} - \frac{i\omega}{1000}Re^{-i\omega t}\right) = K_{\text{bottom}}\left(0 + T\frac{i\omega}{2000}e^{-i\omega t}\right) \tag{32}$$

which implies

$$1 + R = T \tag{33}$$

$$K_{\text{top}}\left(\frac{i\omega}{1000} - \frac{i\omega}{1000}R\right) = K_{\text{bottom}}T\frac{i\omega}{2000}. \tag{34}$$

Recall $K(x, z) = c^{\alpha}(x, z)$. Then,

$$1 + R = T \tag{35}$$

$$i\omega 1000^{\alpha-1} - i\omega 1000^{\alpha-1}R = 2000^{\alpha-1}Ti\omega. \tag{36}$$

Assuming $\omega \neq 0$, Eqn. (35) is equivalent to Eqn. (18) for the 1D case for a velocity jump. As such,

$$R(\alpha) = \frac{2 \cdot 1000^{\alpha} - 2000^{\alpha}}{2 \cdot 1000^{\alpha} + 2000^{\alpha}} \tag{37}$$

$$T(\alpha) = \frac{4 \cdot 1000^{\alpha}}{2 \cdot 1000^{\alpha} + 2000^{\alpha}}. \tag{38}$$

Given that we are considering the plane wave case in 2D where the wave hits the velocity jump at $90°$, it follows that the exact solutions for reflection and transmission coefficients would mimic the 1D case. In fact, the 2D reflection coefficient solution (Eqn. 37) is identical to the 1D reflection coefficient solution (Eqn. 20) for this case.

REFLECTION AND TRANSMISSION COEFFICIENTS FOR 2D VELOCITY RAMP - VARYING α, NORMAL INCIDENCE

Let us now consider the case when we have a velocity ramp in two dimensions. We extend the 1D velocity ramp (Eqn. 10) to

$$c(x, z) = \begin{cases} 1000 & z < 0, x \in \mathbb{R} \\ 1000 + 100z & 0 \leq z \leq 10, x \in \mathbb{R}, \\ 2000 & 10 < z, x \in \mathbb{R}. \end{cases} \tag{39}$$

We are strictly interested in a plane wave hitting the start of the velocity ramp (Eqn. 39) at $90°$. Solving the 2D elastic wave equation (Eqn. 25), we get the solution

$$u(x, z, t) = \begin{cases} e^{i\omega(\pm z/1000 - t)} & z < 0, x \in \mathbb{R}, t \geq 0 \\ (1000 + 100z)^{n}e^{-i\omega t} & 0 \leq z \leq 10, x \in \mathbb{R}, t \geq 0 \\ e^{i\omega(\pm z/2000 - t)} & 10 < z, x \in \mathbb{R}, t \geq 0. \end{cases} \tag{40}$$

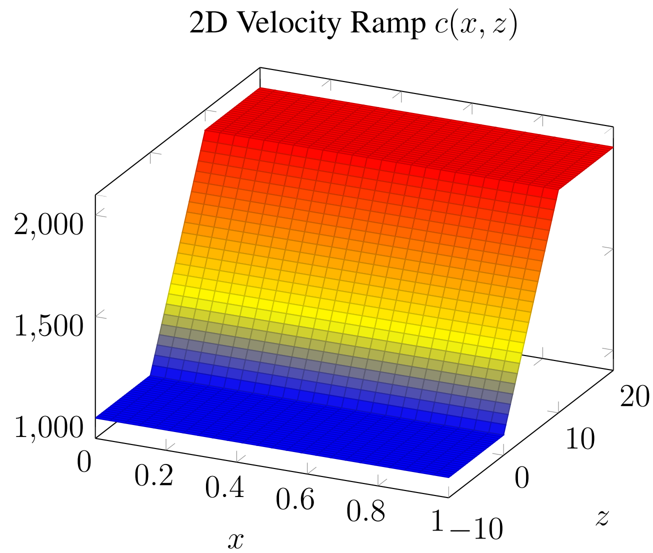

FIG. 5: A velocity field with a linear ramp from 1000 m/s to 2000 m/s over 10 m, in 2D.

where $n = (1 - \alpha) \pm \sqrt{(1 - \alpha)^2/4 - \omega^2}$. Then, the regional solutions for this case are

$$u_{\text{top}}(x, z, t) = e^{i\omega(z/1000 - t)} + Re^{i\omega(-z/1000 - t)} \tag{41}$$

$$u_{\text{trans}}(x, z, t) = A(1000 + 100z)^{n_1} e^{-i\omega t} + B(1000 + 100z)^{n_2} e^{-i\omega t} \tag{42}$$

$$u_{\text{bottom}}(x, z, t) = Te^{i\omega(z/2000 - t)} \tag{43}$$

As in the previous cases, we wish to impose continuity of displacement and continuity of force; however, in this case, we need to enforce these conditions at $z = 0$ and $z = 10$. As such, we have the continuity conditions:

$$u_{\text{top}} = u_{\text{trans}} \quad \text{at } z = 0, \tag{44}$$

$$u_{\text{trans}} = u_{\text{bottom}} \quad \text{at } z = 10, \tag{45}$$

$$K_{\text{top}} \nabla(u_{\text{top}}) = K_{\text{trans}} \nabla(u_{\text{trans}}) \quad \text{at } z = 0 \tag{46}$$

$$K_{\text{trans}} \nabla(u_{\text{trans}}) = K_{\text{bottom}} \nabla(u_{\text{bottom}}) \quad \text{at } z = 10 \tag{47}$$

which yield

$$1 + R = c_1^{n_1} A + c_1^{n_2} B \tag{48}$$

$$c_2^{n_1} A + c_2^{n_2} B = Te^{10i\omega/c_2} \tag{49}$$

$$c_1^{\alpha} \left(\frac{i\omega}{c_1} - \frac{i\omega}{c_1} R \right) = c_1^{\alpha} (100 A n_1 c_1^{n_1 - 1} + 100 B n_2 (c_1)^{n_2 - 1}) \tag{50}$$

$$c_2^{\alpha} (100 A n_1 c_2^{n_1 - 1} + 100 B n_2 c_2^{n_2 - 1}) = c_2^{\alpha} T e^{10i\omega/c_2} \frac{i\omega}{c_2}. \tag{51}$$

where $c_1 = 1000$ and $c_2 = 2000$. We can reduce this to

$$1 + R = c_1^{n_1} A + c_1^{n_2} B \tag{52}$$
$$c_2^{n_1} A + c_2^{n_2} B = T e^{10i\omega/c_2} \tag{53}$$
$$i\omega - i\omega R = 100 n_1 c_1^{n_1} A + 100 n_2 (c_1)^{n_2} B \tag{54}$$
$$100 n_1 c_2^{n_1} A + 100 n_2 c_2^{n_2} B = T i\omega e^{10i\omega/c_2}. \tag{55}$$

Now, we can create the matrix equation $\mathbf{M}x = \mathbf{b}$:

$$\begin{bmatrix} c_1^{n_1} & c_1^{n_2} & -1 & 0 \\ c_2^{n_1} & c_2^{n_2} & 0 & -e^{10i\omega/c2} \\ 100n_1c_1^{n_1} & 100n_2c_1^{n_2} & i\omega & 0 \\ 100n_1c_2^{n_1} & 100n_2c_2^{n_2} & 0 & -i\omega e^{10i\omega/2000} \end{bmatrix} \begin{bmatrix} A \\ B \\ R \\ T \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ i\omega \\ 0 \end{bmatrix}. \tag{56}$$

Using MATLAB to solve $x = \mathbf{M}^{-1}\mathbf{b}$, we get

$$R(\omega, \alpha) = \frac{(\omega + imn_1)(\omega + imn_2)(c_1^{n_1} c_2^{n_2} - c_1^{n_2} c_2^{n_1})}{(\omega^2 + m^2 n_1 n_2)(c_1^{n_1} c_2^{n_2} - c_1^{n_2} c_2^{n_1}) + (imn_2 - imn_1\omega)(c_1^{n_1} c_2^{n_2} + c_1^{n_2} c_2^{n_1})} \tag{57}$$
$$T(\omega, \alpha) = \frac{2 c_2^{n_1} c_2^{n_2} m\omega e^{-(i10\omega)/c_2)} (n_1 - c_2 n_2)}{(i\omega^2 + im^2 n_1 n_2)(c_1^{n_1} c_2^{n_2} - c_1^{n_2} c_2^{n_1}) + (mn_1\omega - mn_2\omega)(c_1^{n_1} c_2^{n_2} + c_1^{n_2} c_2^{n_1})} \tag{58}$$

REFLECTION AND TRANSMISSION COEFFICIENTS FOR 2D VELOCITY RAMP - VARYING α, NON- NORMAL INCIDENCE

In the previous two sections, we considered the relatively simple case of normal incidence for the 2D elastic wave equation when we have a velocity jump or a velocity ramp. In both examples, we saw that the exact solutions for the reflection coefficients of the 2D velocities mirrored their 1D counterparts. Now, we study the non-normal incidence case for the velocity ramp (Eqn. 39). Specifically, this is the case when the wave hits the transition zone at some incidence angle $\theta_1 > 0°$ We will begin by solving the 2D elastic wave equation for $\alpha = 0$ and finding a equation for the reflection coefficients for some arbitrary incidence angle θ_1. Then, we will consider specific cases of θ_1.

We start with a simple wave equation in 2D where the velocity depends on depth z:

$$\frac{1}{c(z)^2} \frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial z^2}. \tag{59}$$

As we noted in the previous paragraph, we can set $\alpha = 0$ in Eqn. (25) to get Eqn. (59). Also, notice that we use the geophysics convention that z is the vertical dimension and increases as you go down.

Recall that the velocity field in this case is constant $c_1 = 1000\ m/s$ in the upper region $z < 0$, constant $c_2 = 2000\ m/s$ in the lower region $z > 10$ deeper than $D = 10$ m, and linear in the transition region $0 < z < 10$ with $c(z) = 1000 + mz$ where $m = (c_2 - c_1)/D = 100$ is the slope for the change in velocity with depth.

Using separation of variables, a basic solution in the upper region will have the form

$$u(x,z,t) = e^{i(k_x x + k_z z + \omega t)} \tag{60}$$

where the real parameters k_x, k_z, ω satisfy the dispersion relation

$$k_x^2 + k_z^2 = \omega^2/c_1^2. \tag{61}$$

The solution in the lower region $z > 10$ will be similar except the dispersion relation will use velocity $2000 m/s$ instead.

The transition region is only slightly harder. Again, using separation of variables $u(x,z,t) = X(x)Z(z)T(t)$, we obtain the usual exponential solutions for $X(x) = \exp(ik_x x)$, $T(t) = \exp(i\omega t)$ and are left with a single ODE for $Z(z)$ in the form

$$\frac{Z''}{Z} + \frac{\omega^2}{c(z)^2} = k_x^2. \tag{62}$$

To avoid any confusion, let us write out the velocity function explicitly, to see the ODE as

$$\frac{Z''}{Z} + \frac{\omega^2}{(1000+100z)^2} = k_x^2. \tag{63}$$

This equation can be solved explicitly using Wolfram Alpha, and the general solution is a linear combination of two Whittaker functions,

$$Z(z) = AM_{0,\beta}(2c(z)k_x/s) + BW_{0,\beta}(2c(z)k_x/s), \tag{64}$$

where M and W are the Whittaker functions and $\beta = \sqrt{100^2 - 4\omega^2}/200$. We see that β is a function of the ratio ω/s which is how the frequency dependence of our reflection coefficients will enter in the problem. (This makes physical sense. The frequency ω, compared to the slope s of the transition zone is what matters to the reflection.)

Notice that the case $k_x = 0$ is special, corresponding to the normal incident case (plane waves that that have no x-dependence). In this case, the solutions are in the form

$$Z(z) = A(1000+100z)^{1/2+\beta} + B(1000+100z)^{1/2-\beta}, \tag{65}$$

using the same parameter β as above.

Now, we are ready to compute the plane wave reflection coefficients. The idea is similar to the previous case: start with a plane wave in the upper region, add in its reflection in the upper region, then match coefficients at the line $z = 0$ to find equations to specify parameters A, B in solution (Eqn. 65). You also need equations at the line $z = 10$ to connect the solution in the transition zone to the plane wave transmitted into the lower region $z > 10$.

We begin by fixing parameters k_x, k_z, ω, all positive and satisfying the relation (Eqn. 61). We set the incident plane wave to be

$$u_{inc} = e^{i(k_x x + k_z z - \omega t)}. \tag{66}$$

Remembering our convention that z points down and that the parameters are positive, this is a wave traveling downwards in the direction of the vector $\mathbf{k} = (k_x, k_z)$.

From physical intuition, we expect the reflected wave to be going up, a mirror reflection of the plane wave:

$$u_{ref} = Re^{i(k_x x - k_z z - \omega t)}, \tag{67}$$

where we flipped the sign in front of the z term to get an up-going wave. We include the reflection coefficient R here, and note it may depend on the parameters k_x, k_z, ω.

So, in the upper region $z < 0$ we write the wave field as

$$u_{\text{top}} = u_{inc} + u_{ref} = e^{i(k_x x + k_z z - \omega t)} + Re^{i(k_x x - k_z z - \omega t)}. \tag{68}$$

Assume that the k_x and the ω parameters stay the same across all regions. So in the transition region we expect

$$u_{\text{trans}} = e^{i(k_x x - \omega t)} \left(AM_{0,\beta}(2c(z)k_x/s) + BW_{0,\beta}(2c(z)k_x/s)\right) \tag{69}$$

and in the lower region $z > 10$ we expect

$$u_{\text{bottom}} = Te^{i(k_x x + k_z' z - \omega t)}, \tag{70}$$

where notice we have a new parameter k_z' to find. (This accounts for the fact that the direction of the plane wave will change when we get into the lower region.)

Before we continue to solve for the reflection coefficient, we should discuss how to solve for k_z'. In fact, all that is required is to use Snell's law. Define θ_1 as the angle of incidence and θ_2 as the angle of refraction. We can then write

$$k_x = \frac{\omega}{c_1}\sin\theta_1, \qquad k_z = \frac{\omega}{c_1}\cos\theta_1, \text{ and} \tag{71}$$

$$k_x' = \frac{\omega}{c_2}\sin\theta_2, \qquad k_z' = \frac{\omega}{c_2}\cos\theta_2. \tag{72}$$

Equating the two k_x's here gives us

$$\frac{\omega}{c_1}\sin\theta_1 = \frac{\omega}{c_2}\sin\theta_2 \tag{73}$$

or in other words

$$\frac{\sin\theta_1}{\sin\theta_2} = \frac{c_1}{c_2}, \tag{74}$$

which is Snell's law.

Now, let us return to finding the reflection coefficient. At $z = 0$, we equate the first two solutions and their derivatives to get

$$e^{i(k_x x + k_z 0 - \omega t)} + Re^{i(k_x x - k_z 0 - \omega t)} = e^{i(k_x x - \omega t)}[AZ_1(0) + BZ_2(0)] \tag{75}$$

$$ik_z e^{i(k_x x + k_z 0 - \omega t)} - ik_z Re^{i(k_x x - k_z 0 - \omega t)} = e^{i(k_x x - \omega t)}[AZ_1'(0) + BZ_2'(0)]. \tag{76}$$

Cancelling the exponentials, we have

$$1 + R = AZ_1(0) + BZ_2(0), \tag{77}$$
$$ik_z(1 - R) = AZ_1'(0) + BZ_2'(0). \tag{78}$$

At the line $z = 10$ we equate the last two solutions and their derivatives to get

$$e^{i(k_x x - \omega t)}[AZ_1(10) + BZ_2(10)] = Te^{i(k_x x + k_z' 10 - \omega t)} \tag{79}$$
$$e^{i(k_x x - \omega t)}[AZ_1'(10) + BZ_2'(10)] = ik_z' Te^{i(k_x x + k_z' 10 - \omega t)}. \tag{80}$$

Again cancelling exponentials, we have

$$AZ_1(10) + BZ_2(10) = Te^{ik_z' 10}, \tag{81}$$
$$AZ_1'(10) + BZ_2'(10) = ik_z' Te^{ik_z' 10}. \tag{82}$$

In summary, we have 4 equations in the 4 unknowns R, T, A, B which are

$$1 + R = AZ_1(0) + BZ_2(0)] \tag{83}$$
$$ik_z(1 - R) = AZ_1'(0) + BZ_2'(0) \tag{84}$$
$$AZ_1(10) + BZ_2(10) = Te^{ik_z' 10} \tag{85}$$
$$AZ_1'(10) + BZ_2'(10) = ik_z' Te^{ik_z' 10} \tag{86}$$

(Recall that k_z' was determined by Snell's law.)

We have used a generic solution $Z_1(z), Z_2(z)$ in the middle region. The point is, we can make different choices for $c(z)$, but the solution procedure for finding the reflection coefficients is the same. In fact, creating a matrix problem $\mathbf{M}x = \mathbf{b}$:

$$\begin{bmatrix} Z_1(0) & Z_2(0) & -1 & 0 \\ Z_1'(0) & Z_2'(0) & ik_z & 0 \\ Z_1(10) & Z_2(10) & 0 & -e^{ik_z' 10} \\ Z_1'(10) & Z_2'(10) & 0 & -ik_z' e^{ik_z' 10} \end{bmatrix} \begin{bmatrix} A \\ B \\ R \\ T \end{bmatrix} = \begin{bmatrix} 1 \\ ik_z \\ 0 \\ 0 \end{bmatrix}. \tag{87}$$

Then we can solve $\mathbf{M}^{-1}\mathbf{b}$ for $x = [A; B; R; T]$. Solving Eqn. (87) in MATLAB, we get

$$R(\omega) = -\frac{1}{N}(Z_1'(0)Z_2'(10) - Z_2'(0)Z_1'(10) - ik_z(Z_1(0)Z_2'(10) - Z_2(0)Z_1'(10)) \tag{88}$$
$$- ik_z'(Z_1'(0)Z_2(10) - Z_2'(0)Z_1(10)) - k_z k_z'(Z_1(0)Z_2(10) - Z_2(0)Z_1(10))) \tag{89}$$
$$T(\omega) = \frac{1}{N}2e^{-i10k_z'} ik_z(Z_1(10)Z_2'(10) - Z_2(10)Z_1'(10)) \tag{90}$$

where

$$N = Z_1'(0)Z_2'(10) - Z_2'(0)Z_1'(10) + ik_z(Z_1(0)Z_2'(10) - Z_2(0)Z_1'(10)) \tag{91}$$
$$- ik_z'(Z_1'(0)Z_2(10) - Z_2'(0)Z_1(10)) + k_z k_z'(Z_1(0)Z_2(10) - Z_2(0)Z_1(10)). \tag{92}$$

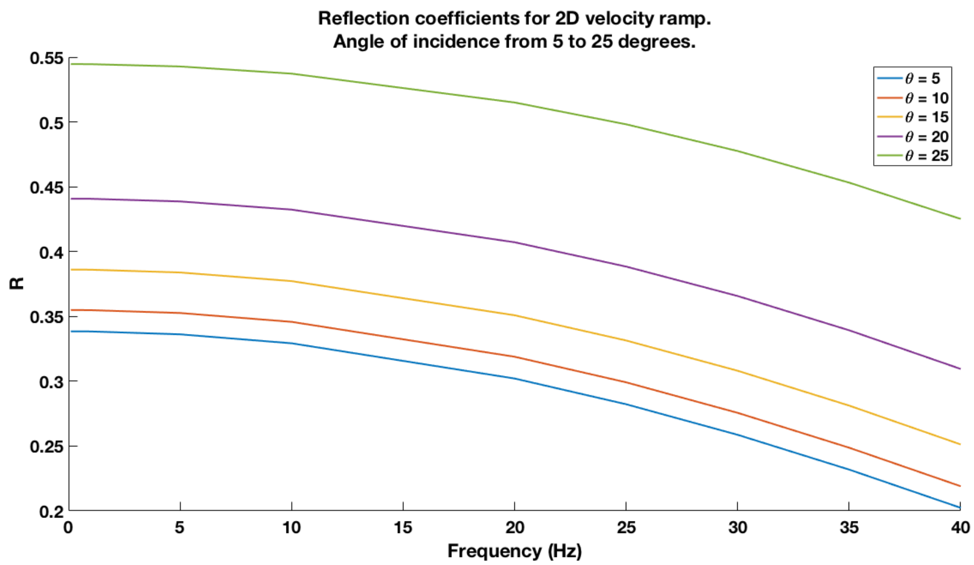

FIG. 6: Reflection coefficients for non-normal incident case for the 2D velocity ramp.

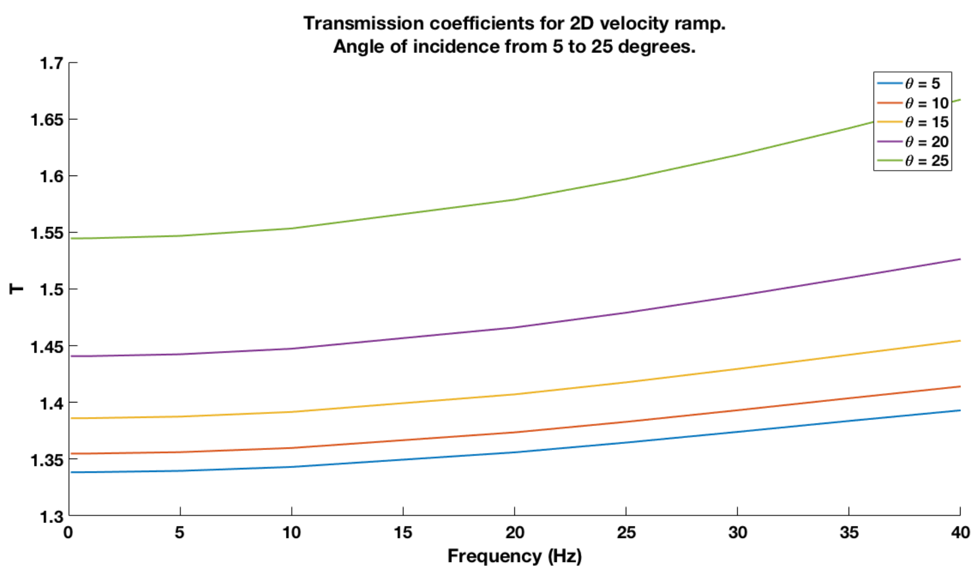

FIG. 7: Transmission coefficients for non-normal incident case for the 2D velocity ramp.

Now, we can apply Eqn. (88) to the solution of Eqn. (59) for the 2D velocity ramp (39) and consider the reflection coefficient for different θ_1. Choosing a range of ω and specific θ_1, we find the remaining parameters k_x, k_z, and k'_z using Eqn. 71.

In Fig. 6, we compare the results of six different values of the incident angle θ_1 for the reflection coefficient. In particular, we consider $\theta_1 = 5°, 10°, 15°, 20°, 25°$, and $30°$. Using Snell's law (Eqn. 74), we can find θ_2. In Fig. 7, we compare the results of six different values of the incident angle θ for the transmission coefficient.

FUTURE WORK

Our next step will be to consider the case when the modulus varies for the non-normal incidence case. From there, we would find the exact solution of the reflection coefficients for general α. Extensions to 3D would be of interest for all velocities studied in this paper.

CONCLUSIONS

We have extended the work of the authors in (Lamoureux et al., 2012) and (Lamoureux et al., 2013) to two dimensions, but first we began by finding an exact solution for the reflection coefficient for the 1D case when a velocity jump is present. Due to the impedance being constant when $\alpha = 0$, we saw that the velocity jump reflected no portion of the wave when $\alpha = 1$. We found the reflection coefficient solutions to the case when a 2D velocity jump was present as well as the case when there was a 2D velocity ramp. In both cases, we considered a plane wave hitting the jump or ramp at normal incidence. For the 2D velocity jump, the exact solution for the reflection coefficient was exactly the same as the 1D case. Finally, we looked at the non-normal incidence case when density varied for the 2D velocity ramp and found an equation for the reflection coefficient with respect to k_x, k_z, ω, and k'_z. Finally, we compared reflection coefficients for different incidence angles θ_1.

ACKNOWLEDGMENTS

We thank the sponsors of CREWES for their support. We also gratefully acknowledge support from NSERC (Natural Science and Engineering Research Council of Canada) through the grant CRDPJ 461179-13 and through a Discovery Grant for the second author.

REFERENCES

Lamoureux, M. P., Gibson, P. C., and Margrave, G. F., 2012, Reflection coefficients through a linear velocity ramp, in 1d: CREWES Research Report, **24**, 1–12.

Lamoureux, M. P., Gibson, P. C., and Margrave, G. F., 2013, Can we see a velocity ramp, in reflection seismic data: CREWES Research Report, **25**, 1–15.