Significance and behaviour of the homogeneous and inhomogeneous components of linearized viscoelastic reflection coefficients

Shahpoor Moradi and Kris Innanen

ABSTRACT

In a recent paper, seismic amplitude-variation-with-offset (AVO) equations describing P-to-P and P-to-S reflections from boundaries separating low-loss viscoelastic media, with account taken for variation in attenuation angle, have been derived. We find that opportunities now present themselves to use these equations to expose a range of relationships between measured amplitudes and subsurface elastic and anelastic properties. This has significant applicability in quantitative interpretation of seismic data in, for instance, reservoir characterization. To facilitate the analysis we decompose the equations into three parts: elastic, homogeneous and inhomogeneous. We show that, for PP modes, the elastic part is sensitive to changes across a reflecting boundary in density and P- and S-wave velocities; the homogeneous part is sensitive to changes in density, S-wave velocity and the P- and S-wave quality factors; and the inhomogeneous part is sensitive to changes in density, and P-and S-wave velocities. The latter term is seen to vanish when the attenuation angle vanishes. Elastic and homogeneous terms are linear with respect to $\sin^2\theta_{\mathrm{P}}$, where θ_{P} is the P-wave incidence angle, however the inhomogeneous term is similarly linear only if normalized by dividing by $\sin\theta_{\mathrm{P}}$. For PS modes, the elastic part is sensitive to changes in density and S-wave velocity; the homogeneous part is sensitive to changes in density, S-wave velocity and the S-wave quality factor; and the inhomogeneous part is sensitive to changes in density and S-wave velocity. This term also vanishes for zero attenuation angle, i.e., in the homogenous limit. For PS modes, the inhomogeneous terms are linear with respect to $\sin^2\theta_{\mathrm{P}}$, however the elastic and homogeneous terms are first and third order in $\sin\theta_{\mathrm{P}}$. A further and key result of this expansion of the wave types allowable in AVO analysis is that, for inhomogeneous PS scattering, the viscoelastic AVO equations predict a non-zero reflectivity at normal incidence. This is a significant deviation from common models of converted wave amplitude analysis.

INTRODUCTION

A primary aim of modern exploration and monitoring seismology is to determine geological and engineering-relevant properties of subsurface hydrocarbon reservoirs from the travel time, phase and amplitude information in seismic reflections. Amplitude-variation-with-offset (AVO) analysis and inversion, in its various forms (Castagna and Backus, 1993; Foster et al., 2010), is a key driving technology in this effort. The problem of robustly analyzing and inverting reflected seismic amplitudes is complex and incompletely solved, requiring (1) integrated seismic acquisition, data processing, and image-forming techniques to produce seismic amplitudes of appropriate fidelity, and (2) accurate, robust, and intelligible formulae and algorithms for modelling and inverting these amplitudes. In the latter domain, research has been active in recent years to understand the limits of standard approximate solutions for reflection amplitudes, connect them with auxiliary geological information to infer increasingly specific reservoir engineering properties cite, and incorporate more complete physics, for instance the anisotropy and/or viscosity arising from the presence of fractures and fluids. In this paper we continue to address the problem of accommodating a maximal amount of the complexity and richness of viscoelastic wave propagation in AVO theory and practice.

Including wave physics beyond the elasticity and isotropy of standard AVO analysis and inversion brings benefit and difficulty. The benefit is an increase in the information derivable from the data. If, for instance, attenuation due to changes in reservoir viscosity influences seismic observations, an extended model will permit that information to be used and this extra rock/fluid property will be at least in principle inferrable. The difficulty is that, for each additional parameter we ask the seismic data to constrain, the more stress is placed on the acquisition-processing-imaging chain to provide accurate data over a wider range of angles and azimuths. As more "difficult" elastic properties are sought, examples of which are density and various anisotropic parameters, often the limits of our mathematical models are reached, and the data variations needed to determine them increasingly tend to occur near the limits of our experimental apertures.

Thus when we extend the reach of seismic amplitude analysis by including a more complete model of wave physics, it is insufficient to simply write down new mathematics and leave it at that. An extension of (say) the Zoeppritz equations cite to incorporate a viscoelastic model, which is part of the work represented in this and the preceding papers, is a starting point only. To activate new technologies wherein attenuative properties of the subsurface can be inferred from seismic measurements requires (1) useable and interpretable approximations in addition to the complex exact equations, (2) an understanding of the behaviour and accuracy of approximations and exact solutions as the quantities related to seismic acquisition reach their limits (e.g., incidence angle and/or maximum source-receiver offset), (3) an understanding of regimes in which data variations caused by new and/or additional parameters are prevalent, and (4) an understanding of if, and how, these data variations can be used to determine simultaneous changes in all of the rock properties as they generally occur in the Earth. The mathematics of the viscoelastic AVO equations, exact and linearized, with attenuation angle incorporated, have been discussed in previous work cite; here we report on developments of these secondary but critical parts of the full problem.

Approximate reflection coefficients for weak contrast interfaces separating elastic isotropic media are well-established. These equations linearly depend on the fractional changes in density, P-wave and S-wave velocities weighted by trigonometric functions of incident angle (Aki and Richards, 2002). In the presence of anelasticity, in which the polarization vector and the ray parameter are complex-valued, reflection coefficients are complex functions (Krebes, 1983, 1984; Ursin and Stovas, 2002; Moradi and Innanen, 2015a, 2016). The corresponding linearized AVO equations not only depend on the changes in elastic properties across the boundary but also depend on the changes in P- and S-wave quality factors weighted by trigonometric functions of incident phase and attenuation angles. The problem of determining exact and approximate reflection and transmission coefficients at a plane interface between two viscoelastic media for homogenous waves was studied by (Ursin and Stovas, 2002). The authors concluded that the approximate PP and PS reflectivities are very similar to the exact solutions of the associated Zoeppritz equations. The same authors generalized the problem to incorporate transversely isotropic viscoelastic problems (Stovas and Ursin, 2003). The effects of attenuation on PP- and PS-wave reflection coefficients for anisotropic viscoelastic media with the main emphasis on transversely isotropic models with a vertical symmetry axis has also been treated (Behura and Tsvankin, 2009a). These authors allowed for inhomogeneity in the waves, assuming that the attenuation angles across a reflecting boundary remain constant, but pointing out that the inhomogeneity angle can make a substantial contribution to the AVO response for strongly attenuative media (Behura and Tsvankin, 2009b).

The related but more general problem of scattering of seismic waves from viscoelastic inclusions in the context of Born approximation has also been recently investigated. A comprehensive mathematical framework for scattering, building from Borcherdt's layered-medium formalism has been developed for the purposes of modeling, processing, and inversion of seismic data exhibiting non-negligible intrinsic attenuation (Moradi and Innanen, 2015b). It was further shown that either independently or beginning from this scattering theory, linearized forms of PP, PS, and SS reflection coefficients for low-contrast interfaces separating two arbitrary low-loss viscoelastic media for arbitrary incident angle are derivable (Moradi and Innanen, 2016, 2015a). These equations relate the AVO response to anelastic parameters. In that work it was shown how the reflectivity depends upon perturbations in elastic properties and on perturbations in quality factors for P- and S-waves. These equations are expected to be of practical importance in the characterization of viscosity and viscosity changes in unconventional reservoirs (see e.g. David Gray, CSEG 2016). One feature of our approach in deriving the linearized AVO equation is that Snell's law and its linearized form is properly accounted for in the linearization, which has typically been assumed to be constant (Behura and Tsvankin, 2009a,b). AVO formulas of this kind represent physically more complete versions of those which underlie anelastic inversion procedures (Innanen, 2011, 2012). Updates to these inversion procedures making use of these more complete expressions is in progress.

In this paper we are concerned with a certain decomposition of the approximate viscoelastic reflection coefficients, and a qualitative and quantitative analysis of the results. Specifically, effects of the attenuation angle and the quality factors will be focused on. We shall see that the attenuation angle has very significant qualitative effect on the AVO responses, and we offer some ideas of the relative importance of the AVO equations for homogeneous and inhomogeneous waves. Another result of this paper is that some insight into common alternative forms for the AVO equations (e.g., the Shuey approximation cite), adjusted for attenuative media, is provided. Again incorporating into these re-written equations the change in attenuation angle across the boundary is unexplored territory. Because we develop results for P-to-S conversions as well as standard P-to-P reflections, our analysis is applicable to linear and nonlinear inversion of multicomponent seismic data (Margrave et al., 2001; Lehocki et al., 2014; Jerez, 2003).

This paper is organized as follows. In section 2 we briefly introduce notation for the complex ray parameter and slowness vector for inhomogeneous waves in low-loss viscoelastic media. In section 3 we apply the Snell's law to decompose the vertical slowness for reflected and transmitted waves. It is shown that the vertical slowness for P- and S-wave is a function of incident attenuation angle. In section 4 we apply the method that we developed in previous section to the decomposition of the exact solutions of the viscoelastic Zoeppritz equations. In section 5 we lay out the viscoelastic version of the Shuey approximation for PP-reflection coefficients, and compare the decomposed exact reflectivities with the approximate ones for two reservoir rock models. Finally we produce some useful approximations for converted PS-wave.

PRELIMINARIES

In viscoelastic media there are three types of waves: P, Type-I S, and Type-II S. These may be homogeneous our inhomogeneous, depending on whether their propagation and attenuation vectors are, respectively, parallel or not (Borcherdt, 2009). Polarization vectors for inhomogeneous P and SI-waves are elliptical; for homogeneous waves they are linear. The elliptical motion reduces to linear motion in the homogeneous limit. The SI-wave is the generalization of the elastic SV wave, and the SII-wave is the generalization of the elastic SH wave, with the former reducing to the latter as attenuation goes to zero. The SII-wave involves linear particle motion perpendicular to the propagation-attenuation plane in both homogeneous and inhomogeneous cases. In reflection problems, if the incident wave is an inhomogeneous P-wave, the reflected wave can be an inhomogeneous P- or SI-wave. For an inhomogeneous wave, the ray parameter and slowness vector not only depend to the phase angle but also on the attenuation angle.

To properly compute linear solutions of the Zoeppritz equations, generalized to accommodate viscoelasticity, to facilitate AVO analysis in the presence of attenuation, we must define polarization and slowness vectors with some care. In a viscoelastic medium, the wavenumber vector is a complex vector whose real part characterizes the direction of wave propagation and whose imaginary part characterizes the attenuation of the wave. This is laid out by, e.g., Borcherdt (2009) who presents a complete theory for seismic waves propagating in layered viscoelastic medium. Borcherdt's formulation predicts a range of transverse, inhomogeneous wave types unique to viscoelastic media (the Type I and II S waves discussed above), including rules for conversion of one type to another during interactions with planar boundaries. As a result of the complexity of the wavenumber vector, slowness and polarization vectors are complex functions. The complex wave-number vector is given by

$$\mathbf{K} = \mathbf{P} - i\mathbf{A}, \tag{1}$$

where the propagation vector $\mathbf{P}$ is perpendicular to the wavefront, and specifies the direction of propagation, and the attenuation vector $\mathbf{A}$ is perpendicular to the plane of constant amplitude, and specifies the direction of maximum attenuation. The angle between these two vectors is always less that 90° and is referred to as the attenuation angle (Fig.1). If the attenuation and propagation vectors are parallel and $\delta = 0$ the wave is homogenous; otherwise it is inhomogeneous. In the case of low-loss viscoelastic media, in which the quality factor $Q^{-1} \ll 1$, the propagation and attenuation vectors can be written (Borcherdt, 2009)

$$\begin{aligned} \mathbf{P} &= \frac{\omega}{V_{\mathrm{E}}}(\mathbf{x}\sin\theta + \mathbf{z}\cos\theta), \\ \mathbf{A} &= \frac{\omega}{V_{\mathrm{E}}}Q^{-1}\sec\delta\left[\mathbf{x}\sin(\theta-\delta) + \mathbf{z}\cos(\theta-\delta)\right], \end{aligned} \tag{2}$$

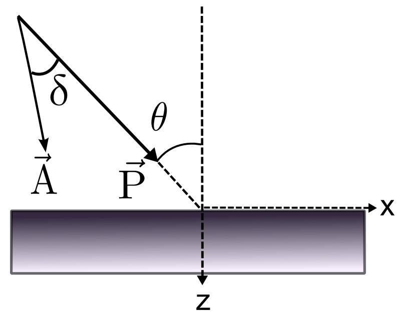

FIG. 1. Incident inhomogeneous wave ($\delta \neq 0$). $\mathbf{P}$ is the propagation vector, $\mathbf{A}$ is the attenuation vector, θ is the incident phase angle and δ is the incident attenuation angle.

where V_{E} is either P-wave or S-wave velocity, θ is the phase angle and δ is the attenuation angle. For an inhomogeneous wave, the ray parameter and the vertical slowness are complex functions which depend on the quality factor and attenuation angles (Moradi and Innanen, 2016). These quantities can be split into elastic, homogeneous and inhomogeneous parts

$$\begin{aligned} p &= p_{\mathrm{E}} + ip_{\mathrm{H}} + ip_{\mathrm{IH}}, \\ q &= q_{\mathrm{E}} + iq_{\mathrm{H}} + iq_{\mathrm{IH}}, \end{aligned} \tag{3}$$

where the components are given by

$$\begin{aligned} p_{\mathrm{E}} &= \frac{\sin\theta}{V_{\mathrm{E}}}, & p_{\mathrm{H}} &= -\frac{Q^{-1}}{2}p_{\mathrm{E}}, & p_{\mathrm{IH}} &= \frac{Q^{-1}}{2}q_{\mathrm{E}}\tan\delta \\ q_{\mathrm{E}} &= \sqrt{V_{\mathrm{E}}^{-2} - p_{\mathrm{E}}^{2}}, & q_{\mathrm{H}} &= -\frac{Q^{-1}}{2}q_{\mathrm{E}}, & q_{\mathrm{IH}} &= -\frac{Q^{-1}}{2}p_{\mathrm{E}}\tan\delta \end{aligned} \tag{4}$$

In the above relations the indexes E, H and IH respectively refer to the elastic, homogeneous and inhomogeneous parts. Reflected and transmitted angles can be obtained in terms of the incident angle from Snell's law. For a low-contrast, two layered medium, the deviation of transmitted angle away from the incident angle is small. Consequently, we can linearize Snell's law to obtain a simple expression of the difference between the incident and transmitted wave in terms of changes in velocity between the layers. Snell's law for viscoelastic materials is discussed by Wennerberg (Wennerberg, 1985) and Borcherdt (Borcherdt, 2009). Importantly for our purposes, since the attenuation angle changes across the boundary, the linearized form of Snell's law also gives us in a simple form the difference in attenuation angle in terms of changes in velocity and quality factor (Moradi and Innanen, 2016).

HOMOGENEOUS AND INHOMOGENEOUS PARTS OF THE SLOWNESSES

To decompose the reflectivity into the contributions from inhomogeneity of wave, first we need to split the slowness vector to elastic, homogeneous and inhomogeneous parts. Let an inhomogeneous P-wave be incident in medium 1 on a horizontal interface. The P-wave transmitted through the interface has vertical slowness

$$q_{\mathrm{P2}} = q_{\mathrm{PE2}} + iq_{\mathrm{PA2}}, \tag{5}$$

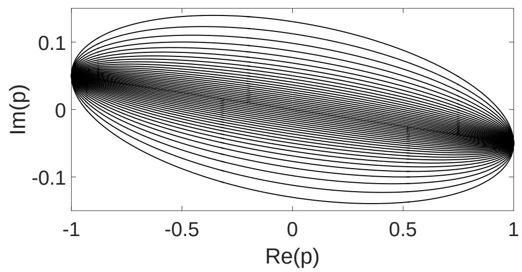

FIG. 2. Plots of the viscoelastic ray parameter p in the complex plane for $Q = 10$ over a range of attenuation angles, ranging from $0°$ to $70°$.

where the elastic and anelastic parts, q_{PE2} and q_{PA2} respectively, are given by

$$q_{\mathrm{PE2}} = \sqrt{V_{\mathrm{PE2}}^{-2} - p_{\mathrm{E}}^2}, \quad \text{and} \tag{6}$$

$$q_{\mathrm{PA2}} = -\frac{Q_{\mathrm{P2}}^{-1}}{2}(q_{\mathrm{PE2}} + p_{\mathrm{E}} \tan \delta_{\mathrm{P2}}). \tag{7}$$

In equation (7), subscript A labels the anelastic part of the vertical slowness. In the same way, for the reflected S-wave we have

$$q_{\mathrm{SE1}} = \sqrt{V_{\mathrm{SE1}}^{-2} - p_{\mathrm{E}}^2} \tag{8}$$

$$q_{\mathrm{SA1}} = -\frac{Q_{\mathrm{S1}}^{-1}}{2}(q_{\mathrm{SE1}} + p_{\mathrm{E}} \tan \delta_{\mathrm{S1}}), \tag{9}$$

and for the transmitted S-wave

$$q_{\mathrm{SE2}} = \sqrt{V_{\mathrm{SE2}}^{-2} - p_{\mathrm{E}}^2} \tag{10}$$

$$q_{\mathrm{SA2}} = -\frac{Q_{\mathrm{S2}}^{-1}}{2}(q_{\mathrm{SE2}} + p_{\mathrm{E}} \tan \delta_{\mathrm{S2}}). \tag{11}$$

To separate the homogeneous and inhomogeneous components of these relations we must invoke Snell's law, because the transmitted and reflected attenuation angles are functions of the incident attenuation angle (see Appendix). We have

$$\tan \delta_{P2} = \frac{1}{q_{\mathrm{PE2}}}\left[p_{\mathrm{E}} - \frac{Q_{P2}}{Q_{P1}}\left(p_{\mathrm{E}} - q_{\mathrm{PE1}} \tan \delta_{P1}\right)\right], \tag{12}$$

where δ_{P2} is the attenuation angle for the transmitted P-wave, and δ_{P1} is the attenuation angle for the incident P-wave. For the reflected S-wave, the attenuation angle δ_{S1} is

$$\tan \delta_{S1} = \frac{1}{q_{\mathrm{SE1}}}\left[p_{\mathrm{E}} - \frac{Q_{S1}}{Q_{P1}}\left(p_{\mathrm{E}} - q_{\mathrm{SE1}} \tan \delta_{P1}\right)\right], \tag{13}$$

and for the transmitted S-wave the attenuation angle δ_{S2} is

$$\tan \delta_{S2} = \frac{1}{q_{\mathrm{SE2}}}\left[p_{\mathrm{E}} - \frac{Q_{S2}}{Q_{P1}}\left(p_{\mathrm{E}} - q_{\mathrm{SE2}} \tan \delta_{P1}\right)\right]. \tag{14}$$

By substituting equation (12) into (7), and (13) into (9), and finally (14) into (11) we can decompose the transmitted P-wave slowness vector into homogenous and inhomogeneous parts,

$$q_{\text{PAH2}} = -\frac{1}{2} Q_{\text{P2}}^{-1} \frac{\cos\theta_{\text{P2}}}{V_{\text{P2}}} - \frac{1}{2} \frac{\sin\theta_{\text{P2}}}{V_{P2}} \tan\theta_{\text{P2}} (Q_{\text{P2}} - Q_{\text{P1}}) \tag{15}$$

$$q_{\text{PAIH2}} = -\frac{1}{2} Q_{\text{P1}}^{-1} \frac{\cos\theta_{P1}}{V_{\text{P1}}} \tan\theta_{P2} \tan\delta_{P1} \tag{16}$$

respectively. Similarly for the reflected S-wave we obtain

$$q_{\text{SAH1}} = -\frac{1}{2} Q_{\text{S1}}^{-1} \frac{\cos\theta_{\text{S1}}}{V_{\text{S1}}} - \frac{1}{2} \frac{\sin\theta_{\text{S1}}}{V_{S1}} \tan\theta_{\text{S1}} (Q_{\text{S1}} - Q_{\text{P1}}) \tag{17}$$

$$q_{\text{SAIH1}} = -\frac{1}{2} Q_{\text{P1}}^{-1} \frac{\cos\theta_{P1}}{V_{\text{P1}}} \tan\theta_{S1} \tan\delta_{P1}, \tag{18}$$

and for the transmitted S-wave

$$q_{\text{SAH2}} = -\frac{1}{2} Q_{\text{S2}}^{-1} \frac{\cos\theta_{\text{S2}}}{V_{\text{S2}}} - \frac{1}{2} \frac{\sin\theta_{\text{S2}}}{V_{P2}} \tan\theta_{\text{S2}} (Q_{\text{S2}} - Q_{\text{P1}}) \tag{19}$$

$$q_{\text{PAIH2}} = -\frac{1}{2} Q_{\text{P1}}^{-1} \frac{\cos\theta_{P1}}{V_{\text{P1}}} \tan\theta_{S2} \tan\delta_{P1}. \tag{20}$$

These relations separate the homogeneous from the inhomogeneous components of the slownesses, in the sense that for a purely homogeneous P-wave, with zero attenuation angle $\delta_{P1} = 0$, the components labelled inhomogeneous vanish. Let us next use this complex Snell's law decomposition procedure for viscoelastic ray parameters and vertical slownesses to analyze the viscoelastic reflectivity.

DECOMPOSITION OF SOLUTIONS OF THE VISCOELASTIC ZOEPPRITZ EQUATIONS

Our interest is to be able to separately analyze and predict behaviour of the homogeneous versus inhomogeneous components of viscoelastic waves having reflected from and transmitted through a planar boundary. Consider two homogeneous viscoelastic half-spaces, in which the upper half-space is characterized by the density ρ_1, P-wave velocity V_{PE}, S-wave velocity V_{SE}, P-wave quality factor Q_{P} and S-wave quality factor Q_{S}. Each of these experiences a jump in transitioning to the lower half-space, where the parameters are labelled with the subscript 2. A plane P-wave incident on the boundary between the two half-spaces generates reflected and transmitted P- and S-waves. Solutions of the purely elastic-isotropic Zoeppritz equations can be straightforwardly extended to correspond to exact PP and PS reflection coefficients in this viscoelastic case. This is done by substituting the complex ray parameter, slowness vector and velocities discussed in the previous section.

These solutions are complicated nonlinear functions of the changes in both elastic and anelastic parameters (Aki and Richards, 2002; Ikelle and Amundsen, 2005; Moradi and

	Shale	Salt	Limestone	Limestone(gas)
V_{PE}(km/s)	3.811	4.537	5.335	5.043
V_{SE}(km/s)	2.263	2.729	2.957	2.957
$\rho(\mathrm{gm/cm}^3)$	2.40	2.005	2.65	2.49

Table 1. Density, P and S-wave velocity used in the numerical tests for shale, salt, limestone and limestone(gas). For all models we assumed the P- and S-wave quality factors as $Q_{S1} = 5$, $Q_{S2} = 7$, $Q_{P1} = 9$ and $Q_{P2} = 11$.

Innanen, 2016):

$$\mathrm{R}_{\mathrm{PP}} = \frac{c_1 d_2 - c_3 d_4}{d_1 d_2 + d_3 d_4}, \tag{21}$$

$$\mathrm{R}_{\mathrm{PS}} = -\left(\frac{V_{\mathrm{P1}}}{V_{\mathrm{S1}}}\right) \frac{c_3 d_1 + c_1 d_3}{d_1 d_2 + d_3 d_4} \tag{22}$$

where

$$d_1 = -2p^2 \Delta M (q_{P1} - q_{P2}) + (\rho_1 q_{P2} + \rho_2 q_{P1}), \tag{23}$$

$$d_3 = -p \left[2\Delta M (q_{P1} q_{S2} + p^2) - \Delta\rho\right], \tag{24}$$

and where $d_2 = d_1$ (but with $q_P \rightarrow q_S$), $d_4 = d_3$ (with $q_P \leftrightarrow q_S$), $c_1 = d_1$ (with $q_{P2} \rightarrow -q_{P2}$), $c_2 = d_2$ (with $q_{S2} \rightarrow -q_{S2}$), $c_3 = d_3$ (with $q_{P1} \rightarrow -q_{P1}$), and $c_4 = d_4$ (with $q_{S1} \rightarrow -q_{S1}$). In above equations $\Delta\rho = \rho_2 - \rho_1$ is the difference between the density in lower and upper media and $\Delta M = \Delta\mu + i\Delta\mu_A$ is the change in the complex modulus across the boundary:

$$\Delta\mu_E = \rho_2 V_{SE2}^2 - \rho_1 V_{SE1}^2 = \mu_{E2} - \mu_{E1}, \tag{25}$$

and

$$\Delta\mu_A = \rho_2 V_{SE2}^2 Q_{S2}^{-1} - \rho_1 V_{SE1}^2 Q_{S1}^{-1} = Q_{S2}^{-1} \mu_{E2} - Q_{S1}^{-1} \mu_{E1}. \tag{26}$$

In Figure 3 the real parts of reflection coefficients for PP and PS modes at boundaries between two viscoelastic half-spaces, with quality factors near and below 10, are plotted. The elastic properties are selected to correspond with natural geological boundaries as discussed below. The coefficients are plotted three times each, with attenuation angle varying from $0°$, to $45°$, and to $70°$; and angles of incidence up to $40°$ are included. The solid line refers to the homogeneous wave with zero attenuation angle; the dotted line is the inhomogeneous wave with moderate incident attenuation angle $\delta_P = 45°$ and the dashed line is the highly attenuative wave with $\delta_P = 70°$. The reflectivity curves in the homogeneous and moderate attenuation cases approach each other near normal incidence, but in general the attenuation angle can be seen to have a significant impact on reflection amplitudes. For the PP-reflection coefficients and shale/salt and shale/limestone models, the attenuation angle has its largest influence for angles greater than $20°$.

The influence of anelastic parameters and attenuation angles on the reflection coefficients in equations (21) and (22) is not easy to analyze in this unaltered form. To address this we decompose the reflectivity into three components, elastic, anelastic homogeneous

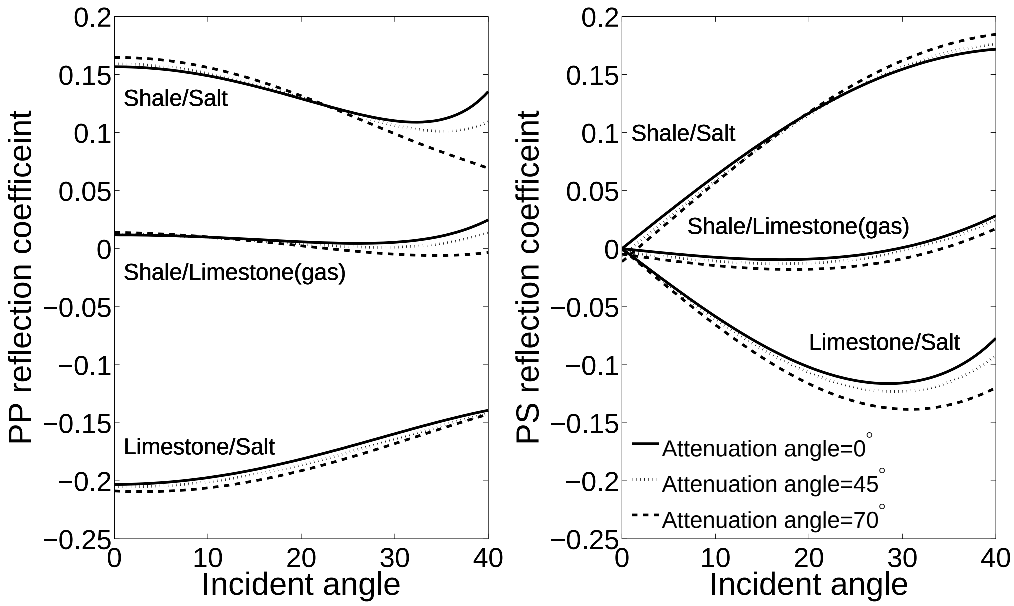

FIG. 3. Comparing the real part of the exact viscoelastic PP-reflectivity for $\delta_P = 0°, 45°, 70°$ for four selected models from table 1.

and anelastic inhomogeneous. We insert the decomposed ray and slowness parameters calculated in previous section into d_i and c_i, obtaining

$$d_j = d_{\mathrm{Ej}} + id_{\mathrm{Hj}} + id_{\mathrm{IHj}}, \tag{27}$$

$$c_j = c_{\mathrm{Ej}} + ic_{\mathrm{H4}} + ic_{\mathrm{IHj}}, \quad j = 1, 2, 3, 4. \tag{28}$$

The detailed form of d_1, for instance, is

$$\begin{aligned} d_1^{\mathrm{E}} =& -2p_E^2\Delta\mu_E(q_{\mathrm{P1}}^{\mathrm{E}} - q_{\mathrm{P2}}^{\mathrm{E}}) + (\rho_2 q_{\mathrm{P1}}^{\mathrm{E}} + \rho_1 q_{\mathrm{P2}}^{\mathrm{E}}) \\ d_1^{\mathrm{H}} =& -2p_E^2\Delta\mu_E(q_{\mathrm{P1}}^{\mathrm{H}} - q_{\mathrm{P2}}^{\mathrm{H}}) - 2(p_E^2\Delta\mu_A + 2p_E p_{\mathrm{H}}\Delta\mu_E)(q_{\mathrm{P1}}^{\mathrm{E}} - q_{\mathrm{P2}}^{\mathrm{E}}) + \rho_2 q_{\mathrm{P1}}^{\mathrm{H}} + \rho_1 q_{\mathrm{P2}}^{\mathrm{H}} \\ d_1^{\mathrm{IH}} =& -2p_E^2\Delta\mu_E(q_{\mathrm{P1}}^{\mathrm{IH}} - q_{\mathrm{P2}}^{\mathrm{IH}}) - 4p_E p_{\mathrm{IH}}\Delta\mu_E(q_{\mathrm{P1}}^{\mathrm{E}} - q_{\mathrm{P2}}^{\mathrm{E}}) + \rho_2 q_{\mathrm{P1}}^{\mathrm{IH}} + \rho_1 q_{\mathrm{P2}}^{\mathrm{IH}}. \end{aligned}$$

More detail of dependency of this parameters to the medium properties can be found in Appendix A. To compute the decomposed reflectivity we note that in low-loss viscoelastic media all terms involving the product of homogeneous and inhomogeneous terms are negligible (for instance, $d^H d^{IH} \approx (d^H)^2 \approx (d^{IH})^2 \approx 0$). Taking into account the low-loss aspect of the media, and inserting (27) to (28) into the (21) and (22), we arrive at

$$\mathrm{R_{PP}} = \mathrm{R_{PP}^{E}} + i\mathrm{R_{PP}^{H}} + i\mathrm{R_{PP}^{IH}}, \tag{29}$$

$$\mathrm{R_{PS}} = \mathrm{R_{PS}^{E}} + i\mathrm{R_{PS}^{H}} + i\mathrm{R_{PS}^{IH}}, \tag{30}$$

where $\mathrm{R_{PP}^{E}}$ is the elastic reflectivity, i.e., the reflectivity in the absence of attenuation:

$$\mathrm{R_{PP}^{E}} = \frac{c_{\mathrm{E1}} d_{\mathrm{E2}} - c_{\mathrm{E3}} d_{\mathrm{E_4}}}{d_{\mathrm{E_1}} d_{\mathrm{E_2}} + d_{\mathrm{E_3}} d_{\mathrm{E_4}}},$$

$\mathrm{R}_{\mathrm{PP}}^{\mathrm{H}}$ is the homogeneous anelastic term, i.e., the term which remains when the attenuation angle is zero,

$$\begin{aligned}\mathrm{R}_{\mathrm{PP}}^{\mathrm{H}} = &- \mathrm{R}_{\mathrm{PP}}^{\mathrm{E}} \frac{d_{\mathrm{E2}}d_{\mathrm{H1}} + d_{\mathrm{E1}}d_{\mathrm{H2}} + d_{\mathrm{E3}}d_{\mathrm{H4}} + d_{\mathrm{E4}}d_{\mathrm{H3}}}{d_{\mathrm{E1}}d_{\mathrm{E2}} + d_{\mathrm{E3}}d_{\mathrm{E4}}} \\ &- \frac{c_{\mathrm{E3}}d_{\mathrm{H4}} + c_{\mathrm{H3}}d_{\mathrm{E4}} - c_{\mathrm{H1}}d_{\mathrm{E2}} - c_{\mathrm{E1}}d_{\mathrm{H2}}}{d_{\mathrm{E1}}d_{\mathrm{E2}} + d_{\mathrm{E3}}d_{\mathrm{E4}}}\end{aligned}$$

and finally $\mathrm{R}_{\mathrm{PP}}^{\mathrm{IH}}$, the inhomogeneous term which appears with non-zero attenuation angle,

$$\begin{aligned}\mathrm{R}_{\mathrm{PP}}^{\mathrm{IH}} = &- \mathrm{R}_{\mathrm{PP}}^{\mathrm{E}} \frac{d_{\mathrm{E2}}d_{\mathrm{IH1}} + d_{\mathrm{E1}}d_{\mathrm{IH2}} + d_{\mathrm{E3}}d_{\mathrm{IH4}} + d_{\mathrm{E4}}d_{\mathrm{IH3}}}{d_{\mathrm{E1}}d_{\mathrm{E2}} + d_{\mathrm{E3}}d_{\mathrm{E4}}} \\ &- \frac{c_{\mathrm{E3}}d_{\mathrm{IH4}} + c_{\mathrm{IH3}}d_{\mathrm{E4}} - c_{\mathrm{IH1}}d_{\mathrm{E2}} - c_{\mathrm{E1}}d_{\mathrm{IH2}}}{d_{\mathrm{E1}}d_{\mathrm{E2}} + d_{\mathrm{E3}}d_{\mathrm{E4}}}.\end{aligned}$$

In the same way three components of PS-reflectivity are

$$\begin{aligned}\mathrm{R}_{\mathrm{PS}}^{\mathrm{E}} &= -\left(\frac{V_{\mathrm{PE1}}}{V_{\mathrm{SE1}}}\right) \frac{c_{\mathrm{E3}}d_{\mathrm{E1}} + c_{\mathrm{E1}}d_{\mathrm{E3}}}{d_{\mathrm{E1}}d_{\mathrm{E2}} + d_{\mathrm{E3}}d_{\mathrm{E4}}} \\ \mathrm{R}_{\mathrm{PS}}^{\mathrm{H}} &= \frac{1}{2}\left(Q_{P1}^{-1} - Q_{S1}^{-1}\right)\mathrm{R}_{\mathrm{PS}}^{\mathrm{E}} \\ &\quad - \mathrm{R}_{\mathrm{PS}}^{\mathrm{E}} \frac{d_{\mathrm{E1}}d_{\mathrm{H2}} + d_{\mathrm{E3}}d_{\mathrm{H4}} + d_{\mathrm{H1}}d_{\mathrm{E2}} + d_{\mathrm{H3}}d_{\mathrm{E4}}}{d_{\mathrm{E1}}d_{\mathrm{E2}} + d_{\mathrm{E3}}d_{\mathrm{E4}}} \\ &\quad - \left(\frac{V_{\mathrm{PE1}}}{V_{\mathrm{SE1}}}\right) \frac{c_{\mathrm{E3}}d_{\mathrm{H1}} + c_{\mathrm{E1}}d_{\mathrm{H3}} + c_{\mathrm{H3}}d_{\mathrm{E1}} + c_{\mathrm{H1}}d_{\mathrm{E3}}}{d_{\mathrm{E1}}d_{\mathrm{E2}} + d_{\mathrm{E3}}d_{\mathrm{E4}}} \\ \mathrm{R}_{\mathrm{PS}}^{\mathrm{IH}} &= -\mathrm{R}_{\mathrm{PS}}^{\mathrm{E}} \frac{d_{\mathrm{E1}}d_{\mathrm{IH2}} + d_{\mathrm{E3}}d_{\mathrm{IH4}} + d_{\mathrm{IH1}}d_{\mathrm{E2}} + d_{\mathrm{IH3}}d_{\mathrm{E4}}}{d_{\mathrm{E1}}d_{\mathrm{E2}} + d_{\mathrm{E3}}d_{\mathrm{E4}}} \\ &\quad - \left(\frac{V_{\mathrm{PE1}}}{V_{\mathrm{SE1}}}\right) \frac{c_{\mathrm{E3}}d_{\mathrm{IH1}} + c_{\mathrm{E1}}d_{\mathrm{IH3}} + c_{\mathrm{IH3}}d_{\mathrm{E1}} + c_{\mathrm{IH1}}d_{\mathrm{E3}}}{d_{\mathrm{E1}}d_{\mathrm{E2}} + d_{\mathrm{E3}}d_{\mathrm{E4}}}.\end{aligned}$$

In summary, the viscoelastic reflection coefficients have real and imaginary parts, with the real part being identical to the solutions of the elastic Zoeppritz equations, and the imaginary part with a more complicated structure, depending on the elastic, anelastic terms as well as the attenuation angle.

THE VISCOELASTIC SHUEY APPROXIMATION

Linearized approximate forms of the solutions to the Zoeppritz equations are typically used in AVO analysis and inversion (Aki and Richards, 2002; Castagna and Backus, 1993; Foster et al., 2010). A range of linearized forms are available, distinguished by their treatment of the plane wave incidence angle and their parameterization of elastic properties and their variation across the reflecting interface; amongst these the Shuey approximation (Shuey, 1985) is one of the most frequently used. Ideally a linearized form (1) only differs from the exact solution by a small amount in the regions (e.g., angle-range) it is employed, (2) provides an intuitive interpretability, (3) leads to stable inversion algorithms, and (4) correctly predicts the main reflection phenomena observed in seismic data. In this section we review and decompose the viscoelastic version of this approximation.

There are two main assumptions made in deriving the equations of linear AVO analysis, firstly, that the relative changes in properties (elastic or anelastic) across the interface are small, and secondly, that the incident angle is well below the critical angle. The small-offset linearized P-to-P reflection coefficient for an inhomogeneous seismic wave reflected from boundary of two isotropic viscoelastic media under the assumption of small contrast interface is given by (Moradi and Innanen, 2016)

$$\mathrm{R}_{\mathrm{PPL}} = \mathrm{R}^{\mathrm{E}}_{\mathrm{PPL}} + i\mathrm{R}^{\mathrm{H}}_{\mathrm{PPL}} + i\mathrm{R}^{\mathrm{IH}}_{\mathrm{PPL}}, \tag{31}$$

where the real part is

$$\mathrm{R}^{\mathrm{E}}_{\mathrm{PPL}} = \frac{1}{2}\left(\frac{\Delta\rho}{\rho} + \frac{1}{\cos^2\theta_{\mathrm{P}}}\frac{\Delta V_{\mathrm{P}}}{V_{\mathrm{P}}}\right) - 2\sin^2\theta_P\left(\frac{V_{\mathrm{S}}}{V_{\mathrm{P}}}\right)^2\left(\frac{\Delta\rho}{\rho} + 2\frac{\Delta V_{\mathrm{S}}}{V_{\mathrm{S}}}\right) \tag{32}$$

the homogeneous-imaginary part $\mathrm{R}^{\mathrm{H}}_{\mathrm{PP}}$ is

$$\mathrm{R}^{\mathrm{H}}_{\mathrm{PPL}} = -2\left(\frac{V_{\mathrm{S}}}{V_{\mathrm{P}}}\right)^2 (Q_{\mathrm{S}}^{-1} - Q_{\mathrm{P}}^{-1})\sin^2\theta_{\mathrm{P}}\left(\frac{\Delta\rho}{\rho} + 2\frac{\Delta V_{\mathrm{S}}}{V_{\mathrm{S}}}\right) \tag{33}$$

$$-\frac{1}{4\cos^2\theta_{\mathrm{P}}}Q_{\mathrm{P}}^{-1}\frac{\Delta Q_{\mathrm{P}}}{Q_{\mathrm{P}}} + 2\sin^2\theta_{\mathrm{P}}\left(\frac{V_{\mathrm{S}}}{V_{\mathrm{P}}}\right)^2 Q_{\mathrm{S}}^{-1}\frac{\Delta Q_{\mathrm{S}}}{Q_{\mathrm{S}}}. \tag{34}$$

and the inhomogeneous-imaginary part $\mathrm{R}^{\mathrm{IH}}_{\mathrm{PP}}$ is

$$\mathrm{R}^{\mathrm{IH}}_{\mathrm{PPL}} = -Q_{\mathrm{P}}^{-1}\tan\delta_{\mathrm{P}}\left[\sin 2\theta_{\mathrm{P}}\left(\frac{V_{\mathrm{S}}}{V_{\mathrm{P}}}\right)^2\left(\frac{\Delta\rho}{\rho} + 2\frac{\Delta V_{\mathrm{S}}}{V_{\mathrm{S}}}\right) - \frac{\tan\theta_{\mathrm{P}}}{2\cos^2\theta_{\mathrm{P}}}\frac{\Delta V_{\mathrm{P}}}{V_{\mathrm{P}}}\right]. \tag{35}$$

Here $\Delta\rho/\rho$ is fractional change in density, with $\Delta\rho = \rho_2 - \rho_1$ and $\rho = (\rho_2 + \rho_1)/2$; $\Delta V_{\mathrm{P}}/V_{\mathrm{P}}$ is fractional change in P-wave velocity, with $\Delta V_{\mathrm{P}} = V_{\mathrm{P2}} - V_{\mathrm{P1}}$ and $V_{\mathrm{P}} = (V_{\mathrm{P2}} + V_{\mathrm{P1}})/2$; $\Delta V_{\mathrm{S}}/V_{\mathrm{S}}$ is fractional change in S-wave velocity, with $\Delta V_{\mathrm{S}} = V_{\mathrm{S2}} - V_{\mathrm{S1}}$ and $V_{\mathrm{S}} = (V_{\mathrm{S2}} + V_{\mathrm{S1}})/2$; $\Delta Q_{\mathrm{P}}/Q_{\mathrm{P}}$ is fractional change in P-wave quality factor, with $\Delta Q_{\mathrm{P}} = Q_{\mathrm{P2}} - Q_{\mathrm{P1}}$ and $Q_{\mathrm{P}} = (Q_{\mathrm{P2}} + Q_{\mathrm{P1}})/2$; $\Delta Q_{\mathrm{S}}/Q_{\mathrm{S}}$ is fractional change in S-wave quality factor, with $\Delta Q_{\mathrm{S}} = Q_{\mathrm{S2}} - Q_{\mathrm{S1}}$ and $Q_{\mathrm{S}} = (Q_{\mathrm{S2}} + Q_{\mathrm{S1}})/2$. In addition $\theta_{\mathrm{P}} = (\theta_{\mathrm{P2}} + \theta_{\mathrm{P1}})/2$ where θ_{P1} is the incident phase angle and θ_{P2} is the transmitted phase angle; $\delta_{\mathrm{P}} = (\delta_{\mathrm{P2}} + \delta_{\mathrm{P1}})/2$ where δ_{P1} is the incident attenuation angle and δ_{P2} is the transmitted attenuation angle. Subscript 1 refers to the upper layer and subscript 2 refers to the lower layer. It can be seen that the anelastic-inhomogeneous term is a function of the fractional changes in density, P- and S-wave velocities.

Equations (31-35) can be rearranged in powers of $\sin\theta_{\mathrm{P}}$ and $\tan\theta_{\mathrm{P}}$:

$$\mathrm{R}_{\mathrm{PPL}}(\theta_{\mathrm{P}}, \delta_{\mathrm{P}}) = \mathrm{R}^{\mathrm{E}}_{\mathrm{PP}}(\theta_{\mathrm{P}}) + i\mathrm{R}^{\mathrm{H}}_{\mathrm{PPL}}(\theta_{\mathrm{P}}) + i\mathrm{R}^{\mathrm{IH}}_{\mathrm{PPL}}(\theta_{\mathrm{P}}, \delta_{\mathrm{P}}), \tag{36}$$

with elastic, anelastic-homogenous and anelastic-inhomogeneous terms given by

$$\mathrm{R}^{\mathrm{E}}_{\mathrm{PPL}}(\theta_{\mathrm{P}}) = A^{\mathrm{E}}_{\mathrm{PP}} + B^{\mathrm{E}}_{\mathrm{PP}}\sin^2\theta_{\mathrm{P}} + C^{\mathrm{E}}_{\mathrm{PP}}(\tan^2\theta_{\mathrm{P}} - \sin^2\theta_P), \tag{37}$$

$$\mathrm{R}^{\mathrm{H}}_{\mathrm{PPL}}(\theta_{\mathrm{P}}) = A^{\mathrm{H}}_{\mathrm{PP}} + B^{\mathrm{H}}_{\mathrm{PP}}\sin^2\theta_{\mathrm{P}} + A^{\mathrm{H}}_{\mathrm{PP}}(\tan^2\theta_{\mathrm{P}} - \sin^2\theta_P), \tag{38}$$

$$\mathrm{R}^{\mathrm{IH}}_{\mathrm{PPL}}(\theta_{\mathrm{P}}, \delta_{\mathrm{P}}) = A^{\mathrm{IH}}_{\mathrm{PP}}\tan\theta_P + B^{\mathrm{IH}}_{\mathrm{PP}}\tan\theta_{\mathrm{P}}\sin^2\theta_{\mathrm{P}} + C^{\mathrm{IH}}_{\mathrm{PP}}\tan\theta_{\mathrm{P}}(\tan^2\theta_{\mathrm{P}} - \sin^2\theta_{\mathrm{P}}), \tag{39}$$

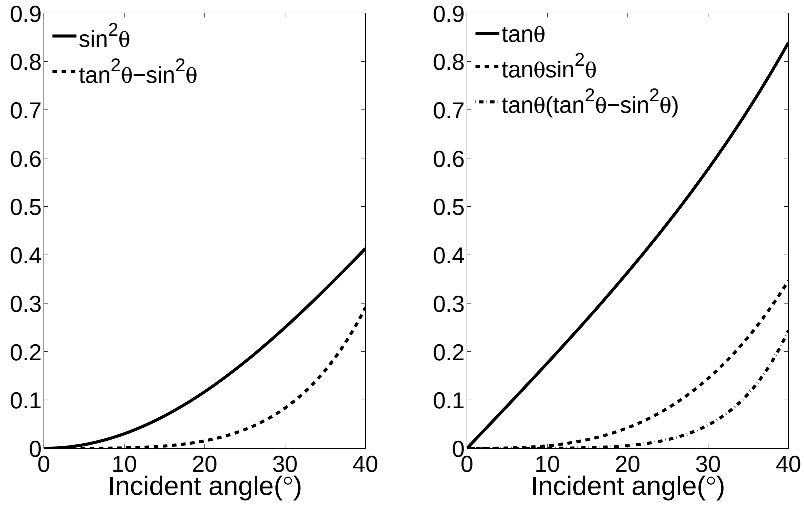

FIG. 4. Comparing the two and three term AVO responses for the decomposed PP reflectivity.

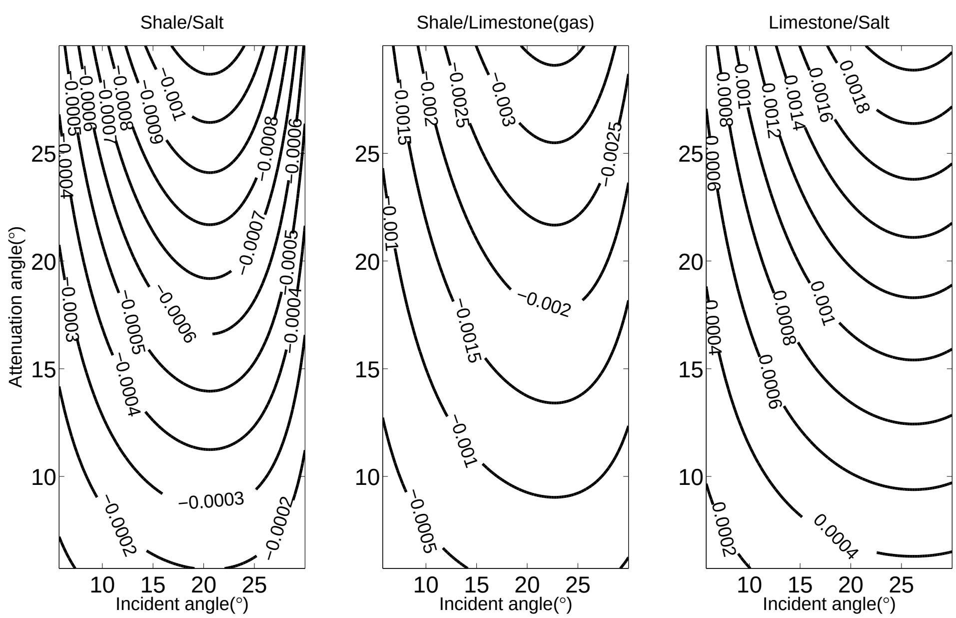

FIG. 5. The maps of the inhomogeneous part of the PP-reflectivity for three models in table 1. The vertical axis indicates the attenuation angle and horizontal axis incident angle.

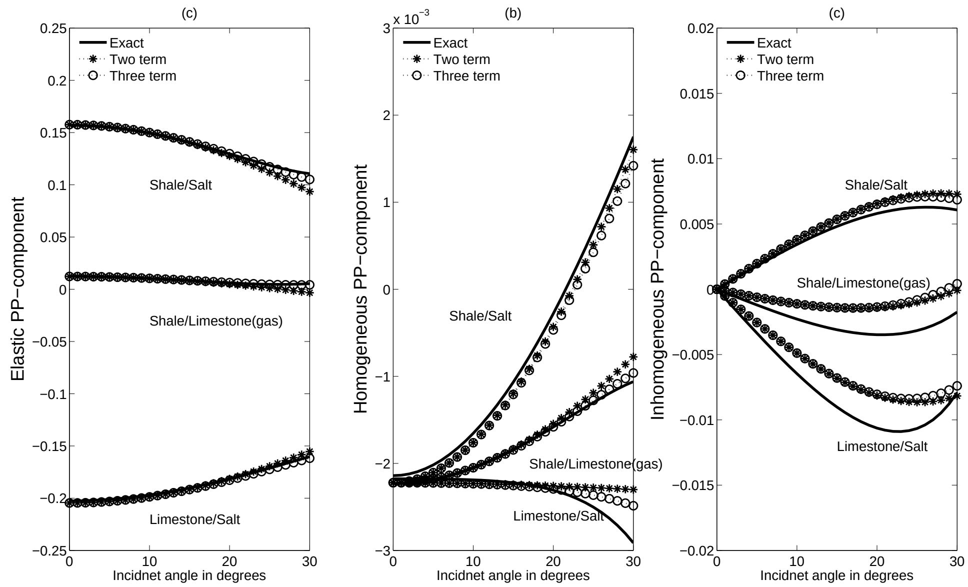

FIG. 6. Components of the PP-reflection coefficients versus incident angle θ_{P} for three two layer mineral models introduced in table 1. Solid line represent the exact reflectivity calculated from the Zoeppritz equation (Eq. 21), star and circle-dot lines respectively are related to the two and three terms (Figs. 6b and 6c). Reflectivity components corresponding to the interface models of Shale/salt, Shale/Limestone(gas) and Limestone/Salt.

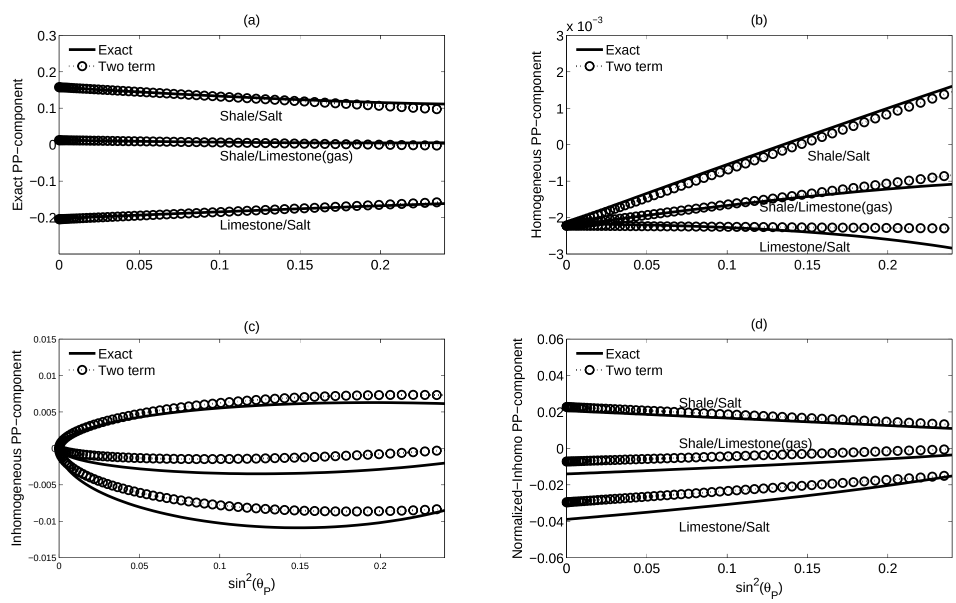

FIG. 7. Components of the PP-reflection coefficients versus $\sin^2\theta_{\mathrm{P}}$ for three two layer mineral models introduced in table 1. Solid line represent the exact reflectivity calculated from the Zoeppritz equation (Eq. 21)and circle-dot lines corresponds to the two term approximation. Reflectivity components corresponding to the interface models of Shale/salt, Shale/Limestone(gas) and Limestone/Salt. Fig.(7d) corresponds to the inhomogeneous components normalized by dividing to $\tan\theta_{\mathrm{P}}$

where the elastic constants are

$$A_{\rm PP}^{\rm E} = \frac{1}{2}\left[\frac{\Delta\rho}{\rho} + \frac{\Delta V_{\rm P}}{V_{\rm P}}\right]$$
$$B_{\rm PP}^{\rm E} = \frac{1}{2}\frac{\Delta V_{\rm P}}{V_{\rm P}} - 2\left(\frac{V_{\rm S}}{V_{\rm P}}\right)^2\left[\frac{\Delta\rho}{\rho} + 2\frac{\Delta V_{\rm S}}{V_{\rm S}}\right]$$
$$C_{\rm PP}^{\rm E} = \frac{1}{2}\frac{\Delta V_{\rm P}}{V_{\rm P}},$$

the homogeneous constants are

$$A_{\rm PP}^{\rm H} = -\frac{1}{4}Q_{\rm P}^{-1}\frac{\Delta Q_{\rm P}}{Q_{\rm P}}$$
$$B_{\rm PP}^{\rm H} = -2\left(\frac{V_{\rm S}}{V_{\rm P}}\right)^2\left[(Q_{\rm P}^{-1} - Q_{\rm S}^{-1})\left(\frac{\Delta\rho}{\rho} + 2\frac{\Delta V_{\rm S}}{V_{\rm S}}\right) + Q_{\rm S}^{-1}\frac{\Delta Q_{\rm S}}{Q_{\rm S}}\right] - \frac{1}{4}Q_{\rm P}^{-1}\frac{\Delta Q_{\rm P}}{Q_{\rm P}},$$

and the inhomogeneous constants are

$$A_{\rm PP}^{\rm IH} = Q_{\rm P}^{-1}\tan\delta_{\rm P}\left[\frac{1}{2}\frac{\Delta V_{\rm P}}{V_{\rm P}} - 2\left(\frac{V_{\rm S}}{V_{\rm P}}\right)^2\left(\frac{\Delta\rho}{\rho} + 2\frac{\Delta V_{\rm S}}{V_{\rm S}}\right)\right]$$
$$B_{\rm PP}^{\rm IH} = Q_{\rm P}^{-1}\tan\delta_{\rm P}\left[\frac{1}{2}\frac{\Delta V_{\rm P}}{V_{\rm P}} + 2\left(\frac{V_{\rm S}}{V_{\rm P}}\right)^2\left(\frac{\Delta\rho}{\rho} + 2\frac{\Delta V_{\rm S}}{V_{\rm S}}\right)\right]$$
$$C_{\rm PP}^{\rm IH} = Q_{\rm P}^{-1}\tan\delta_{\rm P}\frac{1}{2}\left[\frac{\Delta V_{\rm P}}{V_{\rm P}}\right].$$

Equations (37)-(39) are arranged in such a way that successive terms grow in importance as the angle of incidence grows. These equations are the generalization of the Shuey approximation to viscoelastic media. For each of the elastic, homogeneous and inhomogeneous parts, the first term corresponds to reflection coefficient at normal incidence. The second term is called the AVO gradient, and the third term, which becomes important for wide angles of incidence (roughly $\theta_{\rm P} > 30°$) is called the curvature. We note that the inhomogeneous term at normal incidence is zero, indicating that to leading order contributions from the attenuation angle should not be expected in the PP-reflectivity at normal incidence.

The linearized forms also make qualitatively clear aspects of the dependence of the reflection strengths on the physical properties above and below the interface. The elastic part of the reflectivity is sensitive to changes in density, P- and S-wave velocities and has a non zero value for waves at normal incidence. The anelastic-homogeneous term is sensitive to changes in density, S-wave velocity, P-wave quality factor and S-wave quality factor. At normal incidence this term is not zero, but, only a change in P-wave quality factor influences it. The inhomogeneous term is nonzero and sensitive to changes in density, P- and S-wave velocities; it is zero at normal incidence. We show later that the inhomogeneous term is a function of incidence angle, a property shared by the elastic and anelastic converted P-wave.

In Figure (4) we compare the relative importance of the three terms in the AVO responses expressed in equations (37) – (39). For elastic and homogeneous terms the contributions of the curvature at angles $\theta_P < 20°$ is negligible. Increasing the incidence angle, this term becomes relevant beyond $\theta_P > 30°$. The three components of the PP-reflection coefficient are illustrated in Figure (6). The elastic component shows a significant increase in accuracy moving from the two-term approximation (intercept and gradient) to the three-term approximation (including the curvature). The homogeneous component in isolation deviates only slightly from the exact solution for angles up to $20°$ except in the case of the limestone/salt model. In Figure (6c), we observe that the approximate form of the inhomogeneous component of the reflection coefficient deviates significantly from the exact result for all but the shale/salt model with incidence angles up to $20°$.

In Figure 7a, b, and c, we plot the exact versus linearized P-to-P reflectivity for the three single-interface models in Table 1. For elastic and homogeneous parts of the reflectivity, the linearity with respect to $\sin^2\theta_{\rm P}$ can be seen explicitly. The inhomogeneous term $\rm R_{PP}^{IH}$ is not a linear function of $\sin^2\theta_{\rm P}$, but linearity can be enforced through the normalization

$$\rm R_{PPN}^{IH} = \frac{R_{PP}^{IH}}{\tan\theta_P}, \quad \theta_P \neq 0. \tag{40}$$

The exact and approximate inhomogeneous reflectivities, introduced in equation (40), are plotted in Figure 7d. The $\rm R_{PPN}^{IH}$ term is evidently linear with respect to $\sin^2\theta_{\rm P}$ in the range $0 \leq \theta_{\rm P} \leq 25°$. Using the decomposition defined in equation (5), and the terms plotted in Figure 8, we can obtain the zero offset coefficients A and gradient terms B from the exact reflectivity. Contour maps, illustrating the variability of the inhomogeneous component of the P-to-P reflectivity versus phase and attenuation angles, are plotted in Figure 5.

CONVERTED WAVE APPROXIMATIONS

By solving the Zoeppritz equation for two half-spaces involving low-loss viscoelastic media, we can obtain exact expressions for the PP- and PS-reflection coefficients (Moradi and Innanen, 2015a, 2016). To linearize the reflectivities in terms of changes in elastic and anelastic properties, we assume the incidence angle to be smaller that $30°$, and also that the relative change in all elastic/anelastic properties are much less than one. In this paper, to treat the case of the converted wave, we use the appropriate version of Snell's law to obtain an expression for the S-wave attenuation angle, appropriate for small angles of incidence, which is written as a function of the incident phase and attenuation angles (Appendix). The weak-contrast converted-wave reflectivity is then given by

$$\rm R_{PS} = R_{PS}^{E} + \mathit{i}R_{PS}^{H} + \mathit{i}R_{PS}^{IH}, \tag{41}$$

where the real part is

$$\rm R_{PS}^{E} = -\tan\theta_S \frac{1}{2}\frac{V_P}{V_S}\frac{\Delta\rho}{\rho} - \tan\theta_S\cos(\theta_P+\theta_S)\left(\frac{\Delta\rho}{\rho} + 2\frac{\Delta V_S}{V_S}\right), \tag{42}$$

the homogeneous-imaginary part $\rm R_{PP}^{H}$ is

$$\rm R_{PS}^{H} = -\frac{1}{4}\tan\theta_S(Q_P^{-1} - Q_S^{-1})\frac{V_P}{V_S}\frac{\Delta\rho}{\rho} + Q_S^{-1}\tan\theta_S\cos(\theta_P+\theta_S)\frac{\Delta Q_S}{Q_S}, \tag{43}$$

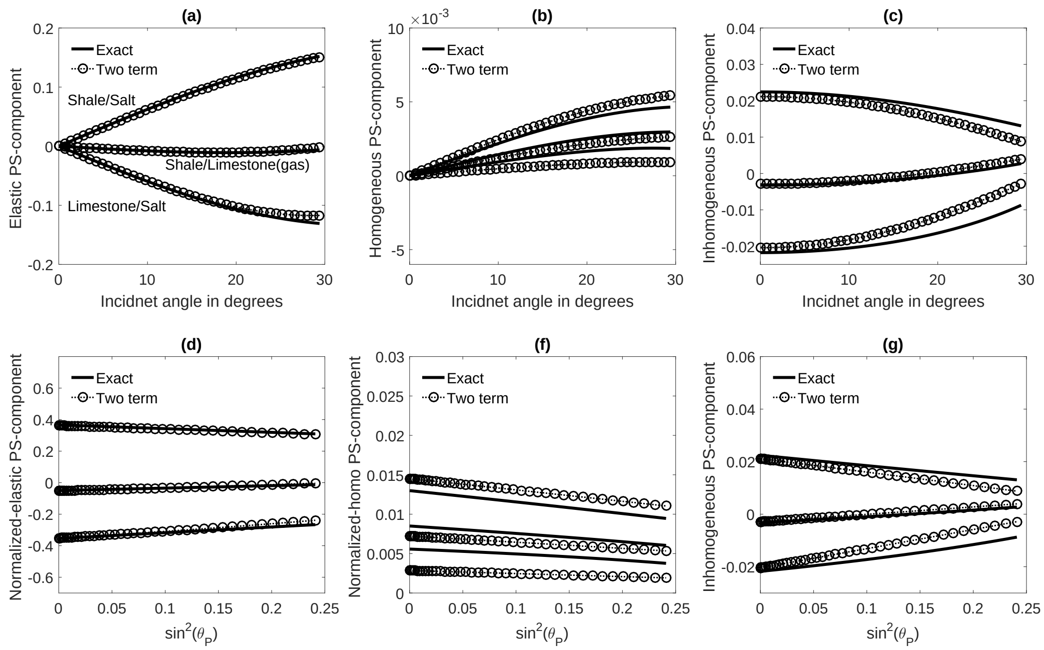

FIG. 8. Components of the PS-reflection coefficients versus incident angle and $\sin^2\theta_{\rm P}$ for three two layer mineral models introduced in table 1. Solid line represent the exact reflectivity calculated from the Zoeppritz equation (Eq. 22)and circle-dot lines corresponds to the two term approximation. Reflectivity components corresponding to the interface models of Shale/salt, Shale/Limestone(gas) and Limestone/Salt. Figures (8d,f) to corresponds to the components normalized by dividing to $\sin\theta_{\rm P}$

and the inhomogeneous-imaginary part $\rm R^{IH}_{PP}$ is

$$\begin{aligned}
\rm R^{IH}_{PS} = &-\frac{1}{4}Q_S^{-1}\tan\delta_S\frac{1}{\cos^2\theta_S}\frac{V_P}{V_S}\frac{\Delta\rho}{\rho} && (44)\\
&-\frac{1}{2}Q_S^{-1}\tan\delta_S\frac{\cos(\theta_P+\theta_S)}{\cos^2\theta_S}\left(\frac{\Delta\rho}{\rho}+2\frac{\Delta V_{\rm S}}{V_{\rm S}}\right) && (45)\\
&+\frac{1}{2}\tan\theta_S\sin(\theta_P+\theta_S)(Q_S^{-1}\tan\delta_S+Q_P^{-1}\tan\delta_P)\left(\frac{\Delta\rho}{\rho}+2\frac{\Delta V_{\rm S}}{V_{\rm S}}\right). && (46)
\end{aligned}$$

This approximate PS-reflectivity is a function of density, S-wave velocity and S-wave quality factors. Snell's law relates the reflected and transmitted phase and attenuation angles to the incident phase and attenuation angles. In order to analyze the converted-wave reflection coefficient properly in the lowest order, using the Snell's law the average S-wave attenuation angle for small angles of incidence is written as a function of incident phase and attenuation angles:

$$Q_S^{-1}\tan\delta_S = \frac{V_S}{V_P}Q_P^{-1}\tan\delta_P \tag{47}$$

$$+\frac{V_S}{V_P}(Q_S^{-1}-Q_P^{-1})\sin\theta_P \tag{48}$$

$$-\frac{1}{2}\frac{V_S}{V_P}\left[1-\left(\frac{V_S}{V_P}\right)^2\right]Q_P^{-1}\tan\delta_P\sin^2\theta_\mathrm{P} \tag{49}$$

$$+\frac{1}{2}\left(\frac{V_S}{V_P}\right)^3(Q_S^{-1}-Q_P^{-1})\sin^3\theta_P. \tag{50}$$

Then, using standard approximations for trigonometric functions for small angles, and collecting the powers of $\sin\theta_\mathrm{P}$, we obtain

$$\mathrm{R_{PS}}(\theta_\mathrm{P},\delta_\mathrm{P}) = \mathrm{R^E_{PS}}(\theta_\mathrm{P}) + i\mathrm{R^H_{PS}}(\theta_\mathrm{P}) + i\mathrm{R^{IH}_{PS}}(\theta_\mathrm{P},\delta_\mathrm{P}), \tag{51}$$

where the elastic, homogenous and inhomogeneous terms are given by

$$\begin{aligned}
\mathrm{R^E_{PS}}(\theta_\mathrm{P}) &= A^\mathrm{E}_\mathrm{PS}\sin\theta_\mathrm{P} + B^\mathrm{E}_\mathrm{PS}\sin^3\theta_\mathrm{P},\\
\mathrm{R^H_{PS}}(\theta_\mathrm{P}) &= A^\mathrm{H}_\mathrm{PS}\sin\theta_\mathrm{P} + B^\mathrm{H}_\mathrm{PS}\sin^3\theta_\mathrm{P},\\
\mathrm{R^{IH}_{PS}}(\theta_\mathrm{P},\delta_\mathrm{P}) &= A^\mathrm{IH}_\mathrm{PS} + B^\mathrm{IH}_\mathrm{PS}\sin^2\theta_P,
\end{aligned}$$

with elastic constants

$$\begin{aligned}
A^\mathrm{E}_\mathrm{PS} =& -\left(\frac{1}{2}+\frac{V_\mathrm{S}}{V_\mathrm{P}}\right)\frac{\Delta\rho}{\rho} - 2\frac{V_\mathrm{S}}{V_\mathrm{P}}\frac{\Delta V_\mathrm{S}}{V_\mathrm{S}},\\
B^\mathrm{E}_\mathrm{PS} =& \frac{V_\mathrm{S}}{V_\mathrm{P}}\left[\left(\frac{1}{2}+\frac{3}{4}\frac{V_\mathrm{S}}{V_\mathrm{P}}\right)\frac{\Delta\rho}{\rho} + 2\left[\frac{1}{2}+\frac{V_\mathrm{S}}{V_\mathrm{P}}\right]\frac{\Delta V_\mathrm{S}}{V_\mathrm{S}}\right],
\end{aligned}$$

homogeneous constants

$$\begin{aligned}
A^\mathrm{H}_\mathrm{PS} =& \frac{V_\mathrm{S}}{V_\mathrm{P}}\left\{Q_S^{-1}\frac{\Delta Q_S}{Q_S} - \frac{1}{2}(Q_S^{-1}-Q_P^{-1})\left(\frac{\Delta\rho}{\rho}+2\frac{\Delta V_\mathrm{S}}{V_\mathrm{S}}\right)\right\},\\
B^\mathrm{H}_\mathrm{PS} =& -\frac{V_\mathrm{S}}{V_\mathrm{P}}\left[\frac{1}{2}+\frac{V_\mathrm{S}}{V_\mathrm{P}}\right]Q_S^{-1}\frac{\Delta Q_S}{Q_S} - \frac{1}{4}\left(\frac{V_\mathrm{S}}{V_\mathrm{P}}\right)^2(Q_S^{-1}-Q_P^{-1})\frac{\Delta\rho}{\rho}\\
&+\frac{1}{4}\frac{V_S}{V_P}\left(1+4\frac{V_\mathrm{S}}{V_\mathrm{P}}\right)(Q_S^{-1}-Q_P^{-1})\left(\frac{\Delta\rho}{\rho}+2\frac{\Delta V_\mathrm{S}}{V_\mathrm{S}}\right),
\end{aligned}$$

and inhomogeneous constants

$$\begin{aligned}A_{\mathrm{PS}}^{\mathrm{IH}} =& -\frac{1}{2}\frac{V_S}{V_P}\left[\left(1+\frac{1}{2}\frac{V_P}{V_S}\right)\frac{\Delta\rho}{\rho}+2\frac{\Delta V_{\mathrm{S}}}{V_{\mathrm{S}}}\right]Q_P^{-1}\tan\delta_P,\\ B_{\mathrm{PS}}^{\mathrm{IH}} =& \frac{1}{8}\left[1-3\left(\frac{V_S}{V_P}\right)^2\right]\frac{\Delta\rho}{\rho}Q_P^{-1}\tan\delta_P\\ &+\frac{V_S}{V_P}\left(1+\frac{3}{2}\frac{V_{\mathrm{S}}}{V_{\mathrm{P}}}\right)\left(\frac{\Delta\rho}{\rho}+2\frac{\Delta V_{\mathrm{S}}}{V_{\mathrm{S}}}\right)Q_P^{-1}\tan\delta_P.\end{aligned}$$

The elastic term is seen to be sensitive to changes in density and S-wave velocity. The anelastic-homogeneous term is likewise seen to be sensitive to changes in density, S-wave velocity and its quality factor. These two terms are zero at normal incidence. The anelastic-inhomogeneous term is affected only by changes in density and S-wave velocity. This term also depends on the incident attenuation angle and is non zero at the normal incidence case; we note that this makes it quite singular in the standard converted wave AVO problem, wherein no contribution at normal incidence is ever predicted.

In Figure 8, we plot the exact versus linearized elastic, anelastic homogeneous and anelastic inhomogeneous terms for the three models in Table 1. We observe that the elastic and homogenous terms are not linear in $\sin^2\theta_P$, and the inhomogeneous terms is. Thus we also define normalized elastic and homogeneous reflectivities as

$$\mathrm{R}_{\mathrm{PSN}}^{\mathrm{E}}(\theta_{\mathrm{P}})=\frac{\mathrm{R}_{\mathrm{PS}}^{\mathrm{E}}(\theta_{\mathrm{P}})}{\sin\theta_{\mathrm{P}}},\qquad \mathrm{R}_{\mathrm{PSN}}^{\mathrm{H}}(\theta_{\mathrm{P}})=\frac{\mathrm{R}_{\mathrm{PS}}^{\mathrm{H}}(\theta_{\mathrm{P}})}{\sin\theta_{\mathrm{P}}}. \tag{52}$$

CONCLUSIONS

The formulation of the amplitude-versus-offset equations for viscoelastic media is of increasing interest and importance, with quantitative interpretation of seismic data being deployed to characterize fluid presence, type, and viscosity in hydrocarbon reservoirs, CO_2 injection sites, and other exploration and monitoring settings. Properly formulated, these equations also provide insights into the character of eventual viscoelastic full waveform inversion algorithms. To date, investigations and analysis of anelastic reflection coefficients have been constructed on the assumption that the attenuation angle is unchanged across the boundary, which cannot be generally justified. We believe that a more fruitful approach approach is to apply an appropriate version of Snell's law in such way that transmitted and reflected attenuation angles are expressed in terms of the incident attenuation angle. This approach allows changes in attenuation angle to be expressed in terms of changes in velocity and quality factors, leading to new terms in the relevant AVO equations with a wider capture of anelastic reflection and transmission phenomena incorporated.

We show how Snell's law can be put to work in order to learn about the homogeneous and inhomogeneous components of complex vertical slowness. We have presented a decomposition of the exact and approximate viscoelastic reflection coefficients to expose the above discussed of the attenuation angle, demonstrating the possibly significant errors resulting from its neglect, in particular in cases of for highly attenuative media.

Linearization of reflection coefficients in viscoelastic media is more complicated than in elastic media in two ways. First, because of seismic amplitude damping, the polarization and slowness vectors are complex, and therefore so is the reflectivity. Second, we have as discussed the perturbation of the attenuation angle across the boundary, as predicted by the viscoelastic Snell's law. Taking into account these facts, the linearized AVO equations include the terms related to the changes in S-wave quality factors and the attenuation angle.

To to understand quantitatively and qualitatively the importance and influence of the attenuation angle, we decompose the reflectivity into three terms, elastic, homogeneous and inhomogeneous. Linearity of the elastic and homogeneous parts are visible; the inhomogeneous part must be normalized to share this feature. In terms of powers of $\sin\theta_{\mathrm{P}}$, the converted PS-wave has four contributions, from zeroth to third order. The extra terms are due to the inhomogeneity of the waves. We examine our AVO equation with three two-half space models. Numerically, we find that the elastic and homogeneous terms are not linear with respect to $\sin^2\theta_{\mathrm{P}}$, however the inhomogeneous term for small angles ($\theta_{\mathrm{P}} < 30°$) is perfectly linear for both exact and approximate cases. The most striking feature of this model of reflections from viscoelastic targets is that a non-zero converted wave at normal incidence is predicted, connected to the attenuation angle.

The result presented in this research indicate that linearized reflection coefficients for inhomogeneous PP-wave match for most inverse schemes the exact reflection coefficients with adequate accuracy. More important, the new approximations and the decomposition of reflectivity into three terms indicate that intercepts and gradients can be used in future research to determine the quality factor and attenuation angle in an appropriate inversion strategy.

ACKNOWLEDGMENTS

The authors thank the sponsors of CREWES for continued support. This work was funded by CREWES industrial sponsors and NSERC (Natural Science and Engineering Research Council of Canada) through the grant CRDPJ 461179-13.

COMPLEX COEFFICIENTS

Decomposition of the complex coefficients in reflection functions (21) and (22) are given by

$$d_1 = d_1^{\mathrm{E}} + i d_1^{\mathrm{H}} + i d_1^{\mathrm{IH}},$$

where

$$\begin{aligned}
d_1^{\mathrm{E}} &= -2p_E^2\Delta\mu_E(q_{\mathrm{P1}}^{\mathrm{E}} - q_{\mathrm{P2}}^{\mathrm{E}}) + (\rho_2 q_{\mathrm{P1}}^{\mathrm{E}} + \rho_1 q_{\mathrm{P2}}^{\mathrm{E}})\\
d_1^{\mathrm{H}} &= -2p_E^2\Delta\mu_E(q_{\mathrm{P1}}^{\mathrm{H}} - q_{\mathrm{P2}}^{\mathrm{H}}) - 2(p_E^2\Delta\mu_A + 2p_E p_{\mathrm{H}}\Delta\mu_E)(q_{\mathrm{P1}}^{\mathrm{E}} - q_{\mathrm{P2}}^{\mathrm{E}}) + \rho_2 q_{\mathrm{P1}}^{\mathrm{H}} + \rho_1 q_{\mathrm{P2}}^{\mathrm{H}}\\
d_1^{\mathrm{IH}} &= -2p_E^2\Delta\mu_E(q_{\mathrm{P1}}^{\mathrm{IH}} - q_{\mathrm{P2}}^{\mathrm{IH}}) - 4p_E p_{\mathrm{IH}}\Delta\mu_E(q_{\mathrm{P1}}^{\mathrm{E}} - q_{\mathrm{P2}}^{\mathrm{E}}) + \rho_2 q_{\mathrm{P1}}^{\mathrm{IH}} + \rho_1 q_{\mathrm{P2}}^{\mathrm{IH}}
\end{aligned}$$

also

$$d_2 = d_2^{\mathrm{E}} + id_2^{\mathrm{H}} + id_2^{\mathrm{IH}},$$

where

$$\begin{aligned}
d_2^{\mathrm{E}} =& -2p_E^2\Delta\mu_E(q_{\mathrm{S}}^{\mathrm{E}} - q_{\mathrm{S2}}^{\mathrm{E}}) + (\rho_2 q_{\mathrm{S1}}^{\mathrm{E}} + \rho_1 q_{\mathrm{S2}}^{\mathrm{E}}) \\
d_2^{\mathrm{H}} =& -2p_E^2\Delta\mu_E(q_{\mathrm{P1}}^{\mathrm{H}} - q_{\mathrm{P2}}^{\mathrm{H}}) - 2(p_E^2\Delta\mu_A + 2p_E p_{\mathrm{H}}\Delta\mu_E)(q_{\mathrm{S1}}^{\mathrm{E}} - q_{\mathrm{S2}}^{\mathrm{E}}) + \rho_2 q_{\mathrm{S1}}^{\mathrm{H}} + \rho_1 q_{\mathrm{S2}}^{\mathrm{H}} \\
d_2^{\mathrm{IH}} =& -2p_E^2\Delta\mu_E(q_{\mathrm{S1}}^{\mathrm{IH}} - q_{\mathrm{S2}}^{\mathrm{IH}}) - 4p_E p_{\mathrm{IH}}\Delta\mu_E(q_{\mathrm{S1}}^{\mathrm{E}} - q_{\mathrm{S2}}^{\mathrm{E}}) + \rho_2 q_{\mathrm{S1}}^{\mathrm{IH}} + \rho_1 q_{\mathrm{S2}}^{\mathrm{IH}}
\end{aligned}$$

also

$$d_3 = d_3^{\mathrm{E}} + id_3^{\mathrm{H}} + id_3^{\mathrm{IH}},$$

where

$$\begin{aligned}
d_3^{\mathrm{E}} =& -p_E\left[2\Delta\mu_E(q_{\mathrm{P1}}^{\mathrm{E}} q_{\mathrm{S2}}^{\mathrm{E}} + p^2) - \Delta\rho\right] \\
d_3^{\mathrm{H}} =& -2(p_{\mathrm{E}}\Delta\mu_A + p_{\mathrm{H}}\Delta\mu_E)(q_{\mathrm{P1}}^{\mathrm{E}} q_{\mathrm{S2}}^{\mathrm{E}} + p_E^2) - 2p_E\Delta\mu_E(2p_{\mathrm{E}}p_{\mathrm{H}} + q_{\mathrm{P1}}^{\mathrm{H}} q_{\mathrm{S2}}^{\mathrm{E}} + q_{\mathrm{P1}}^{\mathrm{E}} q_{\mathrm{S2}}^{\mathrm{H}}) + p_{\mathrm{H}}\Delta\rho \\
d_3^{\mathrm{IH}} =& -2p_{\mathrm{IH}}\Delta\mu_E(q_{\mathrm{P1}}^{\mathrm{E}} q_{\mathrm{S2}}^{\mathrm{E}} + p_E^2) - 2p_E\Delta\mu_E(q_{\mathrm{P1}}^{\mathrm{IH}} q_{\mathrm{S2}}^{\mathrm{E}} + q_{\mathrm{P1}}^{\mathrm{E}} q_{\mathrm{S2}}^{\mathrm{IH}}) + p_{\mathrm{IH}}\Delta\rho
\end{aligned}$$

also

$$d_4 = d_4^{\mathrm{E}} + id_4^{\mathrm{H}} + id_4^{\mathrm{IH}},$$

where

$$\begin{aligned}
d_4^{\mathrm{E}} =& -p_E\left[2\Delta\mu_E(q_{\mathrm{P2}}^{\mathrm{E}} q_{\mathrm{S1}}^{\mathrm{E}} + p^2) - \Delta\rho\right] \\
d_4^{\mathrm{H}} =& -2(p_{\mathrm{E}}\Delta\mu_A + p_{\mathrm{H}}\Delta\mu_E)(q_{\mathrm{P2}}^{\mathrm{E}} q_{\mathrm{S1}}^{\mathrm{E}} + p_E^2) - 2p_E\Delta\mu_E(2p_{\mathrm{E}}p_{\mathrm{H}} + q_{\mathrm{P2}}^{\mathrm{H}} q_{\mathrm{S1}}^{\mathrm{E}} + q_{\mathrm{P2}}^{\mathrm{E}} q_{\mathrm{S1}}^{\mathrm{H}}) + p_{\mathrm{H}}\Delta\rho \\
d_4^{\mathrm{IH}} =& -2p_{\mathrm{IH}}\Delta\mu_E(q_{\mathrm{P2}}^{\mathrm{E}} q_{\mathrm{S1}}^{\mathrm{E}} + p_E^2) - 2p_E\Delta\mu_E(q_{\mathrm{P2}}^{\mathrm{IH}} q_{\mathrm{S1}}^{\mathrm{E}} + q_{\mathrm{P2}}^{\mathrm{E}} q_{\mathrm{S1}}^{\mathrm{IH}}) + p_{\mathrm{IH}}\Delta\rho
\end{aligned}$$

also

$$c_1 = c_1^{\mathrm{E}} + ic_1^{\mathrm{H}} + ic_1^{\mathrm{IH}},$$

where

$$\begin{aligned}
c_1^{\mathrm{E}} =& -2p_E^2\Delta\mu_E(q_{\mathrm{P1}}^{\mathrm{E}} + q_{\mathrm{P2}}^{\mathrm{E}}) + (\rho_2 q_{\mathrm{P1}}^{\mathrm{E}} - \rho_1 q_{\mathrm{P2}}^{\mathrm{E}}) \\
c_1^{\mathrm{H}} =& -2p_E^2\Delta\mu_E(q_{\mathrm{P1}}^{\mathrm{H}} + q_{\mathrm{P2}}^{\mathrm{H}}) - 2(p_E^2\Delta\mu_A + 2p_E p_{\mathrm{H}}\Delta\mu_E)(q_{\mathrm{P1}}^{\mathrm{E}} + q_{\mathrm{P2}}^{\mathrm{E}}) + \rho_2 q_{\mathrm{P1}}^{\mathrm{H}} - \rho_1 q_{\mathrm{P2}}^{\mathrm{H}} \\
c_1^{\mathrm{IH}} =& -2p_E^2\Delta\mu_E(q_{\mathrm{P1}}^{\mathrm{IH}} + q_{\mathrm{P2}}^{\mathrm{IH}}) - 4p_E p_{\mathrm{IH}}\Delta\mu_E(q_{\mathrm{P1}}^{\mathrm{E}} + q_{\mathrm{P2}}^{\mathrm{E}}) + \rho_2 q_{\mathrm{P1}}^{\mathrm{IH}} - \rho_1 q_{\mathrm{P2}}^{\mathrm{IH}}
\end{aligned}$$

also

$$c_2 = c_2^{\mathrm{E}} + ic_2^{\mathrm{H}} + ic_2^{\mathrm{IH}},$$

where

$$
\begin{aligned}
c_2^{\mathrm{E}} =& -2p_E^2\Delta\mu_E(q_{\mathrm{S}}^{\mathrm{E}}+q_{\mathrm{S2}}^{\mathrm{E}})+(\rho_2 q_{\mathrm{S1}}^{\mathrm{E}}-\rho_1 q_{\mathrm{S2}}^{\mathrm{E}})\\
c_2^{\mathrm{H}} =& -2p_E^2\Delta\mu_E(q_{\mathrm{P1}}^{\mathrm{H}}+q_{\mathrm{P2}}^{\mathrm{H}})-2(p_E^2\Delta\mu_A+2p_E p_{\mathrm{H}}\Delta\mu_E)(q_{\mathrm{S1}}^{\mathrm{E}}+q_{\mathrm{S2}}^{\mathrm{E}})+\rho_2 q_{\mathrm{S1}}^{\mathrm{H}}-\rho_1 q_{\mathrm{S2}}^{\mathrm{H}}\\
c_2^{\mathrm{IH}} =& -2p_E^2\Delta\mu_E(q_{\mathrm{S1}}^{\mathrm{IH}}+q_{\mathrm{S2}}^{\mathrm{IH}})-4p_E p_{\mathrm{IH}}\Delta\mu_E(q_{\mathrm{S1}}^{\mathrm{E}}+q_{\mathrm{S2}}^{\mathrm{E}})+\rho_2 q_{\mathrm{S1}}^{\mathrm{IH}}-\rho_1 q_{\mathrm{S2}}^{\mathrm{IH}}
\end{aligned}
$$

also

$$
c_3 = c_3^{\mathrm{E}} + ic_3^{\mathrm{H}} + ic_3^{\mathrm{IH}},
$$

where

$$
\begin{aligned}
c_3^{\mathrm{E}} =& p_E\left[2\Delta\mu_E(q_{\mathrm{P1}}^{\mathrm{E}}q_{\mathrm{S2}}^{\mathrm{E}}-p^2)+\Delta\rho\right]\\
c_3^{\mathrm{H}} =& 2(p_{\mathrm{E}}\Delta\mu_A+p_{\mathrm{H}}\Delta\mu_E)(q_{\mathrm{P1}}^{\mathrm{E}}q_{\mathrm{S2}}^{\mathrm{E}}-p_E^2)-2p_E\Delta\mu_E(2p_{\mathrm{E}}p_{\mathrm{H}}-q_{\mathrm{P1}}^{\mathrm{H}}q_{\mathrm{S2}}^{\mathrm{E}}-q_{\mathrm{P1}}^{\mathrm{E}}q_{\mathrm{S2}}^{\mathrm{H}})+p_{\mathrm{H}}\Delta\rho\\
c_3^{\mathrm{IH}} =& 2p_{\mathrm{IH}}\Delta\mu_E(q_{\mathrm{P1}}^{\mathrm{E}}q_{\mathrm{S2}}^{\mathrm{E}}-p_E^2)+2p_E\Delta\mu_E(q_{\mathrm{P1}}^{\mathrm{IH}}q_{\mathrm{S2}}^{\mathrm{E}}+q_{\mathrm{P1}}^{\mathrm{E}}q_{\mathrm{S2}}^{\mathrm{IH}})+p_{\mathrm{IH}}\Delta\rho
\end{aligned}
$$

also

$$
c_4 = c_4^{\mathrm{E}} + ic_4^{\mathrm{H}} + ic_4^{\mathrm{IH}},
$$

where

$$
\begin{aligned}
c_4^{\mathrm{E}} =& p_E\left[2\Delta\mu_E(q_{\mathrm{P2}}^{\mathrm{E}}q_{\mathrm{S1}}^{\mathrm{E}}+p^2)+\Delta\rho\right]\\
c_4^{\mathrm{H}} =& 2(p_{\mathrm{E}}\Delta\mu_A+p_{\mathrm{H}}\Delta\mu_E)(q_{\mathrm{P2}}^{\mathrm{E}}q_{\mathrm{S1}}^{\mathrm{E}}-p_E^2)-2p_E\Delta\mu_E(2p_{\mathrm{E}}p_{\mathrm{H}}-q_{\mathrm{P2}}^{\mathrm{H}}q_{\mathrm{S1}}^{\mathrm{E}}-q_{\mathrm{P2}}^{\mathrm{E}}q_{\mathrm{S1}}^{\mathrm{H}})+p_{\mathrm{H}}\Delta\rho\\
c_4^{\mathrm{IH}} =& 2p_{\mathrm{IH}}\Delta\mu_E(q_{\mathrm{P2}}^{\mathrm{E}}q_{\mathrm{S1}}^{\mathrm{E}}-p_E^2)+2p_E\Delta\mu_E(q_{\mathrm{P2}}^{\mathrm{IH}}q_{\mathrm{S1}}^{\mathrm{E}}+q_{\mathrm{P2}}^{\mathrm{E}}q_{\mathrm{S1}}^{\mathrm{IH}})+p_{\mathrm{IH}}\Delta\rho
\end{aligned}
$$

TRIGONOMETRIC FUNCTIONS FOR SMALL ANGLES

For small angle of incident θ_{P} we can write

$$
\frac{1}{\cos^2\theta_{\mathrm{P}}} \approx 1+\sin^2\theta_{\mathrm{P}} \tag{53}
$$

$$
\cos\theta_{\mathrm{P}} \approx 1-\frac{1}{2}\sin^2\theta_{\mathrm{P}} \tag{54}
$$

$$
\frac{1}{\cos\theta_{\mathrm{P}}} \approx 1+\frac{1}{2}\sin^2\theta_{\mathrm{P}} \tag{55}
$$

$$
\tan\theta_{\mathrm{P}} \approx \sin\theta_{\mathrm{P}}+\frac{1}{2}\sin^3\theta_{\mathrm{P}} \tag{56}
$$

$$
\sin 2\theta_{\mathrm{P}} \approx 2\sin\theta_{\mathrm{P}}-\sin^3\theta_{\mathrm{P}} \tag{57}
$$

$$
\frac{\tan\theta_{\mathrm{P}}}{\cos^2\theta_{\mathrm{P}}} \approx \sin\theta_{\mathrm{P}}+\frac{3}{2}\sin^3\theta_{\mathrm{P}} \tag{58}
$$

$$\sin\theta_{\mathrm{S}} = \frac{V_{\mathrm{S}}}{V_{\mathrm{P}}}\sin\theta_{\mathrm{P}} \tag{59}$$

$$\cos\theta_{\mathrm{S}} \approx 1 - \frac{1}{2}\left(\frac{V_{\mathrm{S}}}{V_{\mathrm{P}}}\right)^2 \sin^2\theta_{\mathrm{P}} \tag{60}$$

$$\frac{1}{\cos\theta_{\mathrm{S}}} \approx 1 + \frac{1}{2}\left(\frac{V_{\mathrm{S}}}{V_{\mathrm{P}}}\right)^2 \sin^2\theta_{\mathrm{P}} \tag{61}$$

$$\frac{1}{\cos^2\theta_{\mathrm{S}}} \approx 1 + \left(\frac{V_{\mathrm{S}}}{V_{\mathrm{P}}}\right)^2 \sin^2\theta_{\mathrm{P}} \tag{62}$$

$$\tan\theta_{\mathrm{S}} \approx \frac{V_{\mathrm{S}}}{V_{\mathrm{P}}}\sin\theta_{\mathrm{P}}\left(1 + \frac{1}{2}\left(\frac{V_{\mathrm{S}}}{V_{\mathrm{P}}}\right)^2 \sin^2\theta_{\mathrm{P}}\right) \tag{63}$$

$$\cos(\theta_{\mathrm{S}}+\theta_{\mathrm{P}}) \approx 1 - \frac{1}{2}\sin^2\theta_{\mathrm{P}}\left[1 + \frac{V_{\mathrm{S}}}{V_{\mathrm{P}}}\right]^2 \tag{64}$$

$$\sin(\theta_{\mathrm{S}}+\theta_{\mathrm{P}}) \approx \left[1 + \frac{V_{\mathrm{S}}}{V_{\mathrm{P}}}\right]\sin\theta_{\mathrm{P}}\left(1 + \frac{1}{2}\frac{V_{\mathrm{S}}}{V_{\mathrm{P}}}\sin^2\theta_{\mathrm{P}}\right) \tag{65}$$

$$\tan\theta_{\mathrm{S}}\cos(\theta_{\mathrm{S}}+\theta_{\mathrm{P}}) \approx \frac{V_{\mathrm{S}}}{V_{\mathrm{P}}}\sin\theta_{\mathrm{P}}\left(1 - \left[\frac{1}{2} + \frac{V_{\mathrm{S}}}{V_{\mathrm{P}}}\right]\sin^2\theta_{\mathrm{P}}\right) \tag{66}$$

$$\frac{\cos(\theta_{\mathrm{S}}+\theta_{\mathrm{P}})}{\cos^2\theta_{\mathrm{S}}} \approx \frac{V_{\mathrm{S}}}{V_{\mathrm{P}}}\left[1 + \frac{V_{\mathrm{S}}}{V_{\mathrm{P}}}\right]\sin^2\theta_{\mathrm{P}} \tag{67}$$

LINEARIZATION PROCEDURE IN VISCOELASTIC MEDIA

First consider to the perturbations in elastic and anelastic properties. Subscript 1 refers to the upper layer(medium 1) and subscribe 2 refers to the lower layer (medium 2). Δ means difference between the properties in medium 2 and medium 1 and superscript L denotes the linearized form. In the linearization procedure $\Delta^2 = 0$. Properties without index means the average in properties.

Property	Layer 1	Layer2
Density	$\rho - \frac{\Delta\rho}{2}$	$\rho + \frac{\Delta\rho}{2}$
P-wave velocity	$V_{\mathrm{PE}} - \frac{\Delta V_{\mathrm{PE}}}{2}$	$V_{\mathrm{PE}} + \frac{\Delta V_{\mathrm{PE}}}{2}$
S-wave velocity	$V_{\mathrm{SE}} - \frac{\Delta V_{\mathrm{SE}}}{2}$	$V_{\mathrm{SE}} + \frac{\Delta V_{\mathrm{SE}}}{2}$
P-wave quality factor	$Q_{\mathrm{P}} - \frac{\Delta Q_{\mathrm{P}}}{2}$	$Q_{\mathrm{P}} + \frac{\Delta Q_{\mathrm{P}}}{2}$
S-wave quality factor	$Q_{\mathrm{S}} - \frac{\Delta Q_{\mathrm{S}}}{2}$	$Q_{\mathrm{S}} + \frac{\Delta Q_{\mathrm{S}}}{2}$
P-wave phase angle	$\theta_{\mathrm{P}} - \frac{\Delta\theta_{\mathrm{P}}}{2}$	$\theta_{\mathrm{P}} + \frac{\Delta\theta_{\mathrm{P}}}{2}$
S-wave phase angle	$\theta_{\mathrm{S}} - \frac{\Delta\theta_{\mathrm{S}}}{2}$	$\theta_{\mathrm{S}} + \frac{\Delta\theta_{\mathrm{S}}}{2}$
P-wave attenuation angle	$\delta_{\mathrm{P}} - \frac{\Delta\delta_{\mathrm{P}}}{2}$	$\delta_{\mathrm{P}} + \frac{\Delta\delta_{\mathrm{P}}}{2}$
S-wave attenuation angle	$\delta_{\mathrm{S}} - \frac{\Delta\delta_{\mathrm{S}}}{2}$	$\delta_{\mathrm{S}} + \frac{\Delta\delta_{\mathrm{S}}}{2}$

Trigonometric functions in this procedures for attenuation angle are given by

$$\cos\delta_n = \cos\delta_{\mathrm{P}}\left(1 + (-)^{n+1}\tan\delta\frac{\Delta\delta}{2}\right), \tag{68}$$

$$\sin\delta_n = \sin\delta_{\mathrm{P}}\left(1 - (-)^{n+1}\frac{1}{\tan\delta}\frac{\Delta\delta}{2}\right), \tag{69}$$

$$\tan\delta_n = \tan\delta_{\mathrm{P}}\left(1 - (-)^{n+1}\frac{1}{\tan\delta}\Delta\delta\right), \qquad n = 1, 2 \tag{70}$$

(71)

The real part of the Snell's law for P-wave results

$$\frac{\sin\theta_{\mathrm{P1}}}{V_{\mathrm{P1}}} = \frac{\sin\theta_{\mathrm{P2}}}{V_{\mathrm{P2}}}, \tag{72}$$

where θ_{P1} is incident angle and θ_{P2} is transmitted phase angle, using the perturbed term in table () we obtain the change in P-wave phase angle across the boundary in terms of change in the P-wave velocity

$$\Delta\theta_{\mathrm{P}} = \frac{\Delta V_{\mathrm{P}}}{V_{\mathrm{P}}}\tan\theta_{\mathrm{P}}, \tag{73}$$

Imaginary part of the Snell's law for P-wave is given by

$$\frac{Q_{\mathrm{P1}}^{-1}}{V_{\mathrm{P1}}}\left(\sin\theta_{\mathrm{P1}} - \cos\theta_{\mathrm{P1}}\tan\delta_{\mathrm{P1}}\right) = \frac{Q_{\mathrm{P2}}^{-1}}{V_{\mathrm{P2}}}\left(\sin\theta_{\mathrm{P2}} - \cos\theta_{\mathrm{P2}}\tan\delta_{\mathrm{P2}}\right) \tag{74}$$

Using the perturbation terms in table (), we obtain the changes in P-wave attenuation angle across the boundary in terms of changes in phase angle, P-wave velocity and P-wave quality factor

$$\Delta\delta_{\mathrm{P}} = \frac{1}{2}\sin 2\delta_{\mathrm{P}}\left\{\frac{\Delta V_{\mathrm{P}}}{V_{\mathrm{P}}}\frac{1}{\cos^2\theta_{\mathrm{P}}} + \left(1 - \frac{\tan\theta_{\mathrm{P}}}{\tan\delta_{\mathrm{P}}}\right)\frac{\Delta Q_{\mathrm{P}}}{Q_{\mathrm{P}}}\right\}, \tag{75}$$

Let us consider to the linearization of the vertical slowness

$$q_{\mathrm{P}} = q_{\mathrm{PE}} + iq_{\mathrm{PAH}} + iq_{\mathrm{PAIH}},$$

where

$$q_{\mathrm{PE}} = \frac{\cos\theta_{\mathrm{P}}}{V_{\mathrm{P}}}$$

$$q_{\mathrm{PAH}} = -\frac{Q_{\mathrm{P}}^{-1}}{2}\frac{\cos\theta_{\mathrm{P}}}{V_{\mathrm{P}}}$$

$$q_{\mathrm{PAIH}} = -\frac{1}{2}Q_{\mathrm{P}}^{-1}\tan\delta_{\mathrm{P}}\frac{\sin\theta_{\mathrm{P}}}{V_{\mathrm{P}}}$$

To obtain the linearized form of vertical slownesses we note that we have to linearize the summation of homogeneous and inhomogeneous term

$$q_{\mathrm{PAH1}} + q_{\mathrm{PAIH1}} \longrightarrow q_{\mathrm{PAH1}}^{\mathrm{L}} + q_{\mathrm{PAIH1}}^{\mathrm{L}} \tag{76}$$

In other words

$$q_{\mathrm{PAH1}} \neq q^{\mathrm{L}}_{\mathrm{PAH1}}$$
$$q_{\mathrm{PAIH1}} \neq q^{\mathrm{L}}_{\mathrm{PAIH1}}$$

Now the vertical slowness for incident P-wave is given by

$$q_{\mathrm{P1}} = q_{\mathrm{PE1}} + iq_{\mathrm{PAH1}} + iq_{\mathrm{PAIH1}},$$

where

$$q_{\mathrm{PE1}} = \frac{\cos\theta_{\mathrm{P1}}}{V_{\mathrm{P1}}}$$
$$q_{\mathrm{PAH1}} = -\frac{Q^{-1}_{\mathrm{P1}}}{2}\frac{\cos\theta_{\mathrm{P1}}}{V_{\mathrm{P1}}}$$
$$q_{\mathrm{PAIH1}} = -\frac{1}{2}Q^{-1}_{\mathrm{P1}}\tan\delta_{\mathrm{P1}}\frac{\sin\theta_{\mathrm{P1}}}{V_{\mathrm{P1}}}$$

Using the linearized form of the angles and P-wave velocity and P-wave quality factor we have

$$q^{\mathrm{L}}_{\mathrm{PE1}} = q_{\mathrm{PE}}\left(1 + \frac{1}{2\cos^2\theta_{\mathrm{P}}}\frac{\Delta V_{\mathrm{P}}}{V_{\mathrm{P}}}\right)$$
$$q^{\mathrm{L}}_{\mathrm{PAH1}} = q_{\mathrm{PAH}}\left(1 + \frac{1}{2\cos^2\theta_{\mathrm{P}}}\left[\frac{\Delta V_{\mathrm{P}}}{V_{\mathrm{P}}} + \frac{\Delta Q_{\mathrm{P}}}{Q_{\mathrm{P}}}\right]\right)$$
$$q^{\mathrm{L}}_{\mathrm{PAIH1}} = q_{\mathrm{PAIH}}\left(1 - \frac{1}{2\cos^2\theta_{\mathrm{P}}}\frac{\Delta V_{\mathrm{P}}}{V_{\mathrm{P}}}\right)$$

Transmitted P-wave

$$q^{\mathrm{L}}_{\mathrm{PE2}} = q_{\mathrm{PE}}\left(1 - \frac{1}{2\cos^2\theta_{\mathrm{P}}}\frac{\Delta V_{\mathrm{P}}}{V_{\mathrm{P}}}\right)$$
$$q^{\mathrm{L}}_{\mathrm{PAH2}} = q_{\mathrm{PAH}}\left(1 - \frac{1}{2\cos^2\theta_{\mathrm{P}}}\left[\frac{\Delta V_{\mathrm{P}}}{V_{\mathrm{P}}} + \frac{\Delta Q_{\mathrm{P}}}{Q_{\mathrm{P}}}\right]\right)$$
$$q^{\mathrm{L}}_{\mathrm{PAIH2}} = q_{\mathrm{PAIH}}\left(1 + \frac{1}{2\cos^2\theta_{\mathrm{P}}}\frac{\Delta V_{\mathrm{P}}}{V_{\mathrm{P}}}\right)$$

Incident S-wave

$$q^{\mathrm{L}}_{\mathrm{SE1}} = q_{\mathrm{SE}}\left(1 + \frac{1}{2\cos^2\theta_{\mathrm{S}}}\frac{\Delta V_{\mathrm{S}}}{V_{\mathrm{S}}}\right)$$
$$q^{\mathrm{L}}_{\mathrm{SAH1}} = q_{\mathrm{SAH}}\left(1 + \frac{1}{2\cos^2\theta_{\mathrm{S}}}\left[\frac{\Delta V_{\mathrm{S}}}{V_{\mathrm{S}}} + \frac{\Delta Q_{\mathrm{S}}}{Q_{\mathrm{S}}}\right]\right)$$
$$q^{\mathrm{L}}_{\mathrm{SAIH1}} = q_{\mathrm{SAIH}}\left(1 - \frac{1}{2\cos^2\theta_{\mathrm{S}}}\frac{\Delta V_{\mathrm{S}}}{V_{\mathrm{S}}}\right)$$

Transmitted S-wave

$$q^{\rm L}_{\rm SE2} = q_{\rm SE}\left(1 - \frac{1}{2\cos^2\theta_{\rm S}}\frac{\Delta V_{\rm S}}{V_{\rm S}}\right)$$
$$q^{\rm L}_{\rm SAH2} = q_{\rm SAH}\left(1 - \frac{1}{2\cos^2\theta_{\rm S}}\left[\frac{\Delta V_{\rm S}}{V_{\rm S}} + \frac{\Delta Q_{\rm S}}{Q_{\rm S}}\right]\right)$$
$$q^{\rm L}_{\rm SAIH2} = q_{\rm SAIH}\left(1 + \frac{1}{2\cos^2\theta_{\rm S}}\frac{\Delta V_{\rm S}}{V_{\rm S}}\right)$$

The complex shear modulus can be written as a real and imaginary part

$$\mu = \mu_{\rm E} + i\mu_{\rm A}, \tag{77}$$

where $\mu_{\rm E} = \rho V_{\rm S}^2$ and $\mu_{\rm A} = \rho\, Q_{\rm S} V_{\rm S}^2$. Now we linearized the shear modulus

$$\mu_1 = \mu_{\rm E1} + i\mu_{\rm A1}$$
$$\mu_2 = \mu_{\rm E2} + i\mu_{\rm A2}$$

where

$$\mu_{\rm E1} = \mu_{\rm E}\left(1 - \frac{1}{2}\left[\frac{\Delta\rho}{\rho} + 2\frac{\Delta V_{\rm S}}{V_{\rm S}}\right]\right)$$
$$\mu_{\rm E2} = \mu_{\rm E}\left(1 + \frac{1}{2}\left[\frac{\Delta\rho}{\rho} + 2\frac{\Delta V_{\rm S}}{V_{\rm S}}\right]\right)$$
$$\mu_{\rm A1} = \mu_{\rm A}\left(1 - \frac{1}{2}\left[\frac{\Delta\rho}{\rho} + 2\frac{\Delta V_{\rm S}}{V_{\rm S}} - \frac{\Delta Q_{\rm S}}{Q_{\rm S}}\right]\right)$$
$$\mu_{\rm A2} = \mu_{\rm A}\left(1 + \frac{1}{2}\left[\frac{\Delta\rho}{\rho} + 2\frac{\Delta V_{\rm S}}{V_{\rm S}} - \frac{\Delta Q_{\rm S}}{Q_{\rm S}}\right]\right)$$

Now we have

$$\Delta\mu_{\rm E} = \mu_{\rm E}\left[\frac{\Delta\rho}{\rho} + 2\frac{\Delta V_{\rm S}}{V_{\rm S}}\right]$$
$$\Delta\mu_{\rm A} = \mu_{\rm A}\left[\frac{\Delta\rho}{\rho} + 2\frac{\Delta V_{\rm S}}{V_{\rm S}} - \frac{\Delta Q_{\rm S}}{Q_{\rm S}}\right]$$

REFERENCES

Aki, K., and Richards, P. G., 2002, Quantitative Seismology: University Science Books, 2nd edn.

Behura, J., and Tsvankin, I., 2009a, Reflection coefficients in attenuative anisotropic media: Geophysics, **74**, No. 5, WB193–WB202.

Behura, J., and Tsvankin, I., 2009b, Role of the inhomogeneity angle in anisotropic attenuation analysis: Geophysics, **74**, No. 5, WB177–WB191.

Borcherdt, R. D., 2009, Viscoelastic waves in layered media: Cambridge University Press.

Castagna, J., and Backus, M., 1993, Offset dependent reflectivity: Theory and practice of avo analysis: SEG Investigations in Geophysics Series, , No. 8, 345.

Foster, D., Keys, R., and Lane, F., 2010, Interpretation of avo anomalies: Geophysics, **75**, 75A3–75A13.

Ikelle, L. T., and Amundsen, L., 2005, Introduction to petroleum seismology.

Innanen, K. A., 2011, Inversion of the seismic avf/ava signatures of highly attenuative target: Geophysics, **76**, R1–R14.

Innanen, K. A., 2012, Anelastic p-wave, s-wave and converted-wave avo approximations: 74th EAGE Conference and Exhibition, Extended Abstracts, P197.

Jerez, J. R. d. J. C., 2003, An avo method toward direct detection of lithologies combining pp and ps reflection data: Ph.D. thesis, Texas A&M University.

Krebes, E., 1983, The viscoelastic reflection/transmission problem: two special cases: Bulletin of the Seismological Society of America, **73**, No. 6A, 1673–1683.

Krebes, E., 1984, On the reflection and transmission of viscoelastic waves-some numerical results: Geophysics, **49**, No. 8, 1374–1380.

Lehocki, I., Avseth, P., Veggeland, T. et al., 2014, Nonlinear inversion of pp-and ps-reflection data using aki-richards approximation, *in* 2014 SEG Annual Meeting, Society of Exploration Geophysicists.

Margrave, G. F., Stewart, R. R., and Larsen, J. A., 2001, Joint pp and ps seismic inversion: The Leading Edge, **20**, No. 9, 1048–1052.

Moradi, S., and Innanen, K. A., 2015a, Linearized avo in viscoelastic media, *in* 2015 SEG Annual Meeting, Society of Exploration Geophysicists.

Moradi, S., and Innanen, K. A., 2015b, Scattering of homogeneous and inhomogeneous seismic waves in low-loss viscoelastic media: Geophysical Journal International, **202**, No. 3, 1722–1732.

Moradi, S., and Innanen, K. A., 2016, Viscoelastic amplitude variation with offset equations with account taken of jumps in attenuation angle: Geophysics, **81**, No. 3, N17–N29.

Shuey, R., 1985, A simplification of the zoeppritz equations: Geophysics, **50**, No. 4, 609–614.

Stovas, A., and Ursin, B., 2003, Reflection and transmission responses of layered transversely isotropic viscoelastic media: Geophysical Prospecting, **51**, No. 5, 447–477.

Ursin, B., and Stovas, A., 2002, Reflection and transmission responses of a layered isotropic viscoelastic medium: Geophysics, **67**, 307–323.

Wennerberg, L., 1985, Snell's law for viscoelastic materials: Geophys. J. R. astr. Soc., **81**, 13–18.