Theoretical framework of elastic internal multiples prediction based on inverse scattering series

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ABSTRACT

Inverse scattering series has been revealed in extremely powerful capabilities of seismic data processing and inversion, such as full waveform inversion (FWI), direct nonlinear AVO inversion, and surface-related or internal multiples attenuation, due to its property of model independence. Significant benefits have been achieved by performing internal multiple prediction using inverse scattering series in several different domains. Nevertheless, elastic internal multiples remain a significant hazard as multi-component acquisition technology has been widely applied. Unfortunately, for an elastic media, the off-the-shelf internal multiple attenuation is either inadequate or non-existent. By considering an elastic, isotropic and homogeneous media as background, this paper presents a theoretical framework of elastic internal multiple prediction using inverse scattering series.

INTRODUCTION

The requirement of increasing sensitivity in primary amplitude quantitative interpretation is highlight as the mining environment becomes to be more complex, which enhances the role and significance of multiple prediction and removal. However, one might speaks the importance of accurate and robust prediction of multiples may now be on the verge of an even greater upward jump, as full waveform inversion puts forward. However, having detailed information of what event type occurs in the data at specified time will be incentive and critical technology, no matter how full waveform processing becomes in the future, because the residual changes could depend critically on the nature of event (Sun and Innanen, 2016a).

Former research indicate that multiple attenuation is high correlated to its classified type. Take into account the influence of free-surface, multiples can be identified as two major classes, surface-related multiple and interbed multiple. Due to its periodic characteristic in τ - p domain, surface-related multiples can be eliminated in a comfortable manner and many innovative technologies have been developed in different domains, such as predictive deconvolution (Taner, 1980), inverse approach using feedback model (Verschuur, 1991), embedding technique (Liu et al., 2000), inverse data processing (Berkhout and Verschuur, 2005; Berkhout, 2006; Ma et al., 2009). However, the attenuation of the other classical multiple, internal multiple, still remains to be a big challenge, especially on land data, even though much considerable progresses have been made recently.

Kelamis et al. (2002a,b) introduced a boundary-related/layer-related approach to remove internal multiples in the post-stack data (CMP domain). Berkhout and Verschuur (2005) extended the inverse data processing to attenuate internal multiples by considering them as the suppositional surface-related multiples through the boundary-related/layer-

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related approach in common-focus-point (CFP) domain. The same algorithm was applied by Luo et al. (2007) through re-datuming the top of the multiple generators and transforming internal multiples to be 'surface-related'. The common ground of those algorithms is that, as it were, extensive knowledge of subsurface is required;thus if the possibility exists that multiple removal will have to take place with incomplete knowledge of the velocity structure and generators, the ISS approach will be optimal.

By analysing the philosophy of forward scattering series, Araujo et al. (1994) and Weglein et al. (1997, 2003) demonstrated that all possible internal multiples can be reconstructed, in an automatic way, as the combination of those sub-events satisfying a certain criterion, and the processing can be achieved by implementing the inverse scattering series in an appropriate manner. Many incentive research and discussions of inverse scattering series on internal multiple attenuation have been made depending on the variant purposes, (1) correcting predicted amplitude of internal multiples (Zou and Weglein, 2015), and (2) refining the algorithm for certain high priority acquisition styles and environments (Hernandez and Innanen, 2014; Pan et al., 2014; Pan and Innanen, 2015; Pan, 2015; Innanen, 2016b,a; Sun and Innanen, 2014, 2015, 2016a,b), since it was developed by Weglein and collaborators in 1990s.

However, all those research and discussions were on account of one assumption, earth is acoustic, which is unrealistic. As we know, the more realistic the geological model we build is, the more accurate the prediction algorithm becomes to be. In this paper, based on the previous work discussed by Matson (1997), we derive the elastic internal multiple prediction using inverse scattering series from multi-component seismic data, by considering an elastic, isotropic and homogeneous media as background. Even earth is not elastic isotropic medium, this paper could still be a pioneer to the internal multiple prediction on land data. Start with the stress-strain relation for an isotropic elastic medium,

$$\sigma_{ij} = \lambda \mathcal{D} \delta_{ij} + 2\mu e_{ij}, \qquad i, j = 1, 2, 3 \tag{1}$$

where, λ and μ are knowns as *Lamé constants*, $\mathcal{D} = \sum_{k=1}^{3} e_{kk} = \nabla \cdot \mathbf{u}$ is the dilatation.

Euler's equation of motion will reduce to *Cauchy's equation of motion* if the infinitesimal theory of elasticity is considered,

$$\frac{\partial \vec{\boldsymbol{\sigma}}}{\partial \vec{\mathbf{r}}} + \boldsymbol{\mathcal{F}} + \rho\omega^2 \mathbf{u} = 0 \tag{2}$$

Considering stress-strain relation, the first term in the equation of motion can be expanded as, in vector notation,

$$\frac{\partial \vec{\boldsymbol{\sigma}}}{\partial \vec{\mathbf{r}}} = (\lambda + \mu)\nabla(\nabla \cdot \vec{\mathbf{u}}) + (\nabla\lambda)\nabla \cdot \vec{\mathbf{u}} + \mu\nabla^2\vec{\mathbf{u}} + (\nabla\vec{\mathbf{u}} + \vec{\mathbf{u}}\nabla) \cdot (\nabla\mu) \tag{3}$$

Reword the *Cauchy's equation of motion* by substituting the expansion (Eq.3) and leaving out the body forces,

$$(\lambda + \mu)\nabla(\nabla \cdot \vec{\mathbf{u}}) + (\nabla\lambda)\nabla \cdot \vec{\mathbf{u}} + \mu\nabla^2\vec{\mathbf{u}} + (\nabla\vec{\mathbf{u}} + \vec{\mathbf{u}}\nabla) \cdot (\nabla\mu) + \rho\omega^2\mathbf{u} = 0 \tag{4}$$

Based on Eq. (4), the wave equation in frequency domain for an elastic isotropic medium can be derived and expressed in terms of propagating operator (see Appendix A),

$$\mathfrak{L}(\mathbf{r},\omega)\mathbf{u}(\mathbf{r},\omega) = 0 \tag{5}$$

where,

$$\begin{aligned} \mathfrak{L}_{ii} &= \partial_i[(\lambda+\mu)\partial_i] + \sum_k \partial_k(\mu\partial_k) + \rho\omega^2, \\ \mathfrak{L}_{ij} &= \partial_i(\lambda\partial_j) + \partial_j(\mu\partial_i), \quad i,j,k = x,y,z \; and \; j \neq i. \\ \mathbf{u} &= [u_x, u_y, u_z]^T. \end{aligned} \tag{6}$$

ELASTIC SCATTERING POTENTIAL, DIAGONALIZATION AND ROTATION

By adding a delta function as source term, Green's function obeys a similar form, in frequency domain, it can be expressed as,

$$\mathfrak{L}(\mathbf{r},\omega)\boldsymbol{\mathcal{G}}(\mathbf{r},\mathbf{r}_s,\omega) = -\delta(\mathbf{r}-\mathbf{r}_s) \tag{7}$$

Consider an elastic isotropic homogeneous medium as the background medium (which means, λ and μ do not vary with space locations), therefore, wave equation (Eq. 7) can be simplified as,

$$\mathfrak{L}_0(\mathbf{r},\omega)\boldsymbol{\mathcal{G}}_0(\mathbf{r},\mathbf{r}_s,\omega) = -\delta(\mathbf{r}-\mathbf{r}_s) \tag{8}$$

where,

$$\begin{aligned} \mathfrak{L}_{0ii} &= (\lambda_0+\mu_0)\partial_i\partial_i + \mu_0 \sum_k \partial_k\partial_k + \rho_0\omega^2, \\ \mathfrak{L}_{0ij} &= \lambda_0\partial_i\partial_j + \mu_0\partial_j\partial_i, \quad i,j,k = x,y,z \; and \; j \neq i. \end{aligned} \tag{9}$$

with $\gamma = \lambda + 2\mu = \rho\alpha^2$ and $\mu = \rho\beta^2$ (α and β denote the P- and S-wave velocities in background medium), perturbations can be defined as,

$$\begin{aligned} a_\rho &= \frac{\rho-\rho_0}{\rho} = \frac{\rho}{\rho_0} - 1 \approx \frac{\triangle\rho}{\rho} \\ a_\gamma &= \frac{\gamma-\gamma 0}{\rho} = \frac{\gamma}{\gamma 0} - 1 \approx \frac{\triangle\gamma}{\gamma} \\ a_\mu &= \frac{\mu-\mu_0}{\mu} = \frac{\mu}{\mu_0} - 1 \approx \frac{\triangle\mu}{\mu} \end{aligned} \tag{10}$$

An elastic isotropic scattering potential is the difference of wave operator in real and reference mediums, which can be expressed as,

$$\boldsymbol{\mathcal{V}} = \mathfrak{L} - \mathfrak{L}_0 = \begin{pmatrix} \mathcal{V}_{xx} & \mathcal{V}_{xy} & \mathcal{V}_{xz} \\ \mathcal{V}_{yx} & \mathcal{V}_{yy} & \mathcal{V}_{yz} \\ \mathcal{V}_{zx} & \mathcal{V}_{zy} & \mathcal{V}_{zz} \end{pmatrix} \tag{11}$$

where,

$$\begin{aligned}\mathcal{V}_{ii} &= \rho_0\big[\omega^2 a_\rho + \alpha_0^2\partial_i(a_\gamma\partial_i) + \beta_0^2\sum_{j\neq i}\partial_j(a_\mu\partial_j)\big], \quad i,j=x,y,z;\\ \mathcal{V}_{ij} &= \rho_0\big[\alpha_0^2\partial_i(a_\gamma\partial_j) - 2\beta_0^2\partial_i(a_\mu\partial_j) + \beta_0^2\partial_j(a_\mu\partial_i)\big], \quad j\neq i.\end{aligned} \tag{12}$$

In background (elastic, isotropic, and homogeneous) medium, with body force $\boldsymbol{\mathcal{F}}$ included, Eq.(4) can be simplified as,

$$(\lambda+\mu)\nabla(\nabla\cdot\vec{\mathbf{u}}) + \mu\nabla^2\vec{\mathbf{u}} + \boldsymbol{\mathcal{F}} + \rho\omega^2\mathbf{u} = 0 \tag{13}$$

Helmholtz's theorem states that any well-defined vector can be decomposed as the sum of a curl-free component and a divergence-free component. Therefore, we can rewrite the particle displacement $\mathbf{u}$ and the body force $\boldsymbol{\mathcal{F}}$ as,

$$\begin{aligned}\mathbf{u} &= \nabla\boldsymbol{\phi} + \nabla\times\boldsymbol{\psi}\\ \boldsymbol{\mathcal{F}} &= \nabla\boldsymbol{\Phi} + \nabla\times\boldsymbol{\Psi}\end{aligned} \tag{14}$$

where, $\boldsymbol{\phi}$ and $\boldsymbol{\Phi}$ are scalar potentials, $\boldsymbol{\psi}$ and $\boldsymbol{\Psi}$ are vector potentials.

Substitute decompositions of displacement and body force into Eq.(13) and do the math, we have,

$$\nabla[(\lambda+2\mu)\nabla^2\boldsymbol{\phi} + \boldsymbol{\Phi} + \rho\omega^2\boldsymbol{\phi}] + \nabla\times[\mu\nabla^2\boldsymbol{\psi} + \boldsymbol{\Psi} + \rho\omega^2\boldsymbol{\psi}] = 0 \tag{15}$$

P- or S-wave equation can be obtained, by taking a divergence or a curl of Eq.(15), respectively. Therefore, if we define a partial derivatives matrix $\boldsymbol{\Pi}$ (following the notation demonstrated by Stolt and Weglein (2012)), including a divergence and a curl, which is expressed as,

$$\boldsymbol{\Pi} = \begin{pmatrix}\nabla\cdot\\ \nabla\times\end{pmatrix} = \begin{pmatrix}\partial_x & \partial_y & \partial_z\\ 0 & -\partial_z & \partial_y\\ \partial_z & 0 & -\partial_x\\ -\partial_y & \partial_x & 0\end{pmatrix} \tag{16}$$

Then P- and S-wave components can also be separated by acting the derivatives matrix on the particle displacement,

$$\begin{pmatrix}\boldsymbol{\varphi}_P\\ \boldsymbol{\varphi}_S\end{pmatrix} = \boldsymbol{\Pi}\mathbf{u} = \begin{pmatrix}\nabla\cdot\mathbf{u}\\ \nabla\times\mathbf{u}\end{pmatrix} = \begin{pmatrix}\partial_x u_x + \partial_y u_y + \partial_z u_z\\ \partial_y u_z - \partial_z u_y\\ \partial_z u_x - \partial_x u_z\\ \partial_x u_y - \partial_y u_x\end{pmatrix} = \begin{pmatrix}\nabla\cdot\mathbf{u}\\ (\nabla\times\mathbf{u})_x\\ (\nabla\times\mathbf{u})_y\\ (\nabla\times\mathbf{u})_z\end{pmatrix} \tag{17}$$

Beyond that, the inverse of diagonal matrix can be calculated by its transpose by premultiplied an inverse Laplacian operator (See Appendix B). In equation, it's shown as,

$$\boldsymbol{\Pi}^{-1} = \nabla^{-2}\boldsymbol{\Pi}^T \tag{18}$$

Stolt and Weglein (2012) also indicated that the wave operator can be diagonalized into P- and S-wave operators by pre-multiplying the partial derivatives matrix and post-multiplying its inverse. Formally,

$$\mathfrak{L}_{0D} = \begin{pmatrix} \mathfrak{L}_{0P} & 0 & 0 & 0 \\ 0 & \mathfrak{L}_{0S} & 0 & 0 \\ 0 & 0 & \mathfrak{L}_{0S} & 0 \\ 0 & 0 & 0 & \mathfrak{L}_{0S} \end{pmatrix} = \mathbf{\Pi}\mathfrak{L}_0\mathbf{\Pi}^{-1} \tag{19}$$

where, $\mathfrak{L}_P$ and $\mathfrak{L}_S$ are P- and S-wave operators, written as,

$$\begin{aligned} \mathfrak{L}_{0P} &= (\lambda + 2\mu)\nabla^2 + \rho\omega^2 \\ \mathfrak{L}_{0S} &= \mu\nabla^2 + \rho\omega^2 \end{aligned} \tag{20}$$

With the diagonalized wave operator $\mathfrak{L}_{0D}$, the wave equation for an isotropic elastic medium becomes,

$$\mathfrak{L}_{0D}\mathbf{\Pi}\mathbf{u} = \begin{pmatrix} \mathfrak{L}_P\boldsymbol{\varphi}_P \\ \mathfrak{L}_S\boldsymbol{\varphi}_S \end{pmatrix} = 0 \tag{21}$$

Similar equations included diagonalized Green's function for real and reference medium can be expressed as,

$$\boldsymbol{\mathfrak{L}}_D(\mathbf{r},\omega)\boldsymbol{\mathcal{G}}_D(\mathbf{r},\mathbf{r}_s,\omega) = -\delta(\mathbf{r}-\mathbf{r}_s) \tag{22a}$$

$$\boldsymbol{\mathfrak{L}}_{0D}(\mathbf{r},\omega)\boldsymbol{\mathcal{G}}_{0D}(\mathbf{r},\mathbf{r}_s,\omega) = -\delta(\mathbf{r}-\mathbf{r}_s) \tag{22b}$$

where,

$$\boldsymbol{\mathcal{G}}_D = \mathbf{\Pi}\boldsymbol{\mathcal{G}}\mathbf{\Pi}^{-1} = \begin{pmatrix} \mathcal{G}_{PP} & \mathcal{G}_{PS_x} & \mathcal{G}_{PS_y} & \mathcal{G}_{PS_z} \\ \mathcal{G}_{S_xP} & \mathcal{G}_{S_xS_x} & \mathcal{G}_{S_xS_y} & \mathcal{G}_{S_xS_z} \\ \mathcal{G}_{S_yP} & \mathcal{G}_{S_yS_x} & \mathcal{G}_{S_yS_y} & \mathcal{G}_{S_yS_z} \\ \mathcal{G}_{S_zP} & \mathcal{G}_{S_zS_x} & \mathcal{G}_{S_zS_y} & \mathcal{G}_{S_zS_z} \end{pmatrix} \tag{23a}$$

$$\boldsymbol{\mathcal{G}}_{0D} = \mathbf{\Pi}\boldsymbol{\mathcal{G}}_0\mathbf{\Pi}^{-1} = \begin{pmatrix} \mathcal{G}_{0P} & 0 & 0 & 0 \\ 0 & \mathcal{G}_{0S_x} & 0 & 0 \\ 0 & 0 & \mathcal{G}_{0S_y} & 0 \\ 0 & 0 & 0 & \mathcal{G}_{0S_z} \end{pmatrix} \tag{23b}$$

Eq.(22) indicates that,the Green's function can be rewritten into 4×4 matrix with respect to P- and S-wave components, by applying the transformation $\mathbf{\Pi}\boldsymbol{\mathcal{G}}\mathbf{\Pi}^{-1}$, which also works for the propagating operator (Eq. 19). For inhomogeneous isotropic elastic (real) medium, the diagonal elements of $\boldsymbol{\mathcal{G}}_D$ correspond to Green's functions of PP and $x-, y-, z-$ components of SS waves, and off-diagonal terms relate to Green's functions of converted waves from one to another. For homogeneous isotropic elastic (reference) medium, the upper left diagonal term of $\boldsymbol{\mathcal{G}}_{0D}$ is the Green's function for PP-wave, and other diagonal terms correspond to $x-, y-, z-$ components of Green's function for SS wave, and off-diagonal terms are zeros.

One of the disadvantage of the diagonal matrix $\mathbf{\Pi}$ is that it maps a 3D vector into a 4D space with only three independent dimensions present. It does separate P-wave successfully from S-wave components, but it does not work for SV- and SH-waves. To decompose the elastic scattering potential into P-, SH-, SV-modes, Stolt and Weglein (2012) introduced rotation matrices by rotating S-wave components to a local system in which the third (longitudinal) S-wave component is zero. Before the rotation, we have to rewrite the diagonal matrix in terms of P- and S-wave wavenumbers, which is expressed as,

$$\mathbf{\Pi} \to \mathbf{\Pi}_r = \mathrm{i}\begin{pmatrix} k_{Prx} & k_{Pry} & k_{Prz} \\ 0 & -k_{Srz} & k_{Sry} \\ k_{Srz} & 0 & -k_{Srx} \\ -k_{Sry} & k_{Srx} & 0 \end{pmatrix} = \mathrm{i}\begin{pmatrix} \boldsymbol{k}_{Pr}\cdot^{T} \\ \boldsymbol{k}_{Sr}\times \end{pmatrix} \tag{24}$$

and

$$\begin{aligned} \mathbf{\Pi}^{-1} \to (\mathbf{\Pi}^{-1})_i &= \frac{-\mathrm{i}}{\omega^2}\begin{pmatrix} \alpha_0^2 k_{Pix} & 0 & \beta_0^2 k_{Siz} & -\beta_0^2 k_{Siy} \\ \alpha_0^2 k_{Piy} & -\beta_0^2 k_{Siz} & 0 & \beta_0^2 k_{Six} \\ \alpha_0^2 k_{Piz} & \beta_0^2 k_{Siy} & -\beta_0^2 k_{Six} & 0 \end{pmatrix} \\ &= \frac{-\mathrm{i}}{\omega^2}\left[\alpha_0^2 \boldsymbol{k}_{Pi}\cdot \quad \beta_0^2(\boldsymbol{k}_{Si}\times)^T\right] \end{aligned} \tag{25}$$

where,

$$\begin{aligned} k_{Pr} = k_{Pi} &= \frac{\omega}{\alpha_0} \\ k_{Sr} = k_{Si} &= \frac{\omega}{\beta_0} \end{aligned} \tag{26}$$

Correspondingly, 4×4 rotation matrices can be defined for incident and reflected waves,

$$\boldsymbol{E}_i = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & e_{SVix} & e_{SViy} & e_{SViz} \\ 0 & -e_{SHx} & -e_{SHy} & -e_{SHz} \end{pmatrix} = \begin{pmatrix} 1 & \mathbf{0}^T \\ 0 & \hat{\boldsymbol{e}}_{SVi}^T \\ 0 & -\hat{\boldsymbol{e}}_{SH}^T \end{pmatrix} \tag{27}$$

and

$$\boldsymbol{E}_r = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & e_{SVrx} & e_{SVry} & e_{SVrz} \\ 0 & -e_{SHx} & -e_{SHy} & -e_{SHz} \end{pmatrix} = \begin{pmatrix} 1 & \mathbf{0}^T \\ 0 & \hat{\boldsymbol{e}}_{SVr}^T \\ 0 & -\hat{\boldsymbol{e}}_{SH}^T \end{pmatrix} \tag{28}$$

also, we have,

$$\boldsymbol{E}\boldsymbol{E}^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{29}$$

and

$$\boldsymbol{E}^T\boldsymbol{E}\mathbf{\Pi} = \mathbf{\Pi} \tag{30}$$

which states that the inverse of $\boldsymbol{E}$ equals to its transpose.

After applied the rotation matrix, the $x-, y-, z-$ components of S-wave are decomposed into SH- and SV-modes. Involving the orthogonality relations, the combined diagonal and rotation matrices for incident wave,

$$(\mathbf{\Pi}^{-1})_i \boldsymbol{E}_i^{-1} = -\mathrm{i}\left[\frac{\alpha_0^2}{\omega^2}\boldsymbol{k}_{Pi} \quad \frac{\beta_0}{\omega}\hat{\boldsymbol{e}}_{SH} \quad \frac{\beta_0}{\omega}\hat{\boldsymbol{e}}_{SVi}\right] \tag{31}$$

for reflected wave,

$$\boldsymbol{E}_r\boldsymbol{\Pi}_r = \mathrm{i}\begin{bmatrix} \boldsymbol{k}_{Pr}^T \\ \frac{\omega}{\beta_0}\hat{\boldsymbol{e}}_{SH}^T \\ \frac{\omega}{\beta_0}\hat{\boldsymbol{e}}_{SVr}^T \end{bmatrix} \tag{32}$$

Therefore, we have,

$$\boldsymbol{E}_r\boldsymbol{\Pi}_r(\boldsymbol{\Pi}^{-1})_i\boldsymbol{E}_i^{-1} = \mathbf{I} \tag{33}$$

where, **I** is an identity matrix.

In conclusion, the wave equation containing Green's function in P-, SH-, SV-modes can be expressed as,

$$\boldsymbol{\mathcal{L}}(\mathbf{r},\omega)\boldsymbol{G}(\mathbf{r},\mathbf{r}_s,\omega) = -\delta(\mathbf{r}-\mathbf{r}_s) \tag{34a}$$

$$\boldsymbol{\mathcal{L}}_0(\mathbf{r},\omega)\boldsymbol{G}_0(\mathbf{r},\mathbf{r}_s,\omega) = -\delta(\mathbf{r}-\mathbf{r}_s) \tag{34b}$$

where,

$$\boldsymbol{\mathcal{L}} = \boldsymbol{E}_r\boldsymbol{\mathfrak{L}}_D\boldsymbol{E}_i^{-1} = \boldsymbol{E}_r\boldsymbol{\Pi}_r\boldsymbol{\mathfrak{L}}(\boldsymbol{\Pi}^{-1})_i\boldsymbol{E}_i^{-1} \tag{35a}$$

$$\boldsymbol{\mathcal{L}}_0 = \boldsymbol{E}_r\boldsymbol{\mathfrak{L}}_{0D}\boldsymbol{E}_i^{-1} = \boldsymbol{E}_r\boldsymbol{\Pi}_r\boldsymbol{\mathfrak{L}}_0(\boldsymbol{\Pi}^{-1})_i\boldsymbol{E}_i^{-1} \tag{35b}$$

and

$$\mathbf{G} = \boldsymbol{E}_r\boldsymbol{\mathcal{G}}_D\boldsymbol{E}_i^{-1} = \boldsymbol{E}_r\boldsymbol{\Pi}_r\boldsymbol{\mathcal{G}}(\boldsymbol{\Pi}^{-1})_i\boldsymbol{E}_i^{-1} = \begin{pmatrix} \mathbf{G}_{PP} & \mathbf{G}_{PSH} & \mathbf{G}_{PSV} \\ \mathbf{G}_{SHP} & \mathbf{G}_{SHSH} & \mathbf{G}_{SHSV} \\ \mathbf{G}_{SVP} & \mathbf{G}_{SVSH} & \mathbf{G}_{SVSV} \end{pmatrix} \tag{36a}$$

$$\mathbf{G}_0 = \boldsymbol{E}_r\boldsymbol{\mathcal{G}}_{0D}\boldsymbol{E}_i^{-1} = \boldsymbol{E}_r\boldsymbol{\Pi}_r\boldsymbol{\mathcal{G}}_0(\boldsymbol{\Pi}^{-1})_i\boldsymbol{E}_i^{-1} = \begin{pmatrix} \mathbf{G}_{0P} & 0 & 0 \\ 0 & \mathbf{G}_{0S} & 0 \\ 0 & 0 & \mathbf{G}_{0S} \end{pmatrix} \tag{36b}$$

Also, for the scattering potential, a similar form can be achieved,

$$\mathbf{V} = \boldsymbol{E}_r\boldsymbol{\Pi}_r\boldsymbol{\mathcal{V}}(\boldsymbol{\Pi}^{-1})_i\boldsymbol{E}_i^{-1} = \begin{pmatrix} \mathbf{V}_{PP} & \mathbf{V}_{PSH} & \mathbf{V}_{PSV} \\ \mathbf{V}_{SHP} & \mathbf{V}_{SHSH} & \mathbf{V}_{SHSV} \\ \mathbf{V}_{SVP} & \mathbf{V}_{SVSH} & \mathbf{V}_{SVSV} \end{pmatrix} \tag{37}$$

INVERSE SCATTERING SERIES FOR ELASTIC MEDIUM

For an isotropic elastic medium, the wave field can be expressed as a background field adding perturbations. Considering the homogeneous isotropic elastic medium as the background, the Born series for an isotropic elastic medium can be written as,

$$\boldsymbol{\mathcal{G}} = \boldsymbol{\mathcal{G}}_0 + \boldsymbol{\mathcal{G}}_0\boldsymbol{\mathcal{V}}\boldsymbol{\mathcal{G}}_0 + \boldsymbol{\mathcal{G}}_0\boldsymbol{\mathcal{V}}\boldsymbol{\mathcal{G}}_0\boldsymbol{\mathcal{V}}\boldsymbol{\mathcal{G}}_0 + \boldsymbol{\mathcal{G}}_0\boldsymbol{\mathcal{V}}\boldsymbol{G}_0\boldsymbol{\mathcal{V}}\boldsymbol{\mathcal{G}}_0\boldsymbol{\mathcal{V}}\boldsymbol{\mathcal{G}}_0 + ... \tag{38}$$

As discussed above, the Green's Function can be devised into a 3×3 matrix corresponded to P-, SV-, and SH-wave modes with pre- or post-multiplied diagonal matrix $\boldsymbol{\Pi}$

and rotation matrix $\boldsymbol{E}$. Therefore, replacing the Green's function $\boldsymbol{\mathcal{G}}$ and $\boldsymbol{\mathcal{G}}_0$ in Eq. (38), Born series can be rewritten into P-, SH-, and SV-modes,

$$\begin{aligned}(\boldsymbol{\Pi}^{-1})_i\boldsymbol{E}_i^{-1}\mathbf{G}\boldsymbol{E}_r\boldsymbol{\Pi}_r &= (\boldsymbol{\Pi}^{-1})_i\boldsymbol{E}_i^{-1}\mathbf{G}_0\boldsymbol{E}_r\boldsymbol{\Pi}_r + (\boldsymbol{\Pi}^{-1})_i\boldsymbol{E}_i^{-1}\mathbf{G}_0\boldsymbol{E}_r\boldsymbol{\Pi}_r\boldsymbol{\mathcal{V}}(\boldsymbol{\Pi}^{-1})_i\boldsymbol{E}_i^{-1}\mathbf{G}_0\boldsymbol{E}_r\boldsymbol{\Pi}_r\\ &+ (\boldsymbol{\Pi}^{-1})_i\boldsymbol{E}_i^{-1}\mathbf{G}_0\boldsymbol{E}_r\boldsymbol{\Pi}_r\boldsymbol{\mathcal{V}}(\boldsymbol{\Pi}^{-1})_i\boldsymbol{E}_i^{-1}\mathbf{G}_0\boldsymbol{E}_r\boldsymbol{\Pi}_r\boldsymbol{\mathcal{V}}(\boldsymbol{\Pi}^{-1})_i\boldsymbol{E}_i^{-1}\mathbf{G}_0\boldsymbol{E}_r\boldsymbol{\Pi}_r\\ &+ ...\end{aligned} \tag{39}$$

Combining the relationship between diagonal matrix $\boldsymbol{\Pi}$ and rotation matrix $\boldsymbol{E}$ (Eq. 33), pre-multiplying $\boldsymbol{E}_r\boldsymbol{\Pi}_r$ and post-multiplied by $(\boldsymbol{\Pi}^{-1})_i\boldsymbol{E}_i^{-1}$, applying Eq.(37) to transform the elastic scattering potential into P-, SH-, SV-wave modes, Eq.(39) can be simplified as,

$$\mathbf{G} = \mathbf{G}_0 + \mathbf{G}_0\mathbf{V}\mathbf{G}_0 + \mathbf{G}_0\mathbf{V}\mathbf{G}_0\mathbf{V}\mathbf{G}_0 + \mathbf{G}_0\mathbf{V}\mathbf{G}_0\mathbf{V}\mathbf{G}_0\mathbf{V}\mathbf{G}_0 + ... \tag{40}$$

Recall the wave equation for inhomogeneous isotropic elastic medium, containing source term, which can be expressed as,

$$\boldsymbol{\mathfrak{L}}(\mathbf{r},\omega)\mathbf{u}(\mathbf{r},\omega) = \mathbf{f} \tag{41}$$

The displacement field will be decomposed into P-, SH-, SV-wave modes by pre-multiplying $\boldsymbol{E}_r\boldsymbol{\Pi}_r$,

$$\begin{bmatrix}\boldsymbol{\varphi}_P\\ \boldsymbol{\varphi}_{SH}\\ \boldsymbol{\varphi}_{SV}\end{bmatrix} = \boldsymbol{E}_r\boldsymbol{\Pi}_r\mathbf{u} = \boldsymbol{E}_r\boldsymbol{\Pi}_r\boldsymbol{\mathcal{G}}\mathbf{f} = \mathbf{G}\boldsymbol{E}_r\boldsymbol{\Pi}_r\mathbf{f} = \mathbf{G}\mathbf{F} \tag{42}$$

where, $\mathbf{F} = \boldsymbol{E}_r\boldsymbol{\Pi}_r\mathbf{f} = [1,0,0]^T$ if the incidence is a spike of P-wave only.

Incorporating Eq.(42) and Eq.(40), the separated scattering wavefield can be rewritten as, in terms of background and perturbations,

$$\boldsymbol{D}\mathbf{F} = (\mathbf{G}-\mathbf{G}_0)\mathbf{F} = \mathbf{G}_0\mathbf{V}\mathbf{G}_0\mathbf{F} + \mathbf{G}_0\mathbf{V}\mathbf{G}_0\mathbf{V}\mathbf{G}_0\mathbf{F} + \mathbf{G}_0\mathbf{V}\mathbf{G}_0\mathbf{V}\mathbf{G}_0\mathbf{V}\mathbf{G}_0\mathbf{F} + ... \tag{43}$$

with its matrix form,

$$\begin{aligned}&\begin{bmatrix}\boldsymbol{D}_{PP} & \boldsymbol{D}_{PSH} & \boldsymbol{D}_{PSV}\\ \boldsymbol{D}_{SHP} & \boldsymbol{D}_{SHSH} & \boldsymbol{D}_{SHSV}\\ \boldsymbol{D}_{SVP} & \boldsymbol{D}_{SVSH} & \boldsymbol{D}_{SVSV}\end{bmatrix}\mathbf{F}\\ &= \begin{bmatrix}\mathbf{G}_{0P} & 0 & 0\\ 0 & \mathbf{G}_{0S} & 0\\ 0 & 0 & \mathbf{G}_{0S}\end{bmatrix}\begin{bmatrix}\mathbf{V}_{PP} & \mathbf{V}_{PSH} & \mathbf{V}_{PSV}\\ \mathbf{V}_{SHP} & \mathbf{V}_{SHSH} & \mathbf{V}_{SHSV}\\ \mathbf{V}_{SVP} & \mathbf{V}_{SVSH} & \mathbf{V}_{SVSV}\end{bmatrix}\begin{bmatrix}\mathbf{G}_{0P} & 0 & 0\\ 0 & \mathbf{G}_{0S} & 0\\ 0 & 0 & \mathbf{G}_{0S}\end{bmatrix}\mathbf{F}\\ &+ \begin{bmatrix}\mathbf{G}_{0P} & 0 & 0\\ 0 & \mathbf{G}_{0S} & 0\\ 0 & 0 & \mathbf{G}_{0S}\end{bmatrix}\begin{bmatrix}\mathbf{V}_{PP} & \mathbf{V}_{PSH} & \mathbf{V}_{PSV}\\ \mathbf{V}_{SHP} & \mathbf{V}_{SHSH} & \mathbf{V}_{SHSV}\\ \mathbf{V}_{SVP} & \mathbf{V}_{SVSH} & \mathbf{V}_{SVSV}\end{bmatrix}\begin{bmatrix}\mathbf{G}_{0P} & 0 & 0\\ 0 & \mathbf{G}_{0S} & 0\\ 0 & 0 & \mathbf{G}_{0S}\end{bmatrix}\\ &\quad\begin{bmatrix}\mathbf{V}_{PP} & \mathbf{V}_{PSH} & \mathbf{V}_{PSV}\\ \mathbf{V}_{SHP} & \mathbf{V}_{SHSH} & \mathbf{V}_{SHSV}\\ \mathbf{V}_{SVP} & \mathbf{V}_{SVSH} & \mathbf{V}_{SVSV}\end{bmatrix}\begin{bmatrix}\mathbf{G}_{0P} & 0 & 0\\ 0 & \mathbf{G}_{0S} & 0\\ 0 & 0 & \mathbf{G}_{0S}\end{bmatrix}\mathbf{F} + ...\end{aligned} \tag{44}$$

In an effort to avoid confusion and awkward phrasing, we rewrite the Eq.(44) using subscripts,

$$D_{ij} = G_{0i}V_{ij}G_{0j} + G_{0i}V_{ik}G_{0k}V_{kj}G_{0j} + G_{0i}V_{ik}G_{0k}V_{kl}G_{0l}V_{lj}G_{0j} + ... \tag{45}$$

where, the subscripts denote P-, SH-, SV- components for wave propagation or scattering. D_{ij} is an specified element of decomposed measured data related to i, j. And i is the wave mode for reflected wave or on receiver coordinate. j is the mode for incident wave, which means $j = P$ if the incidence is P-wave only. The wave propagation through perturbation mode for an elastic medium is shown in Figure. 1.

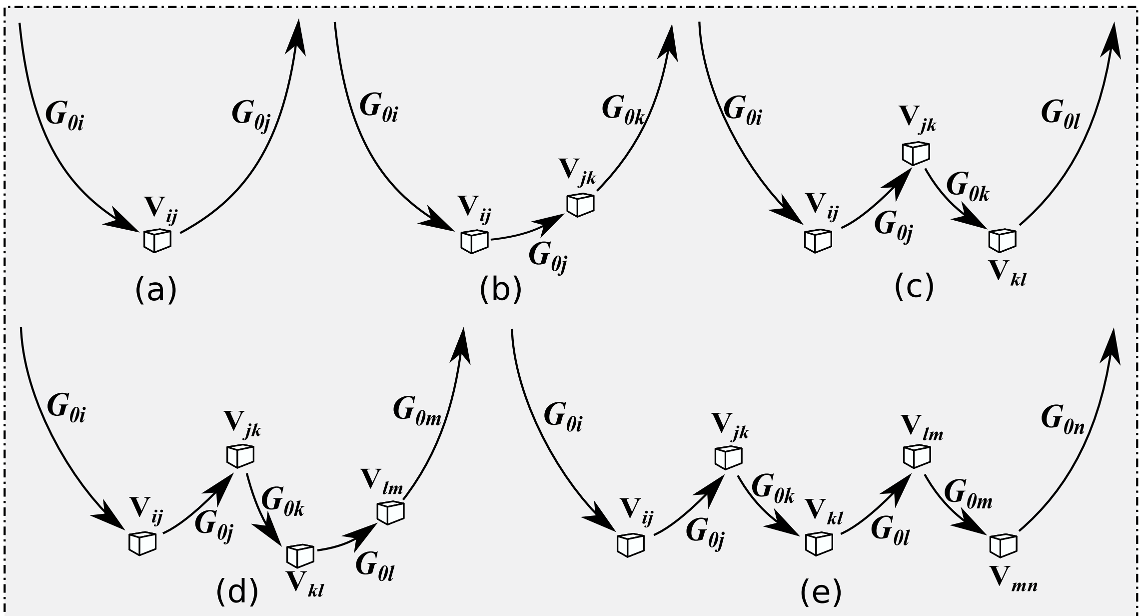

FIG. 1. Wave propagation in perturbation mode. The subscripts (i, j, k, l, m, n) denote the wave mode, $\in \{P, SH, SV\}$.

Similar with acoustic inverse scattering, the scattering potential can be expanded into series by orders,

$$V = V^{(1)} + V^{(2)} + V^{(3)} + ... \tag{46}$$

Substitute this expansion (Eq. 46) into subscripted Born series (Eq. 45), and equate like orders, we have,

$$D_{ij} = G_{0i}V^{(1)}_{ij}G_{0j}, \tag{47a}$$

$$0 = G_{0i}V^{(2)}_{ij}G_{0j} + G_{0i}V^{(1)}_{ik}G_{0k}V^{(1)}_{kj}G_{0j}, \tag{47b}$$

$$0 = G_{0i}V^{(3)}_{ij}G_{0j} + G_{0i}V^{(2)}_{ik}G_{0k}V^{(1)}_{kj}G_{0j} + G_{0i}V^{(1)}_{ik}G_{0k}V^{(2)}_{kj}G_{0j} + G_{0i}V^{(1)}_{ik}G_{0k}V^{(1)}_{kl}G_{0l}V^{(1)}_{lj}G_{0j}, \tag{47c}$$

$$\vdots$$

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Based on Eq. (47a), the first-order of elastic scattering potential can be delineated by the decomposed measured data D_{ij} and Green's function for pure P- or S-waves. Recall

3D Green's function for pure P- or S-wave,

$$G_{0i}(k_{x_1},k_{y_1},z_1,x_2,y_2,z_2,\omega)=\frac{e^{-\mathrm{i}(k_{ix_1}x_2+k_{iy_1}y_2)}e^{\mathrm{i}\nu_{i1}|z_1-z_2|}}{\mathrm{i}2\nu_{i1}} \tag{48a}$$

$$G_{0i}(x_1,y_1,z_1,k_{x_2},k_{y_2},z_2,\omega)=\frac{e^{\mathrm{i}(k_{ix_2}x_1+k_{iy_2}y_1)}e^{\mathrm{i}\nu_{i2}|z_2-z_1|}}{\mathrm{i}2\nu_{i2}} \tag{48b}$$

with

$$\nu_{i1}=\sqrt{\frac{\omega^2}{c_{i0}^2}-k_{ix_1}^2-k_{iy_1}^2} \tag{48c}$$

where, k_{ix_1} and k_{iy_1} are x and y components of wavenumber, ν_{i1} is the vertical component of wavenumber. The subscript 1 means the side of location, i.e., k_{ix_1} is the x component of wavenumber corresponding to location (x_1,y_1,z_1). c_{i0} is the velocity depending on the wave mode i, and $i\in\{P,SH,SV\}$.

Therefore, take an inverse Fourier transform over k_x and k_y, the space-frequency domain 3D Green's function from one location (x_2,y_2,z_2) to another (x_1,y_1,z_1) can be written as,

$$G_{0i}(x_1,y_1,z_1,x_2,y_2,z_2,\omega)=\frac{1}{(2\pi)^2}\iint_{-\infty}^{+\infty}\frac{e^{\mathrm{i}k_{ix_2}(x_1-x_2)}e^{\mathrm{i}k_{iy_2}(y_1-y_2)}e^{\mathrm{i}\nu_{i2}|z_1-z_2|}}{\mathrm{i}2\nu_{i2}}\mathrm{d}k_{ix_2}\mathrm{d}k_{iy_2} \tag{49}$$

Further analysis of Eq. (49) indicates that Green's function can be considered as a superposition of weighted plane wave solution, as follow,

$$\begin{aligned}&G_{0i}(x_g,y_g,z_g,x_s,y_s,z_s,\omega)\\&=\frac{1}{(2\pi)^2}\iint_{-\infty}^{+\infty}\frac{e^{-\mathrm{i}(k_{ix_s}x_s+k_{iy_s}y_s)}}{\mathrm{i}2\nu_{is}}\phi_{0i}(x_g,y_g,z_g,k_{ix_s},k_{iy_s},z_s,\omega)\mathrm{d}k_{ix_s}\mathrm{d}k_{iy_s}\end{aligned} \tag{50}$$

where,

$$\phi_{0i}(x_g,y_g,z_g,k_{ix_s},k_{iy_s}.z_s,\omega)=\mathrm{e}^{\mathrm{i}(k_{ix_s}x_g+k_{iy_s}y_g+\nu_{is}|z_g-z_s|)} \tag{51}$$

Then we have, $\phi_{0i}(x_g,y_g,z_g,k_{ix_s},k_{iy_s}.z_s,\omega)=\mathrm{i}2\nu_{is}G_{0i}(x_g,z_g,k_{ix_s},k_{iy_s},z_s,\omega)$. Substitute this change into the inverse scattering series using reversion (Eq.47),

$$b_{1ij}=G_{0i}V_{ij}^{(1)}\phi_{0j}, \tag{52a}$$

$$0=G_{0i}V_{ij}^{(2)}\phi_{0j}+G_{0i}V_{ik}^{(1)}G_{0k}V_{kj}^{(1)}\phi_{0j}, \tag{52b}$$

$$0=G_{0i}V_{ij}^{(3)}\phi_{0j}+G_{0i}V_{ik}^{(2)}G_{0k}V_{kj}^{(1)}\phi_{0j}+G_{0i}V_{ik}^{(1)}G_{0k}V_{kj}^{(2)}\phi_{0j}+G_{0i}V_{ik}^{(1)}G_{0k}V_{kl}^{(1)}G_{0l}V_{lj}^{(1)}\phi_{0j}, \tag{52c}$$

$$\vdots$$

where, b_{1ij} is the weighted decomposed measured data, and can be calculated by $b_{1ij}(k_{ix_g},k_{iy_g},k_{jx_s},k_{jy_s},\omega)=\mathrm{i}2\nu_{js}D_{ij}(k_{ix_g},k_{iy_g},k_{jx_s},k_{jy_s},\omega)$ and ν_{js} depends on the mode of wave which is determined by j, and $j\in\{P,SH,SV\}$.

Substitute Green's function into Eq.(52a), the first-order of elastic scattering potential V_{ij} can be expressed in terms of the weighted decomposed measured data b_{1ij}. Expand Eq.(52a), we have,

$$
\begin{aligned}
&b_{1ij}(k_{ix_g}, k_{iy_g}, z_g, k_{jx_s}, k_{jy_s}, z_s, \omega) \\
&= \iiint_{-\infty}^{+\infty} \mathrm{d}x_1 \mathrm{d}y_1 \mathrm{d}z_1 \boldsymbol{G}_{0i}(k_{ix_g}, k_{iy_g}, z_g, x_1, y_1, z_1, \omega) V_{ij}^{(1)}(x_1, y_1, z_1) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \times \phi_{0j}(x_1, y_1, z_1, k_{jx_s}, k_{jy_s}, z_s, \omega) \\
&= \iiint_{-\infty}^{+\infty} \mathrm{d}x_1 \mathrm{d}y_1 \mathrm{d}z_1 \frac{e^{-\mathrm{i}(k_{ix_g}x_1 + k_{iy_g}y_1)} e^{\mathrm{i}\nu_{ig}|z_g - z_1|}}{\mathrm{i}2\nu_{ig}} V_{ij}^{(1)}(x_1, y_1, z_1) e^{\mathrm{i}(k_{jx_s}x_1 + k_{jy_s}y_1)} e^{\mathrm{i}\nu_{sj}|z_1 - z_s|} \\
&= \frac{e^{-\mathrm{i}\nu_{ig}z_g} e^{-\mathrm{i}\nu_{js}z_s}}{\mathrm{i}2\nu_{ig}} \iiint_{-\infty}^{+\infty} \mathrm{d}x_1 \mathrm{d}y_1 \mathrm{d}z_1 e^{\mathrm{i}(k_{jx_s} - k_{ix_g})x_1} e^{\mathrm{i}(k_{jy_s} - k_{iy_g})y_1} e^{\mathrm{i}(\nu_{js} + \nu_{ig})z_1} V_{ij}^{(1)}(x_1, y_1, z_1) \\
&= \frac{e^{-\mathrm{i}\nu_{ig}z_g} e^{-\mathrm{i}\nu_{js}z_s}}{\mathrm{i}2\nu_{ig}} \hat{V}_{ij}^{(1)}(k_{jx_s} - k_{ix_g}, k_{jy_s} - k_{iy_g}, \nu_{js} + \nu_{ig}|z_1)
\end{aligned}
\tag{53}
$$

Therefore,

$$
\begin{aligned}
&\hat{V}_{ij}^{(1)}(k_{jx_s} - k_{ix_g}, k_{jy_s} - k_{iy_g}, \nu_{js} + \nu_{ig}|z_1) \\
&\quad = \mathrm{i}2\nu_{ig} e^{\mathrm{i}(\nu_{ig}z_g + \nu_{js}z_s)} b_{1ij}(k_{ix_g}, k_{iy_g}, z_g, k_{jx_s}, k_{jy_s}, z_s, \omega)
\end{aligned}
\tag{54}
$$

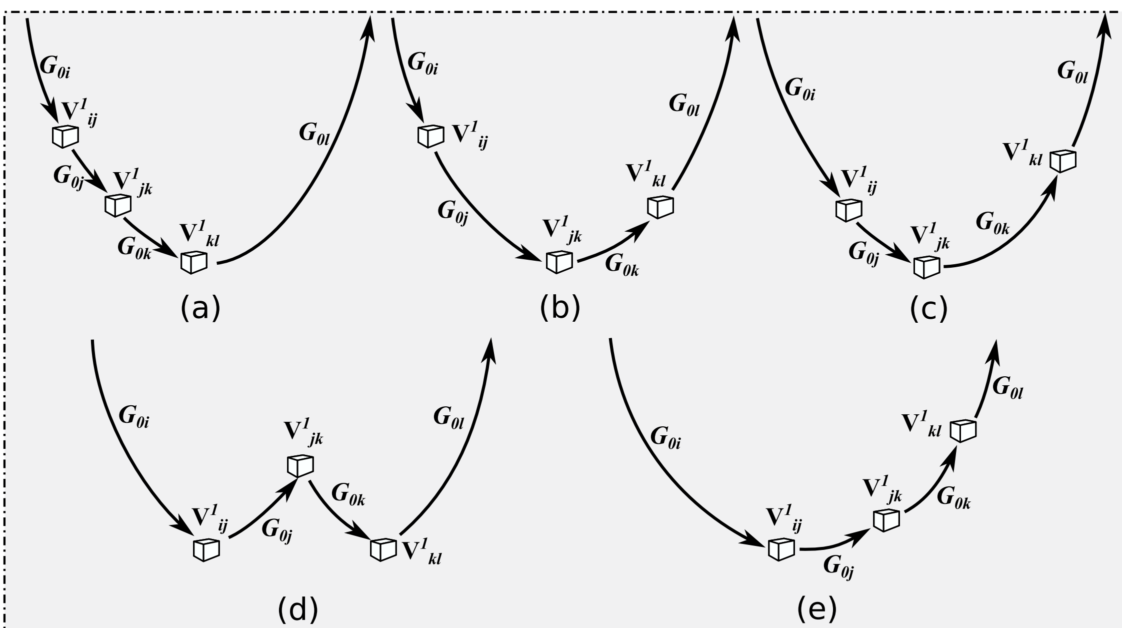

FIG. 2. Contributions of $G_{0i}V_{ij}^{(33)}\phi_{0j}$ depending on variant depth (z_1, z_2, z_3) relations between perturbations. (a) case of $z_1 < z_2 < z_3$, (b) case of $z_1 < z_3 < z_2$, (c) case of $z_3 < z_1 < z_2$, (d) case of $z2 < z_1$ and $z_2 < z_3$, (e) case of $z_3 < z_2 < z_1$.

The 1st-order internal multiple can be generated at least 3 perturbations which satisfy lower-higher-lower relationship in pseudo-depth (depth in reference medium). By analyzing 3rd order in inverse scattering series (Eq.52c), we have,

$$\begin{aligned} G_{0i}V_{ij}^{(3)}\phi_{0j} &= -(G_{0i}V_{ik}^{(2)}G_{0k}V_{kj}^{(1)}\phi_{0j} + G_{0i}V_{ik}^{(1)}G_{0k}V_{kj}^{(2)}\phi_{0j} + G_{0i}V_{ik}^{(1)}G_{0k}V_{kl}^{(1)}G_{0l}V_{lj}^{(1)}\phi_{0j}) \\ &= G_{0i}V_{ij}^{(31)}\phi_{0j} + G_{0i}V_{ij}^{(32)}\phi_{0j} + G_{0i}V_{ij}^{(33)}\phi_{0j} \end{aligned} \tag{55}$$

The first two terms in 3rd-order have no contribution to internal multiple (they only contribute to primary energy, see analysis discussed by Araujo et al. (1994)). The 3rd term $G_{0i}V_{ij}^{(33)}\phi_{0j}$ represents several different wave propagations through three perturbations depending on the variant depth relations between perturbations (Figure 2).

Consider all possible wave propagations involved by $G_{0i}V_{ik}^{(1)}G_{0k}V_{kl}^{(1)}G_{0l}V_{lj}^{(1)}\phi_{0j}$, only one certain wave path, with perturbations satisfying lower-higher-lower relationship in pseudo-depth, has contribution to 1st-order internal multiples, shown in Figure 2d, can be expressed as,

$$\begin{aligned}
&W_{33ij}(k_{ix_g}, k_{iy_g}, z_g, k_{jx_s}, k_{jy_s}, z_s, \omega) \\
&= -\iiint_{-\infty}^{+\infty} \mathrm{d}x_1\mathrm{d}y_1\mathrm{d}z_1 G_{0i}(k_{ix_g}, k_{iy_g}, z_g, x_1, y_1, z_1, \omega)V_{ik}^{(1)}(x_1, y_1, z_1) \\
&\quad\times \iiint_{-\infty}^{z_1} \mathrm{d}x_2\mathrm{d}y_2\mathrm{d}z_2 G_{0k}(x_1, y_1, z_1, x_2, y_2, z_2, \omega)V_{kl}^{(1)}(x_2, y_2, z_2) \\
&\quad\times \iiint_{z_2}^{+\infty} \mathrm{d}x_3\mathrm{d}y_3\mathrm{d}z_3 G_{0l}(x_2, y_2, z_2, x_3, y_3, z_3, \omega) \\
&\quad\times V_{lj}^{(1)}(x_3, y_3, z_3)\phi_{0j}(x_3, y_3, z_3, k_{jx_s}, k_{jy_s}, z_s, \omega) \\
&= -\frac{1}{(2\pi)^4}\iiiint_{-\infty}^{+\infty} \mathrm{d}k_{kx_1}\mathrm{d}k_{ky_1}\mathrm{d}k_{lx_2}\mathrm{d}k_{ly_2}\frac{e^{-\mathrm{i}(\nu_{js}z_s+\nu_{ig}z_g)}}{(\mathrm{i}2\nu_{ig})(\mathrm{i}2\nu_{k1})(\mathrm{i}2\nu_{l2})} \\
&\quad\times \iiint_{-\infty}^{+\infty} \mathrm{d}x_1\mathrm{d}y_1\mathrm{d}z_1 e^{\mathrm{i}(k_{kx_1}-k_{ix_g})x_1}e^{\mathrm{i}(k_{ky_1}-k_{iy_g})y_1}e^{\mathrm{i}(\nu_{k1}+\nu_{ig})z_1}V_{ik}^{(1)}(x_1, y_1, z_1) \\
&\quad\times \iiint_{-\infty}^{z_1} \mathrm{d}x_2\mathrm{d}y_2\mathrm{d}z_2 e^{\mathrm{i}(k_{lx_2}-k_{kx_1})x_2}e^{\mathrm{i}(k_{ly_2}-k_{ky_1})y_2}e^{-\mathrm{i}(\nu_{l2}+\nu_{k1})z_2}V_{kl}^{(1)}(x_2, y_2, z_2) \\
&\quad\times \iiint_{z_2}^{+\infty} \mathrm{d}x_3\mathrm{d}y_3\mathrm{d}z_3 e^{\mathrm{i}(k_{jx_s}-k_{lx_2})x_3}e^{\mathrm{i}(k_{jy_s}-k_{ly_2})y_3}e^{\mathrm{i}(\nu_{js}+\nu_{l2})z_3}V_{lj}^{(1)}(x_3, y_3, z_3) \\
&= -\frac{1}{(2\pi)^4}\iiiint_{-\infty}^{+\infty} \mathrm{d}k_{kx_1}\mathrm{d}k_{ky_1}\mathrm{d}k_{lx_2}\mathrm{d}k_{ly_2}\frac{e^{-\mathrm{i}(\nu_{js}z_s+\nu_{ig}z_g)}}{(\mathrm{i}2\nu_{ig})(\mathrm{i}2\nu_{k1})(\mathrm{i}2\nu_{l2})} \\
&\qquad\qquad \times \hat{V}_{ik}^{(1)}(k_{kx_1}-k_{ix_g}, k_{ky_1}-k_{iy_g}, \nu_{k1}+\nu_{ig}|z_1) \\
&\qquad\qquad \times \hat{V}_{kl}^{(1)}(k_{lx_2}-k_{kx_1}, k_{ly_2}-k_{ky_1}, -\nu_{l2}-\nu_{k1}|z_2<z_1) \\
&\qquad\qquad \times \hat{V}_{lj}^{(1)}(k_{jx_s}-k_{lx_2}, k_{jy_s}-k_{ly_2}, \nu_{js}+\nu_{l2}|z_3>z_2)
\end{aligned} \tag{56}$$

Replacing the elastic scattering potential by the weighted decomposed measured data based on their relations discussed above in Eq.(54), the contributed calculation of elastic internal multiple with three perturbations (Eq.56) can be reword as a function of weighted data, shown as,

$$
\begin{aligned}
&W_{33ij}(k_{ix_g}, k_{iy_g}, z_g, k_{jx_s}, k_{jy_s}, z_s, \omega) \\
&= -\frac{1}{(2\pi)^4} \iiiint_{-\infty}^{+\infty} \mathrm{d}k_{kx_1}\mathrm{d}k_{ky_1}\mathrm{d}k_{lx_2}\mathrm{d}k_{ly_2} e^{\mathrm{i}\nu_{k1}(z_s - z_g)} e^{-\mathrm{i}\nu_{l2}(z_s - z_g)} \\
&\qquad\qquad \times b_{1ik}(k_{ix_g}, k_{iy_g}, k_{kx_1}, k_{ky_1}, \nu_{k1} + \nu_{ig} | z_1) \\
&\qquad\qquad \times b_{1kl}(k_{kx_1}, k_{ky_1}, k_{lx_2}, k_{ly_2}, -\nu_{l2} - \nu_{k1} | z_2 < z_1) \\
&\qquad\qquad \times b_{1lj}(k_{lx_2}, k_{ly_2}, k_{jx_s}, k_{jy_s}, \nu_{js} + \nu_{l2} | z_3 > z_2)
\end{aligned} \tag{57}
$$

An inverse Fourier transform is performed to transfer the weighted data b_{1ij} into pseudo-depth domain. Then, the 1st-leading-order elastic internal multiples prediction algorithm can be obtained, and is written as,

$$
\begin{aligned}
&b_{3ij}(k_{ix_g}, k_{iy_g}, k_{jx_s}, k_{jy_s}, \omega) \\
&= -\frac{1}{(2\pi)^4} \iiiint_{-\infty}^{+\infty} \mathrm{d}k_{kx_1}\mathrm{d}k_{ky_1}\mathrm{d}k_{lx_2}\mathrm{d}k_{ly_2} e^{\mathrm{i}\nu_{k1}(z_s - z_g)} e^{-\mathrm{i}\nu_{l2}(z_s - z_g)} \\
&\qquad \times \int_{-\infty}^{+\infty} \mathrm{d}z_1 e^{\mathrm{i}(\nu_{k1} + \nu_{ig}) z_1} b_{1ik}(k_{ix_g}, k_{iy_g}, k_{kx_1}, k_{ky_1}, z_1) \\
&\qquad \times \int_{-\infty}^{z_1} \mathrm{d}z_2 e^{-\mathrm{i}(\nu_{l2} + \nu_{k1}) z_2} b_{1kl}(k_{kx_1}, k_{ky_1}, k_{lx_2}, k_{ly_2}, z_2) \\
&\qquad \times \int_{z_2}^{+\infty} \mathrm{d}z_3 e^{\mathrm{i}(\nu_{js} + \nu_{l2}) z_2} b_{1lj}(k_{lx_2}, k_{ly_2}, k_{jx_s}, k_{jy_s}, z_3)
\end{aligned} \tag{58}
$$

To reconstruct 2nd-order internal multiples, at least five perturbations have to be involved to calculate the contributions for 2nd-order internal multiples. Simply expand Eq.(57) into five perturbation mode, 2nd-order internal multiples prediction can be expressed as, in vertical wavenumber domain,

$$
\begin{aligned}
&b_{5ij}(k_{ix_g}, k_{iy_g}, k_{jx_s}, k_{jy_s}, \omega) \\
&= -\frac{1}{(2\pi)^8} \iiiint_{-\infty}^{+\infty} \mathrm{d}k_{kx_1}\mathrm{d}k_{ky_1}\mathrm{d}k_{lx_2}\mathrm{d}k_{ly_2} e^{\mathrm{i}\nu_{k1}(z_s - z_g)} e^{-\mathrm{i}\nu_{l2}(z_s - z_g)} \\
&\quad \times \iiiint_{-\infty}^{+\infty} \mathrm{d}k_{mx_3}\mathrm{d}k_{my_3}\mathrm{d}k_{nx_4}\mathrm{d}k_{ny_4} e^{\mathrm{i}\nu_{m3}(z_s - z_g)} e^{-\mathrm{i}\nu_{n4}(z_s - z_g)} \\
&\quad \times b_{1ik}(k_{ix_g}, k_{iy_g}, k_{kx_1}, k_{ky_1}, \nu_{k1} + \nu_{ig} | z_1) \\
&\quad \times b_{1kl}(k_{kx_1}, k_{ky_1}, k_{lx_2}, k_{ly_2}, -\nu_{l2} - \nu_{k1} | z_2 < z_1) \\
&\quad \times b_{1lm}(k_{lx_2}, k_{ly_2}, k_{mx_3}, k_{my_3}, \nu_{m3} + \nu_{l2} | z_3 > z_2) \\
&\quad \times b_{1mn}(k_{mx_3}, k_{my_3}, k_{nx_4}, k_{ny_4}, -\nu_{n4} - \nu_{m3} | z_4 < z_3) \\
&\quad \times b_{1nj}(k_{nx_4}, k_{ny_4}, k_{jx_s}, k_{jy_s}, \nu_{js} + \nu_{n4} | z_5 > z_4)
\end{aligned} \tag{59}
$$

Rewrite Eq.(57) and Eq.(59) as,

$$
\begin{aligned}
&b_{3ij}(k_{ix_g}, k_{iy_g}, k_{jx_s}, k_{jy_s}, \omega) \\
&= -\frac{1}{(2\pi)^4} \iint_{-\infty}^{+\infty} \mathrm{d}k_{kx_1}\mathrm{d}k_{ky_1} e^{\mathrm{i}\nu_{k1}(z_s - z_g)} b_{1ik}(k_{ix_g}, k_{iy_g}, k_{kx_1}, k_{ky_1}, \nu_{k1} + \nu_{ig}|z_1) \\
&\qquad \times A_{3kj}(k_{kx_1}, k_{ky_1}, k_{jx_s}, k_{jy_s}, \nu_{k1} + \nu_{js}|z_1)
\end{aligned} \tag{60a}
$$

$$
\begin{aligned}
&b_{5ij}(k_{ix_g}, k_{iy_g}, k_{jx_s}, k_{jy_s}, \omega) \\
&= -\frac{1}{(2\pi)^8} \iint_{-\infty}^{+\infty} \mathrm{d}k_{kx_1}\mathrm{d}k_{ky_1} e^{\mathrm{i}\nu_{k1}(z_s - z_g)} b_{1ik}(k_{ix_g}, k_{iy_g}, k_{kx_1}, k_{ky_1}, \nu_{k1} + \nu_{ig}|z_1) \\
&\qquad \times A_{5kj}(k_{kx_1}, k_{ky_1}, k_{jx_s}, k_{jy_s}, \nu_{k1} + \nu_{js}|z_1)
\end{aligned} \tag{60b}
$$

where,

$$
\begin{aligned}
&A_{3kj}(k_{kx_1}, k_{ky_1}, k_{jx_s}, k_{jy_s}, \nu_{k1} + \nu_{js}|z_1) \\
&= \iint_{-\infty}^{+\infty} \mathrm{d}k_{lx_2}\mathrm{d}k_{ly_2} e^{-\mathrm{i}\nu_{l2}(z_s - z_g)} b_{1kl}(k_{kx_1}, k_{ky_1}, k_{lx_2}, k_{ly_2}, -\nu_{l2} - \nu_{k1}|z_2 < z_1) \\
&\qquad \times b_{1lj}(k_{lx_2}, k_{ly_2}, k_{jx_s}, k_{jy_s}, \nu_{js} + \nu_{l2}|z_3 > z_2)
\end{aligned} \tag{61a}
$$

$$
\begin{aligned}
&A_{5kj}(k_{kx_1}, k_{ky_1}, k_{jx_s}, k_{jy_s}, \nu_{k1} + \nu_{js}|z_1) \\
&= \iint_{-\infty}^{+\infty} \mathrm{d}k_{lx_2}\mathrm{d}k_{ly_2} e^{-\mathrm{i}\nu_{l2}(z_s - z_g)} b_{1kl}(k_{1k}, k_{2l}, -\nu_{2l} - \nu_{1k}|z_2 < z_1) \\
&\times \iint_{-\infty}^{+\infty} \mathrm{d}k_{mx_3}\mathrm{d}k_{my_3} e^{\mathrm{i}\nu_{m3}(z_s - z_g)} b_{1lm}(k_{lx_2}, k_{ly_2}, k_{mx_3}, k_{my_3}, \nu_{m3} + \nu_{l2}|z_3 > z_2) \\
&\times \iint_{-\infty}^{+\infty} \mathrm{d}k_{nx_4}\mathrm{d}k_{ny_4} e^{-\mathrm{i}\nu_{n4}(z_s - z_g)} b_{1mn}(k_{mx_3}, k_{my_3}, k_{nx_4}, k_{ny_4}, \nu_{n4} + \nu_{m3}|z_4 < z_3) \\
&\qquad \times b_{1nj}(k_{nx_4}, k_{ny_4}, k_{jx_s}, k_{jy_s}, \nu_{js} + \nu_{n4}|z_5 > z_4)
\end{aligned} \tag{61b}
$$

Analogously, nth-leading-order elastic internal multiples prediction algorithm can be expressed as, in vertical wavenumber domain,

$$
\begin{aligned}
&b_{(2n+1)ij}(k_{ix_g}, k_{iy_g}, k_{jx_s}, k_{jy_s}, \omega) \\
&= -\frac{1}{(2\pi)^{2n}} \iint_{-\infty}^{+\infty} \mathrm{d}k_{kx_1}\mathrm{d}k_{ky_1} e^{\mathrm{i}\nu_{k1}(z_s - z_g)} b_{1ik}(k_{ix_g}, k_{iy_g}, k_{kx_1}, k_{ky_1}, \nu_{k1} + \nu_{ig}|z_1) \\
&\qquad \times A_{(2n+1)kj}(k_{kx_1}, k_{ky_1}, k_{jx_s}, k_{jy_s}, \nu_{k1} + \nu_{js}|z_1)
\end{aligned} \tag{62}
$$

where,

$$
\begin{aligned}
&A_{(2n+1)kj}(k_{kx_1}, k_{ky_1}, k_{jx_s}, k_{jy_s}, \nu_{k1} + \nu_{js}|z_1) \\
&= \iint_{-\infty}^{+\infty} \mathrm{d}k_{lx_2}\mathrm{d}k_{ly_2} e^{-\mathrm{i}\nu_{l2}(z_s - z_g)} b_{1kl}(k_{kx_1}, k_{ky_1}, k_{lx_2}, k_{ly_2}, -\nu_{l2} - \nu_{k1}|z_2 < z_1) \\
&\times \iint_{-\infty}^{+\infty} \mathrm{d}k_{mx_3}\mathrm{d}k_{my_3} e^{\mathrm{i}\nu_{m3}(z_s - z_g)} b_{1lm}(k_{lx_2}, k_{ly_2}, k_{mx_3}, k_{my_3}, \nu_{m3} + \nu_{l2}|z_3 > z_2) \\
&\qquad \times A_{(2n-1)mj}(k_{mx_3}, k_{my_3}, k_{jx_s}, k_{jy_s}, \nu_{m3} + \nu_{js}|z_3)
\end{aligned} \tag{63}
$$

Again, take an inverse Fourier transform to perform the input data into pseudo-depth domain, nth-leading-order elastic internal multiples prediction algorithm can be written as, in pseudo-depth domain,

$$\begin{aligned} b_{(2n+1)ij}(k_{ix_g}, k_{iy_g}, k_{jx_s}, k_{jy_s}, \omega) = -\frac{1}{(2\pi)^{2n}} \iint_{-\infty}^{+\infty} \mathrm{d}k_{kx_1} \mathrm{d}k_{ky_1} e^{\mathrm{i}\nu_{k1}(z_s - z_g)} \\ \times \int_{-\infty}^{+\infty} \mathrm{d}z_1 e^{\mathrm{i}(\nu_{k1}+\nu_{ig})z_1} b_{1ik}(k_{ix_g}, k_{iy_g}, k_{kx_1}, k_{ky_1}, z_1) \\ \times A_{(2n+1)kj}(k_{kx_1}, k_{ky_1}, k_{jx_s}, k_{jy_s}, z_1) \end{aligned} \tag{64}$$

where,

$$\begin{aligned} A_{3kj}(k_{kx_1}, k_{ky_1}, k_{jx_s}, k_{jy_s}, z_1) = \iint_{-\infty}^{+\infty} \mathrm{d}k_{lx_2} \mathrm{d}k_{ly_2} e^{-\mathrm{i}\nu_{l2}(z_s - z_g)} \\ \times \int_{-\infty}^{z1} \mathrm{d}z_2 e^{-\mathrm{i}(\nu_{l2}+\nu_{k1})z_2} b_{1kl}(k_{kx_1}, k_{ky_1}, k_{lx_2}, k_{ly_2}, z_2) \\ \times \int_{z_2}^{+\infty} \mathrm{d}z_3 e^{\mathrm{i}(\nu_{sj}+\nu_{2l})z_2} b_{1lj}(k_{lx_2}, k_{ly_2}, k_{jx_s}, k_{jy_s}, z_3) \end{aligned} \tag{65a}$$

$$\begin{aligned} A_{(2n+1)kj}(k_{kx_1}, k_{ky_1}, k_{jx_s}, k_{jy_s}, z_1) = \iint_{-\infty}^{+\infty} \mathrm{d}k_{lx_2} \mathrm{d}k_{ly_2} e^{-\mathrm{i}\nu_{l2}(z_s - z_g)} \\ \times \int_{-\infty}^{z1} \mathrm{d}z_2 e^{-\mathrm{i}(\nu_{l2}+\nu_{k1})z_2} b_{1kl}(k_{kx_1}, k_{ky_1}, k_{lx_2}, k_{ly_2}, z_2) \\ \times \int_{-\infty}^{+\infty} \mathrm{d}k_{mx_3} \mathrm{d}k_{my_3} e^{\mathrm{i}\nu_{m3}(z_s - z_g)} \\ \times \int_{z_2}^{+\infty} \mathrm{d}z_3 e^{\mathrm{i}(\nu_{m3}+\nu_{l2})z_3} b_{1lm}(k_{lx_2}, k_{ly_2}, k_{mx_3}, k_{my_3}, z_3) \\ \times A_{(2n-1)mj}(k_{mx_3}, k_{my_3}, k_{jx_s}, k_{jy_s}, z_3) \end{aligned} \tag{65b}$$

Here, the letter subscripts denote the modes of wave $\{P, SH, SV\}$, and the number subscripts describe the locations.

CONCLUSIONS

Elastic internal multiples attenuation becomes to be a high priority problem to be solved in seismic data processing as the special significance of unconventional plays increasing rapidly, where the sophisticated quantitative interpretation is required. However, the existed internal multiple prediction algorithm, either needs extensive knowledge of subsurface or is not appropriate for an elastic medium. Begin with stress-strain relations in an elastic isotropic medium, by considering a homogeneous isotropic elastic medium as the reference, the wave equation can be written in terms of separated P- and S-wave mode in background with elastic perturbations, based on forward scattering series. Beyond that, the elastic internal multiples can be reconstructed by weighted decomposed data depending on inverse scattering series as it does in an acoustic medium. Finally, we describe the theoretical framework of the inverse scattering series leading order internal multiples prediction algorithm for elastic isotropic media. The equivalent prediction algorithm can also be achieved in other different domains, only if the ordering of sub-events in domain parameter is the same as ordering in actual (or pseudo-) depth.

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APPENDIX A: ISOTROPIC ELASTIC WAVE EQUATION

For an isotropic elastic medium, stress-strain relation can be written as, for $i, j = 1, 2, 3,$

$$\sigma_{ij} = \lambda \mathcal{D} \delta_{ij} + 2\mu e_{ij}, \tag{A.1}$$

And recall equation of motion,

$$\sum_{j=1}^{3} \frac{\partial \sigma_{ij}}{\partial r_j} + f_i = \rho \frac{\partial^2 u_i}{\partial t^2} \tag{A.2}$$

Based on stress-strain relation (Eq.A.1), expand the first term of equation of motion (Eq.A.2), we have,

$$\begin{aligned}\sum_{j=1}^{3} \frac{\partial \sigma_{ij}}{\partial r_j} &= \lambda \sum_{j=1}^{3} \frac{\partial \mathcal{D}}{\partial r_j} \delta_{ij} + \mathcal{D} \sum_{j=1}^{3} \frac{\partial \lambda}{\partial r_j} \delta_{ij} + 2\mu \sum_{j=1}^{3} \frac{\partial e_{ij}}{\partial r_j} + 2 \sum_{j=1}^{3} \frac{\partial \mu}{\partial r_j} e_{ij} \\ &= \lambda \frac{\partial \mathcal{D}}{\partial r_i} + \mathcal{D} \frac{\partial \lambda}{\partial r_i} + \mu \sum_{j=1}^{3} \frac{\partial^2 u_i}{\partial r_j^2} + \mu \frac{1}{\partial r_i} \sum_{j=1}^{3} \frac{\partial u_j}{r_j} + \sum_{j=1}^{3} (\frac{\partial u_i}{\partial r_j} + \frac{\partial u_j}{\partial r_i}) \frac{\partial \mu}{\partial r_j}\end{aligned} \tag{A.3}$$

Rewrite Eq.(A.3) into vector form,

$$\begin{aligned}\frac{\partial \vec{\boldsymbol{\sigma}}}{\partial \vec{\mathbf{r}}} &= \lambda \nabla(\nabla \cdot \vec{\mathbf{u}}) + (\nabla \lambda) \nabla \cdot \vec{\mathbf{u}} + \mu \nabla^2 \vec{\mathbf{u}} + \mu \nabla(\nabla \cdot \vec{\mathbf{u}}) + (\nabla \mu)^T \cdot (\nabla \vec{\mathbf{u}} + \vec{\mathbf{u}} \nabla) \\ &= (\lambda + \mu) \nabla(\nabla \cdot \vec{\mathbf{u}}) + (\nabla \lambda) \nabla \cdot \vec{\mathbf{u}} + \mu \nabla^2 \vec{\mathbf{u}} + (\nabla \vec{\mathbf{u}} + \vec{\mathbf{u}} \nabla) \cdot (\nabla \mu) \\ &= (\lambda + \mu) \mathrm{grad\ div} \vec{\mathbf{u}} + (\mathrm{grad} \lambda)(\mathrm{div} \vec{\mathbf{u}}) + \mu \nabla^2 \vec{\mathbf{u}} + (\nabla \vec{\mathbf{u}} + \vec{\mathbf{u}} \nabla) \cdot \nabla \mu\end{aligned} \tag{A.4}$$

Substitute Eq.(A.4) into the equation of motion, assume the body forces are negligible and take the Fourier transform,

$$(\lambda + \mu) \nabla(\nabla \cdot \vec{\mathbf{u}}) + (\nabla \lambda) \nabla \cdot \vec{\mathbf{u}} + \mu \nabla^2 \vec{\mathbf{u}} + (\nabla \vec{\mathbf{u}} + \vec{\mathbf{u}} \nabla) \cdot (\nabla \mu) + \rho \omega^2 \mathbf{u} = 0 \tag{A.5}$$

Here,

$$\begin{aligned}\nabla \lambda &= \begin{bmatrix} \frac{\partial \lambda}{\partial x} & \frac{\partial \lambda}{\partial y} & \frac{\partial \lambda}{\partial z} \end{bmatrix}^T \\ \nabla \mu &= \begin{bmatrix} \frac{\partial \mu}{\partial x} & \frac{\partial \mu}{\partial y} & \frac{\partial \mu}{\partial z} \end{bmatrix}^T\end{aligned} \tag{A.6}$$

$$\nabla \cdot \vec{\mathbf{u}} = \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} = \begin{bmatrix} \partial_x & \partial_y & \partial_z \end{bmatrix} \begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix} \tag{A.7}$$

$$\nabla \times \vec{\mathbf{u}} = \begin{bmatrix} \dfrac{\partial u_z}{\partial y} - \dfrac{\partial u_y}{\partial z} \\ \dfrac{\partial u_x}{\partial z} - \dfrac{\partial u_z}{\partial x} \\ \dfrac{\partial u_y}{\partial x} - \dfrac{\partial u_x}{\partial y} \end{bmatrix} = \begin{bmatrix} 0 & -\partial_z & \partial_y \\ \partial_z & 0 & -\partial_x \\ -\partial_y & \partial_x & 0 \end{bmatrix} \begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix} \tag{A.8}$$

$$\nabla(\nabla \cdot \vec{\mathbf{u}}) = \begin{bmatrix} \partial_x^2 & \partial_x\partial_y & \partial_x\partial_z \\ \partial_y\partial_x & \partial_y^2 & \partial_y\partial_z \\ \partial_z\partial_x & \partial_z\partial_y & \partial_z^2 \end{bmatrix} \begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix} \tag{A.9}$$

$$\nabla \times (\nabla \times \vec{\mathbf{u}}) = \begin{bmatrix} -\partial_y^2 - \partial_z^2 & \partial_y\partial_x & \partial_z\partial_x \\ \partial_x\partial_y & -\partial_x^2 - \partial_z^2 & \partial_z\partial_y \\ \partial_x\partial_z & \partial_y\partial_z & -\partial_x^2 - \partial_y^2 \end{bmatrix} \begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix} \tag{A.10}$$

$$\begin{aligned} \nabla^2\vec{\mathbf{u}} &= \nabla(\nabla \cdot \vec{\mathbf{u}}) - \nabla \times (\nabla \times \vec{\mathbf{u}}) \\ &= \begin{bmatrix} \partial_x^2 + \partial_y^2 + \partial_z^2 & \partial_x\partial_y - \partial_y\partial_x & \partial_x\partial_z - \partial_z\partial_x \\ \partial_y\partial_x - \partial_x\partial_y & \partial_x^2 + \partial_y^2 + \partial_z^2 & \partial_y\partial_z - \partial_z\partial_y \\ \partial_z\partial_x - \partial_x\partial_z & \partial_z\partial_y - \partial_y\partial_z & \partial_x^2 + \partial_y^2 + \partial_z^2 \end{bmatrix} \begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix} \end{aligned} \tag{A.11}$$

$$\nabla\vec{\mathbf{u}} + \vec{\mathbf{u}}\nabla = \begin{bmatrix} 2\partial_x u_x & \partial_y u_x + \partial_x u_y & \partial_z u_x + \partial_x u_z \\ \partial_x u_y + \partial_y u_x & 2\partial_y u_y & \partial_y u_z + \partial_z u_y \\ \partial_x u_z + \partial_z u_x & \partial_z u_y + \partial_y u_z & 2\partial_z u_z \end{bmatrix} \tag{A.12}$$

$$\begin{aligned} (\nabla\vec{\mathbf{u}} + \vec{\mathbf{u}}\nabla) \cdot \nabla\mu &= \begin{bmatrix} 2\partial_x u_x & \partial_y u_x + \partial_x u_y & \partial_z u_x + \partial_x u_z \\ \partial_x u_y + \partial_y u_x & 2\partial_y u_y & \partial_y u_z + \partial_z u_y \\ \partial_x u_z + \partial_z u_x & \partial_z u_y + \partial_y u_z & 2\partial_z u_z \end{bmatrix} \begin{bmatrix} \partial_x\mu \\ \partial_y\mu \\ \partial_z\mu \end{bmatrix} \\ &= \begin{bmatrix} (\partial_x\mu)\partial_x + \sum_k(\partial_k\mu)\partial_k & (\partial_y\mu)\partial_x & (\partial_z\mu)\partial_x \\ (\partial_x\mu)\partial_y & (\partial_y\mu)\partial_y + \sum_k(\partial_k\mu)\partial_k & (\partial_z\mu)\partial_y \\ (\partial_x\mu)\partial_z & (\partial_y\mu)\partial_z & (\partial_z\mu)\partial_z + \sum_k(\partial_k\mu)\partial_k \end{bmatrix} \begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix} \end{aligned} \tag{A.13}$$

Substitute those expression (from Eq.A.6 to Eq.A.13) into Eq.(A.5), then the equation of motion can be rewritten into wave operator form,

$$\mathfrak{L}(\mathbf{r}, \omega)\mathbf{u}(\mathbf{r}, \omega) = 0 \tag{A.14}$$

where,

$$\begin{aligned} \mathfrak{L}_{ii} &= \partial_i[(\lambda + 2\mu)\partial_i] + \sum_{j \neq i} \partial_j(\mu\partial_j) + \rho\omega^2, \quad i, j = x, y, z; \\ \mathfrak{L}_{ij} &= \partial_i(\lambda\partial_j) + \partial_j(\mu\partial_i), \quad j \neq i; \\ \mathbf{u} &= [u_x, u_y, u_z]^T. \end{aligned} \tag{A.15}$$

with $\gamma = \lambda + 2\mu = \rho\alpha^2$, $\mu = \rho\beta^2$.

APPENDIX B: DIAGONALIZATION OPERATORS

Define a partial derivative matrix,

$$\mathbf{\Pi} = \begin{pmatrix} \nabla\cdot \\ \nabla\times \end{pmatrix} = \begin{pmatrix} \partial_x & \partial_y & \partial_z \\ 0 & -\partial_z & \partial_y \\ \partial_z & 0 & -\partial_x \\ -\partial_y & \partial_x & 0 \end{pmatrix} \tag{B.1}$$

By pre-multiplying its transpose, produces the Laplacian ∇^2 times a 3D unit operator,

$$\mathbf{\Pi}^T\mathbf{\Pi} = \begin{pmatrix} \nabla^T\cdot & -\nabla\times \end{pmatrix} \begin{pmatrix} \nabla\cdot \\ \nabla\times \end{pmatrix} = [\nabla(\nabla\cdot) - \nabla\times(\nabla\times)]\mathbf{I} = \begin{pmatrix} \nabla^2 & 0 & 0 \\ 0 & \nabla^2 & 0 \\ 0 & 0 & \nabla^2 \end{pmatrix} \tag{B.2}$$

If $\mathbf{\Pi}$ is post-multiplied by its transpose,

$$\begin{aligned} \mathbf{\Pi}\,\mathbf{\Pi}^T &= \begin{pmatrix} \nabla\cdot \\ \nabla\times \end{pmatrix} \begin{pmatrix} \nabla^T\cdot & -\nabla\times \end{pmatrix} = \begin{pmatrix} \nabla^2 & \mathbf{0} \\ \mathbf{0} & -\nabla\times(\nabla\times) \end{pmatrix} \\ &= \begin{pmatrix} \nabla^2 & 0 & 0 & 0 \\ 0 & \partial_y^2+\partial_z^2 & -\partial_y\partial_x & -\partial_z\partial_x \\ 0 & -\partial_x\partial_y & \partial_x^2+\partial_z^2 & -\partial_z\partial_y \\ 0 & -\partial_x\partial_z & -\partial_y\partial_z & \partial_x^2+\partial_y^2 \end{pmatrix} \end{aligned} \tag{B.3}$$

Operating on the P- and S-wave components vector, we have,

$$\begin{aligned} \mathbf{\Pi}\,\mathbf{\Pi}^T \begin{pmatrix} \boldsymbol{\varphi}_P \\ \boldsymbol{\varphi}_S \end{pmatrix} &= \begin{pmatrix} \nabla^2 & \mathbf{0} \\ \mathbf{0} & -\nabla\times(\nabla\times) \end{pmatrix} \begin{pmatrix} \boldsymbol{\varphi}_P \\ \boldsymbol{\varphi}_S \end{pmatrix} \\ &= \begin{pmatrix} \nabla^2\boldsymbol{\varphi}_P \\ \nabla^2\boldsymbol{\varphi}_S - \nabla(\nabla\cdot\boldsymbol{\varphi}_S) \end{pmatrix} \end{aligned} \tag{B.4}$$

Here, $\boldsymbol{\varphi}_S = \nabla\times\mathbf{u}$, which means $\nabla(\nabla\cdot\boldsymbol{\varphi}_S) = 0$. Therefore, under this condition, the Eq.(B.4) can be written as,

$$\mathbf{\Pi}\,\mathbf{\Pi}^T \begin{pmatrix} \boldsymbol{\varphi}_P \\ \boldsymbol{\varphi}_S \end{pmatrix} = \nabla^2 \begin{pmatrix} \boldsymbol{\varphi}_P \\ \boldsymbol{\varphi}_S \end{pmatrix} \tag{B.5}$$

This implies that, under some assumptions, the inverse of partial derivative matrix can be written as the multiplication of its transpose with the inverse of the Laplacian,

$$\mathbf{\Pi}^{-1} = \nabla^{-2}\mathbf{\Pi}^T \tag{B.6}$$

Define P- and S-wave operators, $\mathfrak{L}_P$ and $\mathfrak{L}_S$ satisfy,

$$\begin{aligned} \mathfrak{L}_{0P} &= (\lambda+2\mu)\nabla^2 + \rho\omega^2 \\ \mathfrak{L}_{0S} &= \mu\nabla^2 + \rho\omega^2 \end{aligned} \tag{B.7}$$

Using P- and S-wave operators, the wave operator for a homogeneous isotropic elastic medium $\mathfrak{L}_0(\mathbf{x}, \omega)$, which means derivatives of λ and μ can be neglected, can be rewritten as,

$$\mathfrak{L}_0 = \begin{pmatrix} (\lambda+\mu)\partial_x^2 + \mathfrak{L}_{0S} & (\lambda+\mu)\partial_y\partial_x & (\lambda+\mu)\partial_z\partial_x \\ (\lambda+\mu)\partial_x\partial_y & (\lambda+\mu)\partial_y^2 + \mathfrak{L}_{0S} & (\lambda+\mu)\partial_z\partial_y \\ (\lambda+\mu)\partial_x\partial_z & (\lambda+\mu)\partial_y\partial_z & (\lambda+\mu)\partial_z^2 + \mathfrak{L}_{0S} \end{pmatrix} \tag{B.8}$$

Therefore, we have,

$$\mathbf{\Pi}\mathfrak{L}_0 = \begin{pmatrix} \partial_x\mathfrak{L}_{0P} & \partial_y\mathfrak{L}_{0P} & \partial_z\mathfrak{L}_{0P} \\ 0 & -\partial_z\mathfrak{L}_{0S} & \partial_y\mathfrak{L}_{0P} \\ \partial_z\mathfrak{L}_{0S} & 0 & -\partial_z\mathfrak{L}_{0S} \\ -\partial_y\mathfrak{L}_{0S} & \partial_x\mathfrak{L}_{0S} & 0 \end{pmatrix} \tag{B.9}$$

It is worth to note that, the right hand side of Eq.(B.9) can be considered as the multiplication of the diagonalized matrix and the operator $\mathbf{\Pi}$ as follow,

$$\begin{pmatrix} \partial_x\mathfrak{L}_{0P} & \partial_y\mathfrak{L}_{0P} & \partial_z\mathfrak{L}_{0P} \\ 0 & -\partial_z\mathfrak{L}_{0S} & \partial_y\mathfrak{L}_{0P} \\ \partial_z\mathfrak{L}_{0S} & 0 & -\partial_z\mathfrak{L}_{0S} \\ -\partial_y\mathfrak{L}_{0S} & \partial_x\mathfrak{L}_{0S} & 0 \end{pmatrix} = \mathfrak{L}_{0D}\mathbf{\Pi} \tag{B.10}$$

where,

$$\mathfrak{L}_{0D} = \begin{pmatrix} \mathfrak{L}_{0P} & 0 & 0 & 0 \\ 0 & \mathfrak{L}_{0S} & 0 & 0 \\ 0 & 0 & \mathfrak{L}_{0S} & 0 \\ 0 & 0 & 0 & \mathfrak{L}_{0S} \end{pmatrix} \tag{B.11}$$

Combine Eq.(B.9) and Eq.(B.10), which indicates the wave operator $\mathfrak{L}_0$ can be diagonalizable into P- and S-wave operators,

$$\mathfrak{L}_{0D} = \mathbf{\Pi}\mathfrak{L}_0\mathbf{\Pi}^{-1} \tag{B.12}$$

Therefore, by pre-multiplying a partial derivative matrix, wave equation for an elastic isotropic homogeneous medium (Eq. A.14) is rewritten as,

$$\mathbf{\Pi}\mathfrak{L}_0(\mathbf{\Pi}^{-1}\mathbf{\Pi})\mathbf{u} = 0 \tag{B.13}$$

Replacing the displacement into the P- and S-wave components, the above equation can be expressed as,

$$\mathfrak{L}_{0D}\begin{pmatrix} \boldsymbol{\varphi}_P \\ \boldsymbol{\varphi}_S \end{pmatrix} = 0 \tag{B.14}$$

where, $\mathbf{\Pi}\mathbf{u} = \begin{pmatrix} \boldsymbol{\varphi}_P \\ \boldsymbol{\varphi}_S \end{pmatrix}$.

One should note that the above transformation does not diagonalize the wave equation Stolt and Weglein (2012). However, the interpretation of the elements in separated P- and S- wave components is still reasonable, where off-diagonal terms in $\mathfrak{L}_{0D}$ denote wave mode conversion from one to another.

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