Gabor multipliers revisited

Michael P. Lamoureux* and Heather K. Hardeman-Vooys*

ABSTRACT

Time-frequency methods have proven to be valuable in seismic data processing as localized Fourier transforms can accurately analyse the nonstationary characteristics of data in response to the non-uniform geology of the region under survey. For instance, Gabor deconvolution by way of Gabor multipliers is an effective way of extending stationary Weiner decon and spectral whitening to the nonstationary seismic domain.

We propose the continuous wavelet transform and its multipliers as an improvement over Gabor methods, using the logarithmically-spaced frequency bins in the CWT to improve resolution and control in the lower frequency range of the seismic signal.

INTRODUCTION

Seismic signals are not stationary. Their character changes as time evolves in response to the physics of the recorded seismic events as a wave propagates through the earth. These changes are due to a variety of physical phenomena: attenuation caused by spherical spreading of the wavefield, rapid loss of high frequencies due to stratigraphic filtering and anelastic or friction losses (Q-effects), and of course difference in physical rock properties for different regions of the subsurface where the wave is propagating.

A wide variety of mathematical methods have been developed to analyse nonstationary signals. This includes the general category of time-frequency analysis (see e.g. Gröchenig (2001), Qian (2002)), the extensively developed area of wavelets methods (see Boggess and Narcowich (2001), Burrus et al. (1998), Daubechies et al. (1986), Mallat (1999), Strang and Nguyen (1997), Walnut (2002), among others), more specialized Gabor methods (see Feichtinger and Strohmer (1998), Feichtinger and Nowak (2003), Gabor (1946), Margrave et al. (2011)), the general area of pseudo-differential operators and Fourier integral operators (Feichtinger et al. (2008), Treves (1980), Wong (2014),) and many more. The fundamental idea in all these approaches is to encode information about the signal (or operator on signals) that is localized in both time and frequency.

Many years back, collaborators of the first author developed some novel deconvolution algorithms for seismic data processing, based on the Gabor transform and related Gabor multipliers (see Margrave and Lamoureux (2001), Margrave et al. (2003), Margrave et al. (2011)). This approach extends the stationary methods of Wiener deconvolution, or spectral whitening, to the nonstationary situation that is appropriate for seismic signals.

The basic approach to Gabor deconvolution is to transform a univariate signal $s = s(t)$ to a two-variable function in time and frequency, say $G_s(t, f)$ and modify it by multiplying with some function $\alpha(t, f)$. The resulting product $\alpha(t, f)G_s(t, f)$ represents a modifica-

*University of Calgary

tion of the original signal s in the time-frequency domain, and by applying the inverse transform, we obtain a suitably modified signal $s'(t)$ back in the time domain.

This function $\alpha(t, f)$ is called a Gabor multiplier, and it contains all the mathematics representing the physics of the process we wish to simulate by nonstationary filtering, convolution, or what have you. There is a close relationship between the Gabor multiplier and pseudodifferential operators (Gibson et al. (2013)) which can guide the careful construction of these multipliers.

It has been observed by these researchers that a weakness of the Gabor transform and Gabor multipliers is their limited resolution at low frequencies, given the fixed size of the analysis window. A potentially promising approach is suggested by the continuous wavelet transform, which is like a Gabor transform with variable-sided windows.

In this paper, we explore the use of the continuous wavelet transform for non-stationary modification of signals with an application to seismic signal deconvolution.

NOTE ON TERMINOLOGY

Throughout this paper, we will be discussing the continuous wavelet transform and its properties. The term "wavelet" in this context should not be confused with the seismic source wavelet that is commonly understood in geophysics as representing the physical source of a seismic wave, often initialized by a dynamic blast, weight drop, or vibroseis device. In this paper, the wavelet is a mathematical object, typically a windowed sinusoid, which is used in the definition of the wavelet transform.

This can be particular confusing in seismic data processing, as one of the goals of seismic deconvolution is to remove artifacts caused by the source wavelet. In this paper, the wavelet we are discussing is the mathematical one.

NONSTATIONARY REFLECTIVITY MODEL

In Margrave et al. (2003), the authors develop the Gabor multiplier method for deconvolution by beginning with Kjartansson's earlier work (Kjartansson, 1979) on representing the seismic data problems in the form of a pseudodifferential operator. Beginning with the reflectivity series $r(t)$ as a function of time t, we multiply by a function $\alpha(t, f)$ which encodes the physics of wave propagation. We then apply essentially a Fourier transform to the product αr and then multiply by $\widehat{w}(f)$, the Fourier transform of the seismic source. The result is the Fourier transform of the recorded seismic signal $s(t)$. This is expressed in the following formula,

$$\widehat{s}(f) = \widehat{w}(f) \int_{-\infty}^{\infty} \alpha(t, f) r(t) e^{-2\pi i f t} dt,$$

where the "hat" notation $\widehat{s}(f), \widehat{w}(f)$ indicates the Fourier transform of signals $s(t), w(t)$ respectively.

In the case of the commonly assumed "constant Q" model (Kjartansson, 1979), α is determined by its magnitude as a decaying exponential

$$|\alpha(t,f)| = \exp\left[\frac{-\pi t|f|}{Q(t)}\right].$$

The minimum phase condition on α determines its complex phase as the Hilbert transform of the log amplitude, so we have

$$\alpha(t,f) = \exp\left[\frac{-\pi t}{Q(t)}(|f| + iH(|f|))\right],$$

where H denotes the Hilbert transform. The Hilbert transform of the absolute value function $|f|$ can be computed directly or we can use the results of Gibson et al. (2013) to recover α as the unique outer analytic function $F(z) = \alpha(t,z)$ in the upper half complex plane that agrees in amplitude $|F(z=f)| = |\alpha(t,f)|$ on the real frequency axis $z = f + 0i$. The complex logarithm $\log(z)$ has the property that its imaginary component is zero along the positive real axis, and constant π on the negative real axis, so we can use this to create the appropriate outer function agreeing with $|f|$ on the real axis. Thus we obtain the exact expression for $\alpha(t,z)$ as

$$\alpha(t,z) = \exp\left[\frac{-2tz}{Q}(i\log(z) + \pi/2)\right].$$

The phase of α is given by the $|f|\ln(|f|)$ term with

$$phase(\alpha(t,f)) = \exp[\frac{-2it|f|\ln(|f|)}{Q}].$$

To verify these derivations, we compute the inverse Fourier transform for α to see some minimum phase pulses at various times, as shown in Figure 1. We can also use α in the reflectivity model described above to generate a synthetic seismic signal as shown in Figure 2. As expected, the seismic signal shows the characteristic decay in amplitude and broadening of the reflectivity spikes as time advances.

GABOR AND WAVELET TRANSFORMS

Both the Gabor and wavelet transforms are closely related to the usual Fourier transform, except they focus on local data in a signal rather than the full signal all at once. The Gabor transform is a particular version of the short-time Fourier transform, which is obtained by localizing a signal through windowing, then applying the Fourier transform to each windowed slice of the signal. Given a signal $r = r(t)$ as a function of time, the (continuous) Gabor transform with Gaussian windows is defined as

$$G_r(t,f) = \int_{-\infty}^{\infty} r(t')e^{-(t-t')^2/\sigma^2}e^{-2\pi i t' f}dt',$$

where σ here is a fixed parameter which sets the width of the Gaussian window. In summary, this definition shows the Gabor transform of signal r is a function of two variables,

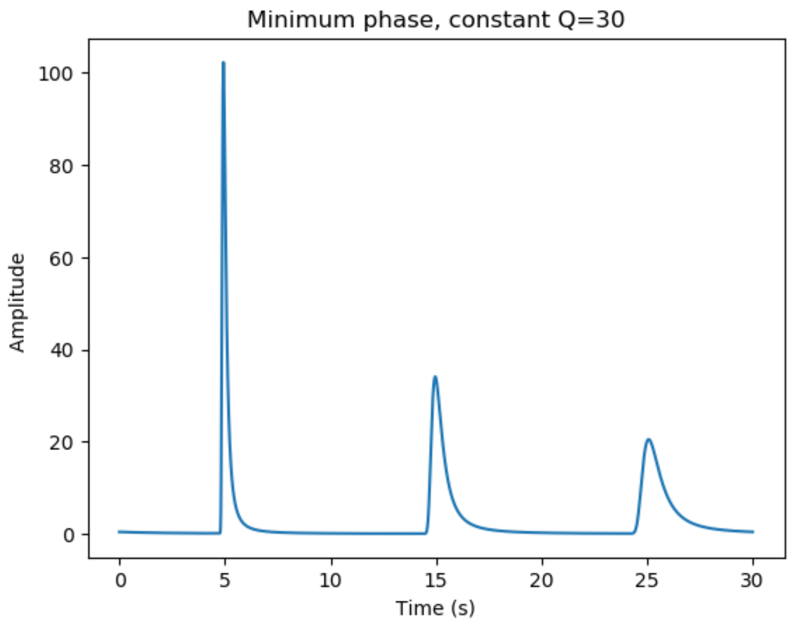

FIG. 1. Minimum phase impulses with constant Q attenuation.

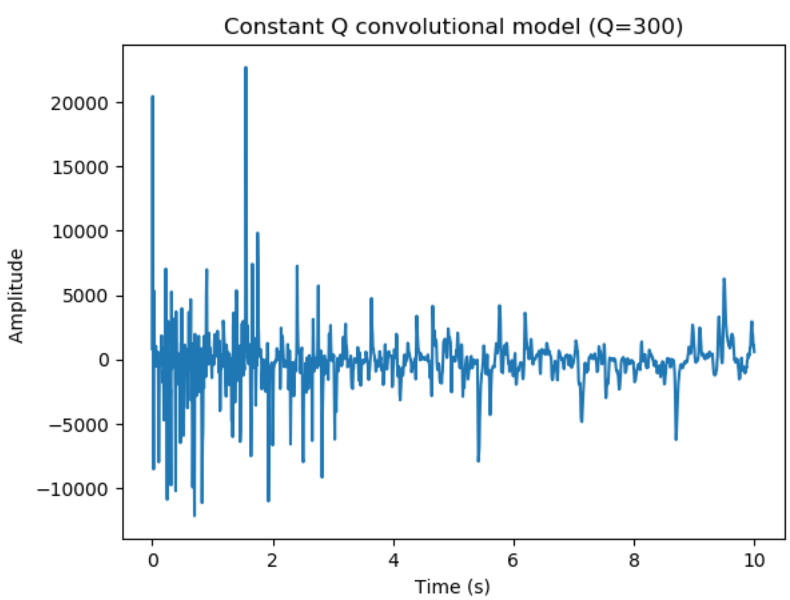

FIG. 2. Convolutional model results with constant Q attenuation.

time t and frequency f, and represents the frequency content of the signal near point in time indicated by t. Note that the window width σ is fixed in the Gabor transform.

A particular version of the continuous wavelet transform (CWT) can be derived from the Gabor transform by allowing the width of the window to scale with frequency, so we replace σ by σ/f in the above definition. In this case the window is wide for low frequencies, and narrow for high frequencies, which makes intuitive sense since we need a wide window to capture many cycles of a low frequency waveform. We then obtain the formula for a variable window time-frequency transform as

$$S_r(t,f) = \int_{-\infty}^{\infty} r(t')e^{-(t-t')^2 f^2/\sigma^2} e^{-2\pi i t' f} dt'.$$

This is known as the S-transform (Stockwell et al. (1996)). To transform this into a continuous wavelet transform (see also Gibson et al. (2006)), we scale by a factor $\sqrt{f}e^{2\pi i t}$ to obtain the wavelet transform definition

$$\begin{aligned} W_r(t,f) &= \sqrt{|f|}e^{2\pi i t} \int_{-\infty}^{\infty} r(t')e^{-(t-t')^2 f^2/\sigma^2} e^{-2\pi i t' f} dt' && (1) \\ &= \sqrt{|f|} \int_{-\infty}^{\infty} r(t')e^{-(t-t')^2 f^2/\sigma^2} e^{2\pi i (t-t') f} dt' && (2) \\ &= (r * \phi_f)(t), && (3) \end{aligned}$$

where this is the convolution of the relectivity signal r with the scaled version $\phi_f(t)$ of the complex Morlet wavelet

$$\phi(t) = e^{-t^2/\sigma^2} e^{2\pi i t}.$$

These transforms are invertible and can be applied to discrete, sampled signals in a straightforward manner. It is important to point out, though, that the fixed window size in the Gabor transform is a weakness – one has to choose between wide windows for frequency resolution at long wavelengths or narrow windows for temporal precision with high frequency events. The continuous wavelet transform does not have this restriction. When sampling in the frequency space, we can assign a logarithmic scale to the frequency and pull out useful information.

Throughout the examples in this paper, we use a logarithmic frequency scale for the continuous wavelet transform and its inverse, to explore whether this gives us the necessary control at low frequencies.

Figure 3 gives a simple demonstration of the comparison between Gabor transform with linear frequency scale, and wavelet transform with a logarithmically spaced frequency scale. We show here both the Gabor and wavelet transforms of a signal which is the sum of two linear sinusoidal sweeps from 0 to 100 Hz, and 0 to 400 Hz, respectively. By assigning a logarithmic scale to the wavelet frequencies, we can see in more detail the changing content at the lower frequencies. A similar assignment in the Gabor transform does not help to resolve details any more than the linear spacing shown as the fixed window size is too small relative to the long wavelength signal components.

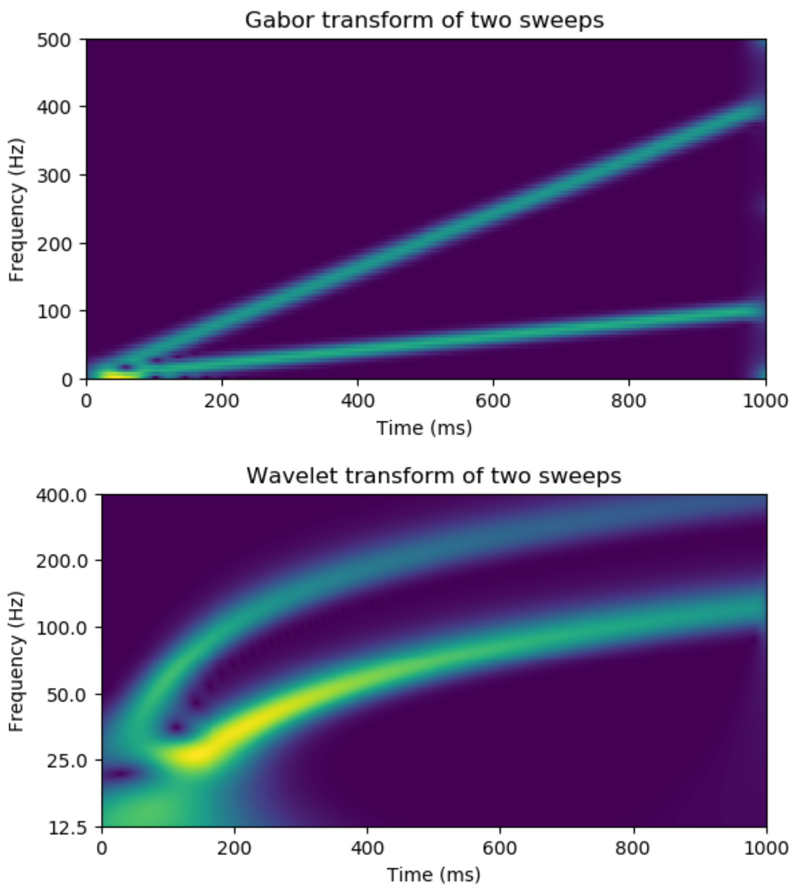

FIG. 3. Comparison of Gabor and Wavelet transforms. Note the resolution at low frequencies.

NONSTATIONARY DECONVOLUTION IN WAVELET DOMAIN

In this section, we explore how to use the wavelet transform to deconvolve a seismic signal that was generated using the nonstationary reflectivity model of Kjartansson (Kjartansson (1979)). We first begin with a simple reflectivity series consisting of six spikes, separated in time. We then consider a more general, randomly generated reflectivity series.

First example. We generate six spikes in the constant Q model using the pseudodifferential operator approach with function $\alpha(t, f)$ as the symbol for the operator. The result is shown in the top half of Figure 4, indicating six (Q-decaying) spikes separated by half second intervals. Notice the amplitude decay in the spread-out spikes and their expanding width. This is characteristic of the constant-Q model of Kjartansson (1979), and matches the images produced in his 1979 paper.

The continuous wavelet transform for these decaying spikes is shown in Figure 5. To deconvolve, we take the wavelet transform and multiply by

$$m(t, f) = \frac{1}{\alpha(t, f)}.$$

We note that when this denominator is close to zero, we numerically limit the size of m. The resulting deconvolution is shown in the bottom half of Figure 4. We note the deconvolution successfully reassigns the broad peaks in the seismic signal to (bandlimited) spikes at the proper time-locations in the signal.

For a more challenging test, in Figure 6, we have a simulated seismic signal (constant Q = 100) and its corresponding wavelet transform. Note in the wavelet transform image, we can see the loss of energy at high frequencies and large time values, which is characteristic of Q decay.

Figure 7 shows the result of the wavelet deconvolution and its comparison to the original reflectivity. They certainly both look spiky and show some reasonable similiarities, but to be fair, this is not enough of a test to show that the deconvolution is working well.

FUTURE WORK

This is very much a work in progress. More work is needed to get the form of the multiplier more precise, and more work to verify that this works correctly on even simulated seismic signals. We would develop a mathematical description of the explicit connection between the symbols in a pseudodifferential operator and the wavelet multiplier function $\alpha(t, f)$, analogous to the results for Gabor multipliers, as in Gibson et al. (2013). We then would like to design more accurate multipliers and apply to real seismic signals. We also need to include the seismic source signature in the model, and derive it from the recorded data.

We then need to apply this to real seismic data in a normal processing flow to see if it shows any particular advantages.

Finally, the continuous wavelet transform is rather slow. If this technique is useful,

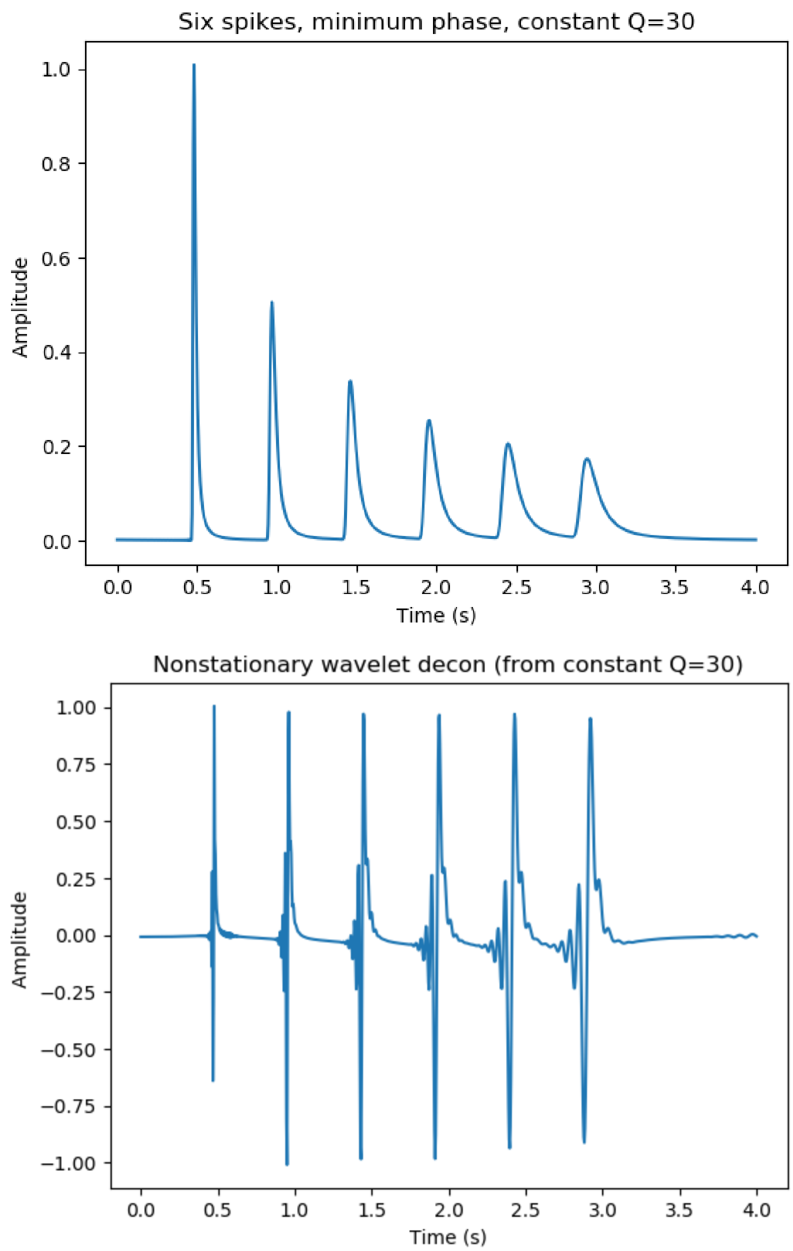

FIG. 4. Six spikes in a constant Q medium and the result of nonstationary wavelet devconvolution.

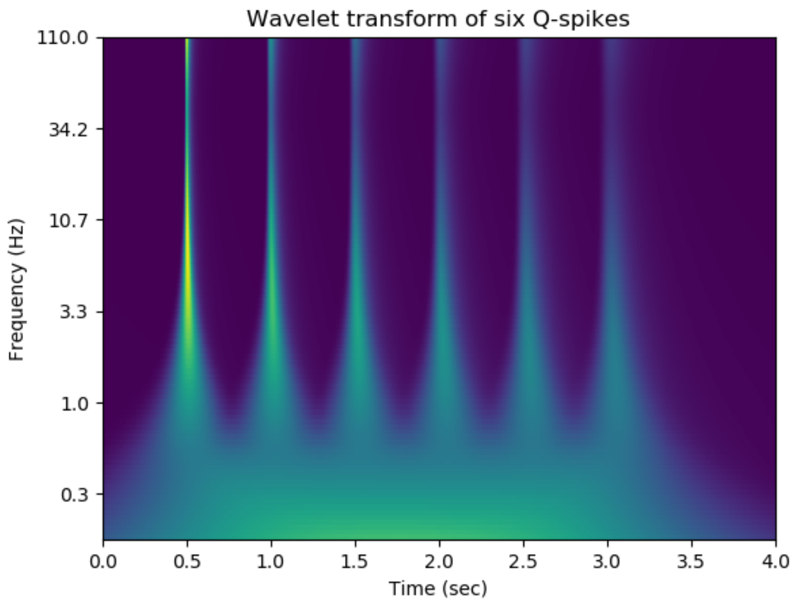

FIG. 5. For reference, the CWT of the six spikes in constant Q medium.

we will have to speed up the algorithm. We do have some ideas along these lines, taking advantage of the discrete transforms in wavelet theory and utilizing spectral downsampling.

CONCLUSIONS

We have shown that a continuous wavelet multiplier can deconvolve simple spikes in a constant-Q model for wave attenuation, and shows promise for deconvolving a reflectivity series in a constant-Q model. Much work is still needed to be done.

ACKNOWLEDGMENTS

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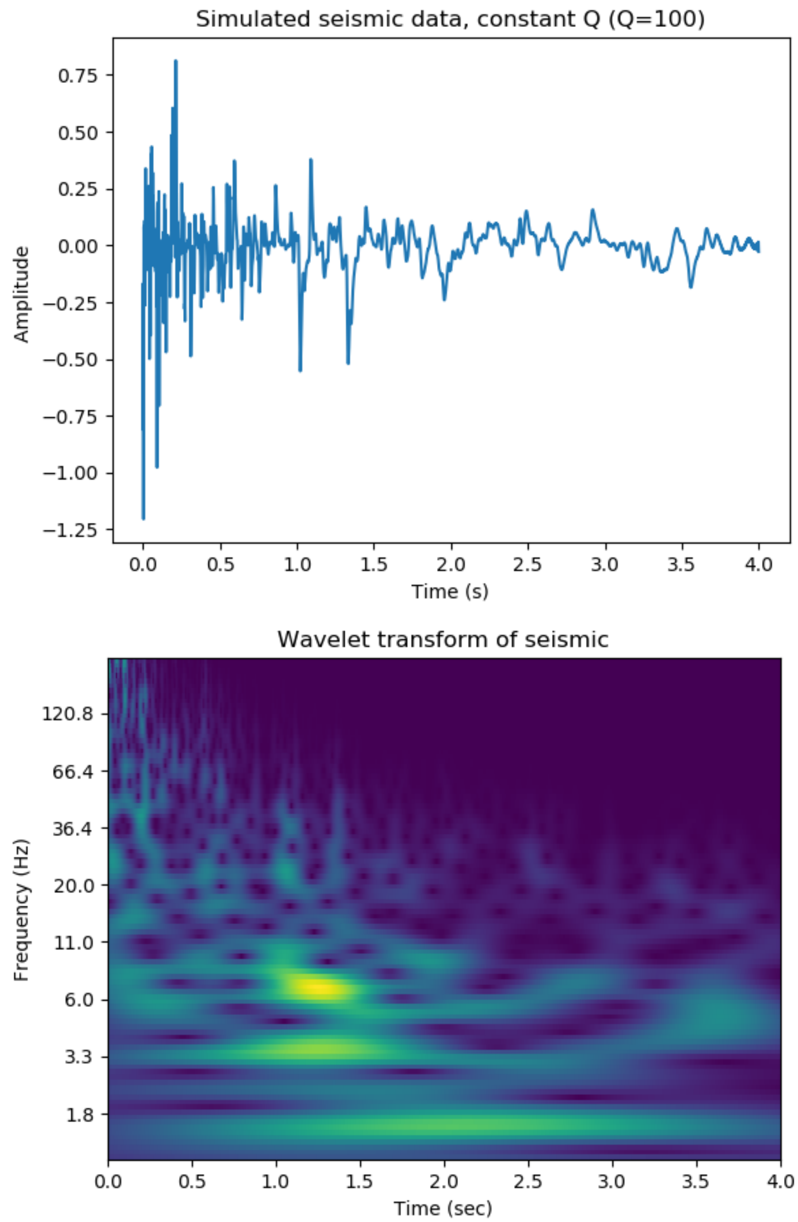

FIG. 6. Simulated constant Q seismic data, and CWT. Note the energy decay in higher frequencies.

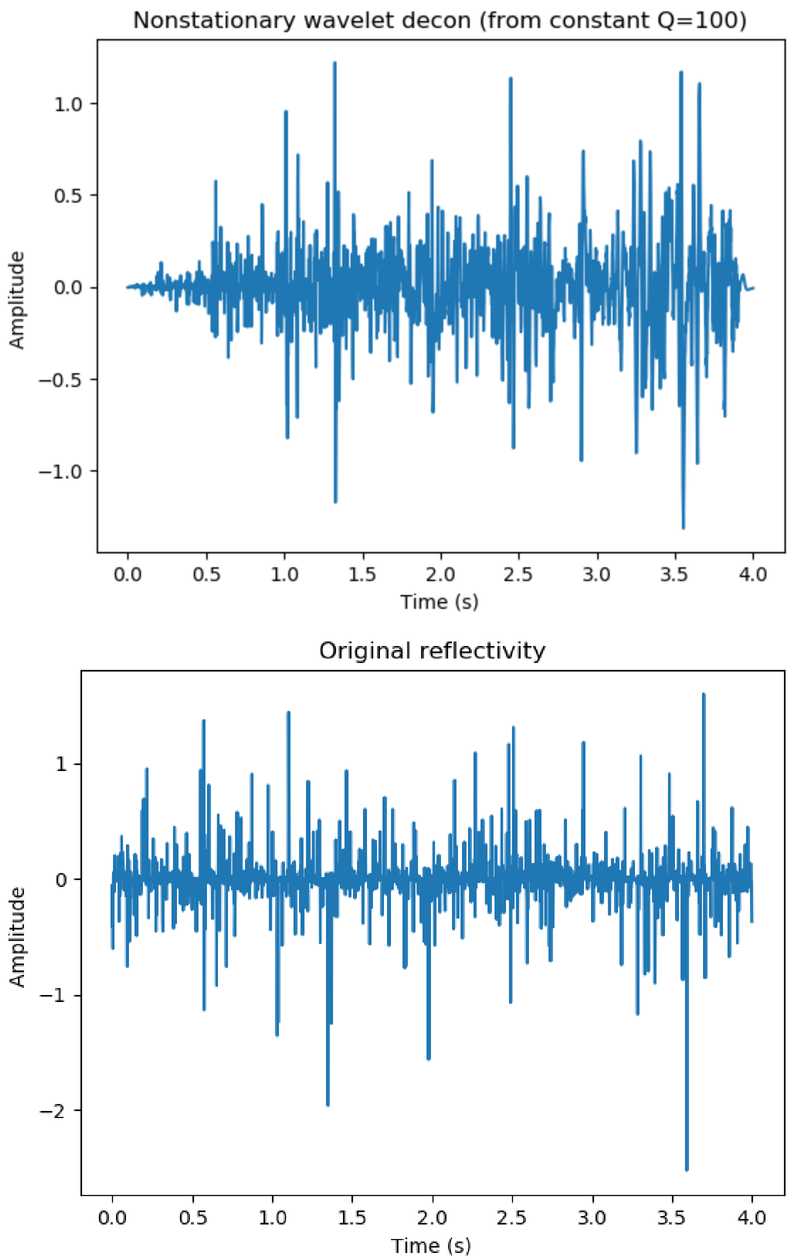

FIG. 7. The results of wavelet deconvolution and the original reflectivity series.

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