Application of misfit-based model space coordinate system design to seismic AVO inversion

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ABSTRACT

The standard re-expressions used in AVO analysis and inversion (e.g., from velocity-density to modulus-density, or from the Aki-Richards approximation to the Shuey approximation, etc.) are formally coordinate transforms between oblique-rectilinear (i.e., non-Cartesian) coordinate systems. Alternative coordinate transforms leading to favourable updating properties represent in this sense a valid suite of AVO re-expressions. In a low-dimensional model space like that of AVO (which involves dimensionalities in the low single digits), analytic forms for transformation matrices to systems in which the Hessian operator is an identity matrix can be found. These imply new AVO approximations, within which updates in AVO inversion require no 2nd order objective function information. This may have consequences both for iterative linear AVO inversion algorithms and "weighted stack" algorithms, the latter of which can be based on much simpler weights.

INTRODUCTION

In a companion report we have introduced the idea of overlaying oblique and scaled coordinate systems onto model spaces, which are so designed as to confer favourable convergence properties on least-squares inversion. In this report we will apply the idea to a low-dimensional seismic inverse problem, Amplitude Variation with Offset (AVO) inversion. AVO inversion (e.g., Castagna and Backus, 1993) is the process by which seismic reflection data amplitudes at multiple angles are used to simultaneously estimate the jumps in elastic medium properties at one or more subsurface boundaries. Except in rare cases, AVO analysis and inversion involve linearizations of the relationship between reflected seismic amplitudes and parameter "jumps", or perturbations, across reflecting boundaries. We will start with these linearized forms in our development.

AVO APPROXIMATIONS

AVO analysis and inversion is based on linearized approximate solutions of the Zoeppritz equations. The basic form is the Aki-Richards approximation (Aki and Richards, 2002), which for $R_{\rm PP}$ is

$$R_{\rm PP}(\theta) \approx \frac{1}{2}\left(1+\tan^2\theta\right)\frac{\Delta V_P}{V_P} - \gamma\sin^2\theta\frac{\Delta V_S}{V_S} + \frac{1}{2}\left(1-\gamma\sin^2\theta\right)\frac{\Delta\rho}{\rho}, \tag{1}$$

where $\gamma = 4(^{V_S}/_{V_P})^2$, and where θ is either the angle of incidence of the plane P-wave, or the average of the incident and transmitted angle. A wide range of re-expressions, which take equation (1) as a starting point, are in use. Many of these involve changing the physical variables that experience a contrast at the boundary. For instance, a form based on the P-wave and S-wave impedances $I_P = \rho V_P$ and $I_S = \rho V_S$, and the relations

$$\frac{\Delta I_P}{I_P} \approx \frac{\Delta V_P}{V_P} + \frac{\Delta\rho}{\rho}, \quad \frac{\Delta I_S}{I_S} \approx \frac{\Delta V_S}{V_S} + \frac{\Delta\rho}{\rho}, \tag{2}$$

is often used (e.g., Larsen, 1999):

$$R_{\mathrm{PP}}(\theta) \approx \frac{1}{2}\left(1+\tan^2\theta\right)\frac{\Delta I_P}{I_P} - \gamma\sin^2\theta\frac{\Delta I_S}{I_S} + \frac{1}{2}\left(\gamma\sin^2\theta - \tan^2\theta\right)\frac{\Delta\rho}{\rho}. \tag{3}$$

It is also common to re-express the Aki-Richards approximation with an emphasis on the form of the coefficients, rather than the perturbations alone. The Shuey approximation (e.g., Castagna and Backus, 1993),

$$R_{\mathrm{PP}}(\theta) \approx \frac{1}{2}\frac{\Delta S_1}{S_1} + \frac{1}{2}\sin^2\theta\frac{\Delta S_2}{S_2} + \frac{1}{2}\left(\tan^2\theta - \sin^2\theta\right)\frac{\Delta S_3}{S_3}, \tag{4}$$

where

$$\frac{\Delta S_1}{S_1} = \frac{\Delta V_P}{V_P} + \frac{\Delta\rho}{\rho}, \quad \frac{\Delta S_2}{S_2} = \frac{\Delta V_P}{V_P} - 2\gamma\frac{\Delta V_S}{V_S} - \gamma\frac{\Delta\rho}{\rho}, \text{ and } \frac{\Delta S_3}{S_3} = \frac{\Delta V_P}{V_P}, \tag{5}$$

is of this type. Here, the first term dominates near $\theta \approx 0°$, the second in an intermediate range, and the third where θ is large enough for $\tan\theta$ and $\sin\theta$ to differ significantly. By choosing this formulation, in other words, we allow ourselves to focus on available data and which parameters are derived from which angle ranges.

AVO approximations exist for each of four reflection coefficient types: R_{PP}, R_{PS}, R_{SP} and R_{SS}. In converted wave AVO analysis and inversion, combinations of R_{PP} and R_{PS} have been shown to lead to model inferences with greater accuracy and stability than those based on R_{PP} alone (e.g., Larsen, 1999). An impedance formulation, with the density eliminated by assuming it varies in a known manner with P-wave velocity, involves simultaneous quantification of these two reflection coefficients as

$$\begin{aligned} R_{\mathrm{PP}}(\theta) &\approx A(\theta)\frac{\Delta I_P}{I_P} + B(\theta)\frac{\Delta I_S}{I_S}, \text{ and} \\ R_{\mathrm{PS}}(\theta,\varphi) &\approx C(\theta,\varphi)\frac{\Delta I_P}{I_P} + D(\theta,\varphi)\frac{\Delta I_S}{I_S}, \end{aligned} \tag{6}$$

where

$$A = \frac{1}{2}\left(1+\tan^2\theta\right),\ B = -\gamma\sin^2\theta,\ C = -\frac{V_P\tan\varphi}{V_S}\left(1+E\right),\ D = -\frac{V_P\tan\varphi}{10V_S}E, \tag{7}$$

and where $E = 2\sin^2\varphi - 2({}^{V_S}\!/\!{}_{V_P})\cos\theta\cos\varphi$, and φ is the shear wave angle.

In each of the cases above, the reflection coefficient is an inner product between two vectors, one a vector of coefficients and the other a vector of model parameters (or unknowns, in AVO inversion). For instance, equation (1) can be written

$$R_{\mathrm{PP}}(\theta) \approx \left[\frac{1}{2}\left(1+\tan^2\theta\right), -\gamma\sin^2\theta, \frac{1}{2}\left(1-\gamma\sin^2\theta\right)\right]\left[\begin{array}{c} {}^{\Delta V_P}\!/\!{}_{V_P} \\ {}^{\Delta V_S}\!/\!{}_{V_S} \\ {}^{\Delta\rho}\!/\!{}_{\rho} \end{array}\right]. \tag{8}$$

Later we will make use of the fact that the ordering of the elements of these vectors is not important, provided any switching of elements in one vector is matched by a corresponding switch in the other. For instance, R_{PP} is evidently symmetric under the exchange

$$R_{\text{PP}}(\theta) \approx \left[-\gamma \sin^2\theta, \frac{1}{2}\left(1+\tan^2\theta\right), \frac{1}{2}\left(1-\gamma\sin^2\theta\right) \right] \begin{bmatrix} \Delta V_S/V_S \\ \Delta V_P/V_P \\ \Delta\rho/\rho \end{bmatrix}. \tag{9}$$

This inner product form and this symmetry hold across all AVO linear approximations.

AVO RE-PARAMETERIZATION AS A COORDINATE TRANSFORM

We assume that the set of model parameters $[\Delta V_P/V_P, \Delta V_S/V_S, \Delta\rho/\rho]^T$ has the transformation properties of a contravariant vector. Then, if the coefficient vector is treated as a covariant vector, the reflection coefficient as the inner product of the two can be considered to be a proper scalar or invariant. In this section we will show how AVO re-parameterizations can be expressed in terms of coordinate transformations from one oblique-rectilinear system to another. To do this, we will make use of the relationships between vectors in two coordinate systems, say s and r (see, e.g., Innanen, 2020a). Given any one transformation matrix, e.g., t_μ^ν, these rules give the others:

$$\begin{aligned} &s^\nu = t_\mu^\nu r^\mu : && \text{from } r \text{ to } s\text{, contravariant} \\ &r^\nu = (t^{-1})_\mu^\nu s^\mu : && \text{from } s \text{ to } r\text{, contravariant} \\ &\phi_\mu(r) = t_\mu^\nu \phi_\nu(s) : && \text{from } s \text{ to } r\text{, covariant} \\ &\phi_\mu(s) = (t^{-1})_\mu^\nu \phi_\nu(r) : && \text{from } s \text{ to } r\text{, covariant.} \end{aligned} \tag{10}$$

Suppose that we are characterizing the reflection coefficient R_{PP} arising from a specific contrast in elastic properties. The three velocity/density perturbations associated with this contrast are the contravariant components of a vector in a coordinate system labelled s:

$$s^\mu = \begin{bmatrix} s^1 \\ s^2 \\ s^3 \end{bmatrix} = \begin{bmatrix} \Delta V_P/V_P \\ \Delta V_S/V_S \\ \Delta\rho/\rho \end{bmatrix}, \tag{11}$$

and the coefficients are the covariant components of a vector $F_\mu^{(s)}(\theta)$:

$$F_\mu^{(s)}(\theta) = \begin{bmatrix} F_1^{(s)}(\theta) \\ F_2^{(s)}(\theta) \\ F_3^{(s)}(\theta) \end{bmatrix} = \begin{bmatrix} 1/2\,(1+\tan^2\theta) \\ -\gamma\sin^2\theta \\ 1/2\left(1-\gamma\sin^2\theta\right) \end{bmatrix}. \tag{12}$$

Here the indices 1, 2 and 3 have been assigned to P-wave velocity, S-wave velocity, and density respectively, but, as pointed out above, any re-assignment is permissible, provided it occur within both s^μ and F_μ. R_{PP} is then the scalar product

$$R_{\text{PP}}(\theta) = F_\mu^{(s)}(\theta) s^\mu. \tag{13}$$

Suppose next that the impedance perturbations associated with the same medium are the contravariant components of a vector in a different coordinate system labelled r:

$$r^{\mu} = \begin{bmatrix} r^1 \\ r^2 \\ r^3 \end{bmatrix} = \begin{bmatrix} {}^{\Delta I_P}\!/_{I_P} \\ {}^{\Delta I_S}\!/_{I_S} \\ {}^{\Delta\rho}\!/_{\rho} \end{bmatrix}. \tag{14}$$

The reflection coefficient, which is an invariant in our setup, is then given by

$$R_{\text{PP}}(\theta) = F^{(r)}_{\mu}(\theta) r^{\mu}. \tag{15}$$

If the re-parameterization is legitimately an oblique-rectilinear coordinate transform, this $F^{(r)}_{\mu}(\theta)$ must arise from its $F^{(s)}_{\mu}(\theta)$ counterpart via transform rules for covariant vectors, and the perturbation vector via rules for contravariant vectors. Inspection of equation (2) allows us to infer the transformation matrix taking a contravariant vector in the s system to its counterpart in the r system:

$$r^{\mu} = \begin{bmatrix} r^1 \\ r^2 \\ r^3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s^1 \\ s^2 \\ s^3 \end{bmatrix}. \tag{16}$$

From the rules summarized in equation (10), the transformation matrix that must be applied to $F^{(r)}_{\mu}$ in order to obtain $F^{(s)}_{\mu}$ is the transpose of the inverse of the matrix in (16). The inverse, computed analytically, is

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}. \tag{17}$$

Applying the transpose of this matrix to the Aki-Richards coefficients, we obtain

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} {}^1\!/_2(1+\tan^2\theta) \\ -\gamma\sin^2\theta \\ {}^1\!/_2(1-\gamma\sin^2\theta) \end{bmatrix} = \begin{bmatrix} {}^1\!/_2\left(1+\tan^2\theta\right) \\ -\gamma\sin^2\theta \\ {}^1\!/_2\left(\gamma\sin^2\theta - \tan^2\theta\right) \end{bmatrix}. \tag{18}$$

This correctly reproduces the coefficients in equation (3), confirming that this type of re-parameterization is a bona fide oblique coordinate transform. We can quickly confirm this by considering the Shuey approximation as another example. Let the r system instead contain model vectors of the form

$$r^{\mu} = \begin{bmatrix} r^1 \\ r^2 \\ r^3 \end{bmatrix} = \begin{bmatrix} {}^{\Delta S_1}\!/_{S_1} \\ {}^{\Delta S_2}\!/_{S_2} \\ {}^{\Delta S_3}\!/_{S_3} \end{bmatrix}. \tag{19}$$

We treat this vector is a transformation of the Aki-Richards vector s^{μ}, through a matrix that can be constructed by inspection of equations (5):

$$\begin{bmatrix} r^1 \\ r^2 \\ r^3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & -2\gamma & -\gamma \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} s^1 \\ s^2 \\ s^3 \end{bmatrix}. \tag{20}$$

This matrix is of the second type in (10). It follows that the matrix taking a covariant vector from the r system to the s system is the transpose of the inverse of this matrix. Computing this analytically, and applying it to the Aki-Richards coefficients, we obtain the coefficients of $F_\mu^{(r)}(\theta)$:

$$\begin{bmatrix} 0 & -1/2 & 1 \\ 0 & -1/2\gamma & 0 \\ 1 & 1/2\gamma + 1/2 & -1 \end{bmatrix} \begin{bmatrix} 1/2(1+\tan^2\theta) \\ -\gamma\sin^2\theta \\ 1/2(1-\gamma\sin^2\theta) \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2\sin^2\theta \\ 1/2\left(\tan^2\theta - \sin^2\theta\right) \end{bmatrix}, \tag{21}$$

which, by comparison with equation (4), is observed to correctly reproduce the Shuey coefficients.

Converted wave AVO, or any approach using more than one reflection type, requires us to introduce multiple scalar products. In the impedance formulation examined by Larsen (1999), for instance, we have

$$\begin{aligned} R_{\mathrm{PP}}(\theta) &\approx F_\mu^{(s)}(\theta)s^\mu \\ R_{\mathrm{PS}}(\theta,\varphi) &\approx G_\mu^{(s)}(\theta,\varphi)s^\mu, \end{aligned} \tag{22}$$

both involving the same model vector

$$s^\mu = \begin{bmatrix} \Delta I_P/I_P \\ \Delta I_S/I_S \end{bmatrix}, \tag{23}$$

where

$$F_1^{(s)}(\theta) = A(\theta),\ F_2^{(s)}(\theta) = B(\theta),\ G_1^{(s)}(\theta,\varphi) = C(\theta,\varphi),\ G_2^{(s)}(\theta,\varphi) = D(\theta,\varphi). \tag{24}$$

Here a single transformation comprising a 2×2 matrix,

$$t_\mu^\nu = \begin{bmatrix} t_1^1 & t_2^1 \\ t_1^2 & t_2^2 \end{bmatrix}, \tag{25}$$

would in general be applied separately to both equations. It may be acceptable to reduce the data, e.g., by forming a linear combination

$$R(\theta,\varphi) = q_1 R_{\mathrm{PP}}(\theta) + q_2 R_{\mathrm{PS}}(\theta,\varphi) \approx \left(q_1 F_\mu^{(s)}(\theta) + q_2 G_\mu^{(s)}(\theta,\varphi)\right)s^\mu, \tag{26}$$

recovering a single scalar product form, which transforms via (25).

THE HESSIAN IN LEAST-SQUARES AVO INVERSION

The Hessian matrix associated with a least-squares objective function plays a very important role in practical AVO inversion. The inverse of the Hessian appears explicitly as a pre-multiplier of the gradient in an iterative scheme for nonlinear AVO inversion; however, it also appears implicitly in the weights of all linearized inversion schemes (which are much more commonly applied). The linear inverse scheme is essentially equivalent to a single

Gauss-Newton update within the nonlinear/iterative scheme, so we can discuss the latter without loss of generality.

Suppose we were to assume a linearized forward model for deconvolved PP reflection data as a function of time t and incidence angle θ. "Predicted" data associated with a particular set of velocity and density perturbations are, within this scheme,

$$\begin{aligned} R_{\text{PP}}^{\text{pred}}(\theta,t) \approx & \frac{1}{2}\left(1+\tan^2\theta(t)\right)\frac{\Delta V_P(t)}{V_P} - \gamma(t)\sin^2\theta(t)\frac{\Delta V_S(t)}{V_S} \\ & + \frac{1}{2}\left(1-\gamma(t)\sin^2\theta(t)\right)\frac{\Delta\rho(t)}{\rho}. \end{aligned} \tag{27}$$

Time values are treated independently of one another, so we can drop the t dependence, remembering that the problem will be solved once per time point in practice. Assigning the coefficients to the covariant components of a vector F_μ and the perturbations to the contravariant components of a vector s^μ, we have

$$R_{\text{PP}}^{\text{pred}}(\theta) = F_\mu^{(s)}(\theta)s^\mu. \tag{28}$$

Given M angles of input data, the least-squares objective function is then

$$\Phi(s) = \frac{1}{2}\sum_{j=1}^{M}\left(F_\mu^{(s)}(\theta_j)s^\mu - R_{\text{PP}}(\theta_j)\right)^2 = s^\mu\phi_{\mu\nu}(s)s^\nu + s^\mu g_\mu(s) + \chi, \tag{29}$$

where

$$\phi_{\mu\nu}(s) = \frac{1}{2}\sum_j F_\mu^{(s)}F_\nu^{(s)},\ \varphi_\mu(s) = -\sum_j F_\mu^{(s)}R_{\text{PP}},\ \text{and}\ \chi = \frac{1}{2}\sum_j R_{\text{PP}}^2. \tag{30}$$

Because we have defined the model unknown as a contravariant vector, for Φ to be a scalar, $\phi_{\mu\nu}$ must be a second rank covariant tensor, φ_μ must be a covariant vector, and χ must be scalar. The first of these, $\phi_{\mu\nu}$, is the Hessian:

$$\phi_{\mu\nu}(s) = \begin{bmatrix} (1/2)\sum_j F_1^{(s)}F_1^{(s)} & (1/2)\sum_j F_1^{(s)}F_2^{(s)} & (1/2)\sum_j F_1^{(s)}F_3^{(s)} \\ (1/2)\sum_j F_2^{(s)}F_1^{(s)} & (1/2)\sum_j F_2^{(s)}F_2^{(s)} & (1/2)\sum_j F_2^{(s)}F_3^{(s)} \\ (1/2)\sum_j F_3^{(s)}F_1^{(s)} & (1/2)\sum_j F_3^{(s)}F_2^{(s)} & (1/2)\sum_j F_3^{(s)}F_3^{(s)} \end{bmatrix}. \tag{31}$$

The rules in (10) apply to tensors as well as vectors; if t_μ^ν is a transformation matrix via which a contravariant vector in an r system goes over into an s system, then a rank-2 covariant tensor transforms as

$$\phi_{\mu\nu}(r) = t_\mu^\lambda\,\phi_{\lambda\sigma}(s)\,t_\nu^\sigma. \tag{32}$$

To give another example, for simultaneous PP and PS inversion, the least-squares objective function takes the form

$$\Phi(s) = \frac{1}{2}\sum_{j=1}^{M}\left(F_\mu^{(s)}(\theta_j)s^\mu - R_{\text{PP}}(\theta_j)\right)^2 + \frac{1}{2}\sum_{j=1}^{M}\left(G_\mu^{(s)}(\theta_j,\varphi_j)s^\mu - R_{\text{PS}}(\theta_j,\varphi_j)\right)^2, \tag{33}$$

where θ_j and φ_j are the average P-wave and S-wave ray angles associated with the jth offset in the data, which we assume is a known relationship. The Hessian matrix in this case is

$$\phi_{\mu\nu}(s) = \frac{1}{2}\sum_{j=1}^{M}\left(F_\mu(\theta_j)F_\nu(\theta_j) + G_\mu(\theta_j,\varphi_j)G_\nu(\theta_j,\varphi_j)\right), \tag{34}$$

where

$$\begin{aligned} F_1(\theta) &= \frac{1}{2}\left(1+\tan^2\theta\right),\ F_2(\theta) = -\gamma\sin^2\theta, \\ G_1(\theta,\varphi) &= -\frac{V_P\tan\varphi}{V_S}\left(1+\Gamma(\theta,\varphi)\right),\ G_2(\theta,\varphi) = -\frac{V_P\tan\varphi}{10V_S}\Gamma(\theta,\varphi), \end{aligned} \tag{35}$$

and where $\Gamma = 2\sin^2\varphi - 2(^{V_S}/_{V_P})\cos\theta\cos\varphi$.

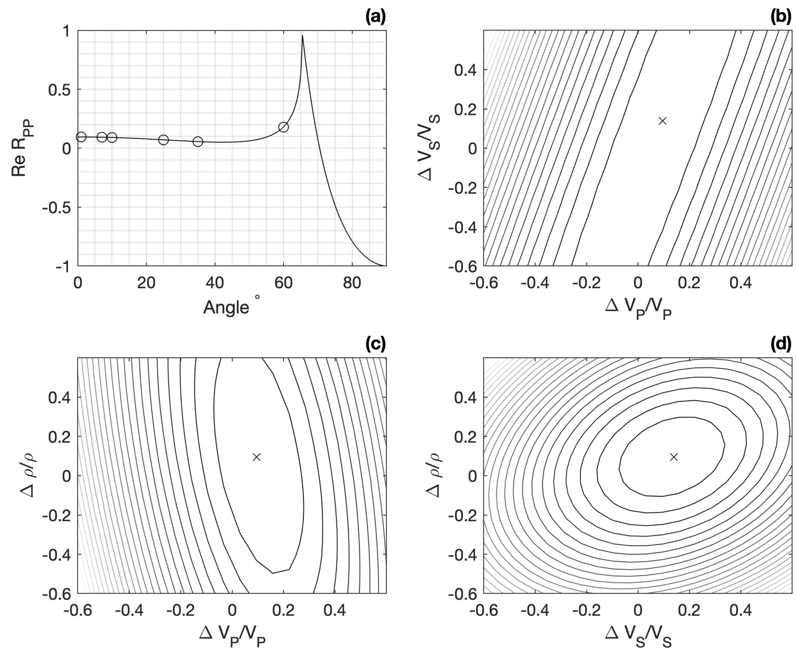

FIG. 1. Objective function for standard 3-term AVO inversion. (a) The real part of R_{PP} is plotted as a function of incident angle θ. The discrete set of data amplitudes are plotted as circles. (b)-(d) 2D slices through the 3D objective function are taken, each time holding one perturbation fixed at its true value. In (b) the slice is at constant $\Delta\rho/\rho = \Delta\rho/\rho|_{\text{true}}$; in panel (c) the slice is at constant $\Delta V_S/V_S = \Delta V_S/V_S|_{\text{true}}$; and in panel (d) the slice is at constant $\Delta V_P/V_P = \Delta V_P/V_P|_{\text{true}}$. The true model is plotted with an x in these three panels.

AVO inversion (like all multiparameter inversion) is an un-mixing process, where the influences of 2-3 independent perturbations on a set of reflection amplitudes are separated. This only happens to the extent permitted by (1) the physics describing the mixture, and (2) the available data. The shape and character of the Hessian matrix is informative about whether (1) and (2) will lead to successful unmixing or not. For problems with N unknowns, the Hessian is an $N\times N$ matrix conferring onto the objective function an ellipsoidal symmetry*. Difficulties with un-mixing of parameters, as well as convergence problems and uncertainty, appear as eccentricities in the ellipsoid and misalignments between the axes of the ellipsoid and those of the coordinate axes of model space in a given parameterization. A spherically-symmetric objective function has a scaled Kronecker delta function as a Hessian; this represents the best case scenario, with little chance of confusion or cross-talk between parameters during inversion.

To illustrate this, consider a standard 3-term AVO problem in the velocity/density parameterization. Suppose the P-wave reflection coefficient of a boundary between medium 1 (with properties V_{P_1} = 2.0 km/s, V_{S_1} = 1.0 km/s, ρ_1 = 2000 kg/m^3) and medium 2 (with properties V_{P_2} = 2.2 km/s, V_{S_2} = 1.15 km/s, ρ_2 = 2200 kg/m^3) is measured at $M = 6$ angles, $\{1°, 7°, 10°, 25°, 35°, 60°\}$. The data and the objective function associated with this problem are plotted in the four panels of Figure 1. The basin of attraction as projected onto these three planes is elongated along some preferred axes, indicating significant uncertainty in these directions; it is also misaligned, meaning the axes of the ellipsoid are oblique to the model space coordinate axes.

The above example was created assuming the availability of high-angle data (i.e., $\theta = 60°$). In practice, a maximum angle of roughly $\theta = 30°$ is more common. Overall the expectation will be that the uncertainty will increase given data on a smaller angle range. The objective function associated with data on an angle range $\{1°, 7°, 10°, 30°\}$ is plotted in Figure 2 (all other quantities, and the figure design, are unchanged). The elongation of the basin of attraction, or, equivalently, the eccentricities of the ellipsoid, have increased noticeably in both slices pertaining to the density. Density is notoriously difficult to determine from short-offset (small angle) reflection amplitude data; this increase in eccentricity of the objective function, via the Hessian $\phi_{\mu\nu}$, is a hallmark of that difficulty. Interestingly, given the correct density perturbation, in panel (b) it transpires that the smaller angle range slightly improves the distinguishability of the P- and S-wave velocities.

AVO RE-PARAMETERIZATIONS BASED ON DATA MISFIT

We have thus far (1) identified with each re-parameterization of the AVO problem a transform between oblique-rectilinear coordinate systems, and (2) focused on the obliquity and eccentricity of the objective function implied by the Hessian matrix as a strong indicator of the ease or difficulty of simultaneous multiparameter inversion. We next undertake a search for new parameterizations, designed by enforcing favourable properties on the resulting Hessian matrices.

We have developed algorithms by which transformation matrices whose action on a Hessian satisfies certain constraints (Innanen, 2020b); one of these allows us to find an AVO re-parameterization whose least-squares gradient always points directly to the minimum, regardless of starting point. There are many such matrices, distinguished by their lower

*We use the terms ellipsoid and sphere throughout this paper; by these words, we mean more specifically "N dimensional hyperellipsoid", or "N dimensional hyperspheroid". In each case the dimension of the sphere or ellipsoid to be visualized is equal to the number of model parameters being inverted for.

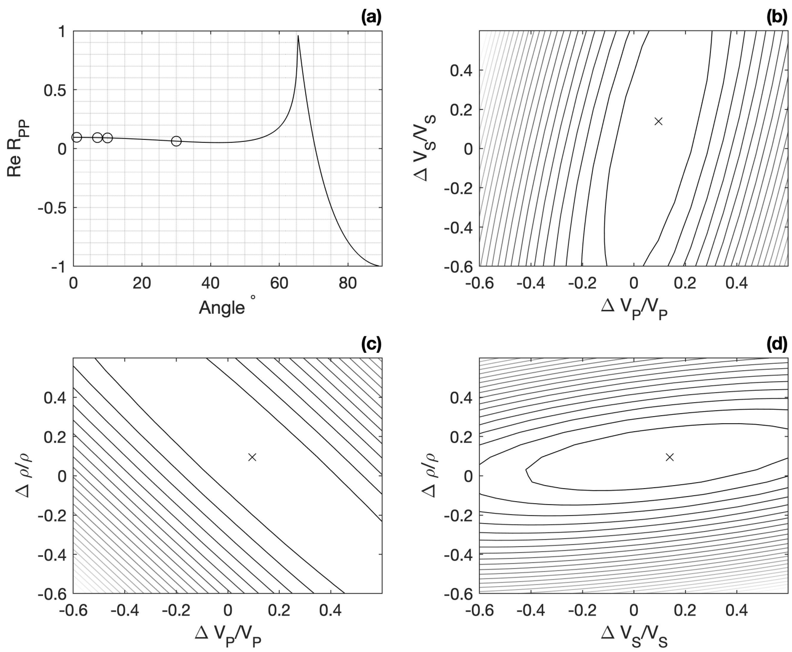

FIG. 2. Objective function with all variables the same as those in Figure 1 except with an angle range of $\{1°, 7°, 10°, 30°\}$.

triangular values. The simplest, in which the lower triangular entries are all zero, is

$$t_\mu^\nu = \begin{bmatrix} t_1^1 & t_2^1 & t_3^1 \\ 0 & t_2^2 & t_3^2 \\ 0 & 0 & t_3^3 \end{bmatrix}. \tag{36}$$

The other 6 elements are determined by enforcing the 6 constraints

$$\phi_{\mu\nu}(r) = t_\mu^\lambda\, \phi_{\lambda\sigma}(s)\, t_\nu^\sigma = \delta_{\mu\nu}, \tag{37}$$

where $\delta_{\mu\nu}$ is the Kronecker delta, and where $\phi_{\lambda\sigma}(s)$ is the Hessian derived from the Aki-Richards approximation in equation (31). The solution is

$$t_\mu^\nu = \begin{bmatrix} \psi_1 & \alpha_2\psi_2 & \alpha_3\psi_3 \\ 0 & \psi_2 & \beta_3\psi_3 \\ 0 & 0 & \psi_3 \end{bmatrix}, \tag{38}$$

where (suppressing the s dependence)

$$\alpha_2 = -\frac{\phi_{12}}{\phi_{11}},\ \alpha_3 = \frac{\phi_{22}\phi_{13} - \phi_{12}\phi_{23}}{\phi_{12}^2 - \phi_{11}\phi_{22}},\ \beta_3 = \frac{\phi_{11}\phi_{23} - \phi_{12}\phi_{13}}{\phi_{12}^2 - \phi_{11}\phi_{22}} \tag{39}$$

and

$$\begin{aligned} \psi_1 &= \phi_{11}^{-1/2},\ \psi_2 = \left(\phi_{22} - {\phi_{12}^2}/{\phi_{11}}\right)^{-1/2}, \\ \psi_3 &= \left\{ [\alpha_3, \beta_3, 1] \begin{bmatrix} \phi_{11} & \phi_{12} & \phi_{13} \\ \phi_{12} & \phi_{22} & \phi_{23} \\ \phi_{13} & \phi_{23} & \phi_{33} \end{bmatrix} \begin{bmatrix} \alpha_3 \\ \beta_3 \\ 1 \end{bmatrix} \right\}^{-1/2}. \end{aligned} \tag{40}$$

The transformation matrix of a contravariant vector from the s system to the r system occurs via the inverse of this t_μ^ν, which is

$$(t_\mu^\nu)^{-1} = \begin{bmatrix} 1/\psi_1 & -\alpha_2/\psi_1 & (\alpha_2\beta_3 - \alpha_3)/\psi_1 \\ 0 & 1/\psi_2 & -\beta_3/\psi_2 \\ 0 & 0 & 1/\psi_3 \end{bmatrix}. \tag{41}$$

This means the AVO approximation in the new coordinate system is

$$R_{\text{PP}}(\theta) = F_1^{(r)}(\theta) r^1 + F_2^{(r)}(\theta) r^2 + F_3^{(r)}(\theta) r^3, \tag{42}$$

where $r^\mu = [\Delta V_P/V_P', \Delta V_S/V_S', \Delta\rho/\rho']^T$, and

$$\begin{bmatrix} (\Delta V_P/V_P)' \\ (\Delta V_S/V_S)' \\ (\Delta\rho/\rho)' \end{bmatrix} = \begin{bmatrix} \psi_1 & \alpha_2\psi_2 & \alpha_3\psi_3 \\ 0 & \psi_2 & \beta_3\psi_3 \\ 0 & 0 & \psi_3 \end{bmatrix}^{-1} \begin{bmatrix} (\Delta V_P/V_P) \\ (\Delta V_S/V_S) \\ (\Delta\rho/\rho) \end{bmatrix}, \tag{43}$$

and

$$\begin{bmatrix} F_1^{(r)}(\theta) \\ F_2^{(r)}(\theta) \\ F_3^{(r)}(\theta) \end{bmatrix} = \begin{bmatrix} \psi_1 & \alpha_2\psi_2 & \alpha_3\psi_3 \\ 0 & \psi_2 & \beta_3\psi_3 \\ 0 & 0 & \psi_3 \end{bmatrix}^{T} \begin{bmatrix} F_1(\theta) \\ F_2(\theta) \\ F_3(\theta) \end{bmatrix}. \tag{44}$$

Three-parameter velocity/density AVO, coordinate-aligned Hessian

A further example of a useful transformation based on the Hessian operator is one which removes misalignment of the symmetry axes of the objective function with the coordinate axes. This is a simpler problem, allowing a greater freedom to pre-select transformation matrix elements and a smaller number of constraints to apply. For instance, we may select the diagonal and lower-triangular elements of t_μ^ν to be

$$t_\mu^\nu = \begin{bmatrix} 1 & t_2^1 & t_3^1 \\ 0 & \phi_{11} & t_3^2 \\ 0 & 0 & T_3^3 \end{bmatrix}, \tag{45}$$

where $T_3^3 = \phi_{12}^2 - \phi_{11}\phi_{22}$, and permit the three constraint equations

$$\begin{aligned} \phi_{12}(r) &= t_1^\lambda \, \phi_{\lambda\sigma}(s) \, t_2^\sigma = 0, \\ \phi_{13}(r) &= t_1^\lambda \, \phi_{\lambda\sigma}(s) \, t_3^\sigma = 0, \\ \phi_{23}(r) &= t_2^\lambda \, \phi_{\lambda\sigma}(s) \, t_3^\sigma = 0, \end{aligned} \tag{46}$$

to fix t_2^1, t_3^1 and t_3^2. The solution is

$$t_\mu^\nu = \begin{bmatrix} 1 & -\phi_{12} & \phi_{22}\phi_{13} - \phi_{12}\phi_{23} \\ 0 & \phi_{11} & \phi_{11}\phi_{23} - \phi_{12}\phi_{13} \\ 0 & 0 & \phi_{12}^2 - \phi_{11}\phi_{22} \end{bmatrix}, \tag{47}$$

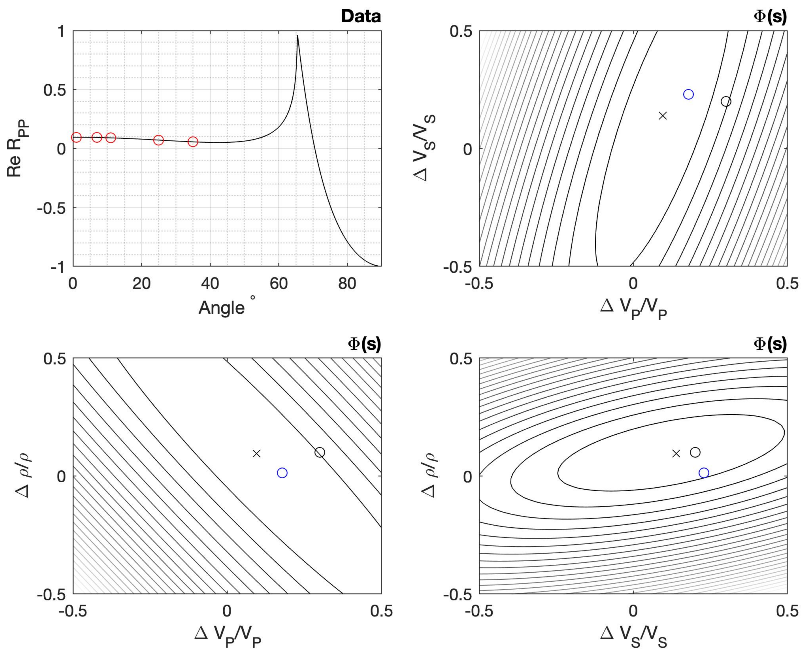

FIG. 3. Objective function based on discrete data in (a). Random initial point is plotted as a black circle; result of an unweighted stack update is plotted as a black circle.

and its inverse, the matrix to be applied to the standard AVO model vector to transform it to the new coordinate system, is

$$(t_{\mu}^{\nu})^{-1} = \begin{bmatrix} 1 & \lambda_1 & (\lambda_1\phi_{12}\phi_{13} - \phi_{22}\phi_{13})/\lambda_2 \\ 0 & 1/\phi_{11} & (\lambda_1\phi_{13} - \phi_{23})/\lambda_2 \\ 0 & 0 & 1/\lambda_2 \end{bmatrix}, \tag{48}$$

where

$$\lambda_1 = \frac{\phi_{12}}{\phi_{11}}, \;\; \lambda_2 = \left(\phi_{12}^2 - \phi_{11}\phi_{22}\right). \tag{49}$$

CONSEQUENCES FOR AVO INVERSION

We will frame the consequences of applying the transformation implied by (47) in terms up steepest-descent optimization steps. It should be emphasized that it is rare for AVO inverse problems to be solved iteratively in this way; normally, a linear weighted sum of the data is invoked. However, it is possible to understand the weights in the stacking process in terms of first and second derivatives of a least-squares objective function, so we lose no applicability discussing it this way. A steepest descent update in a transform domain in which the Hessian is spherically symmetric can be understood as a weighted stack with very mild weighting.

In Figure 3 we return to the projections of the objective function in the standard AVO model space with coordinates $\Delta V_P/V_P$, $\Delta V_S/V_S$ and $\Delta\rho/\rho$. This time we place randomly-chosen starting points as black circles, and minima in those directions as blue circles. Updating beyond this is found to be practically impossible, with the rate of reduction of the objective function between the blue point and the minimum being very small.

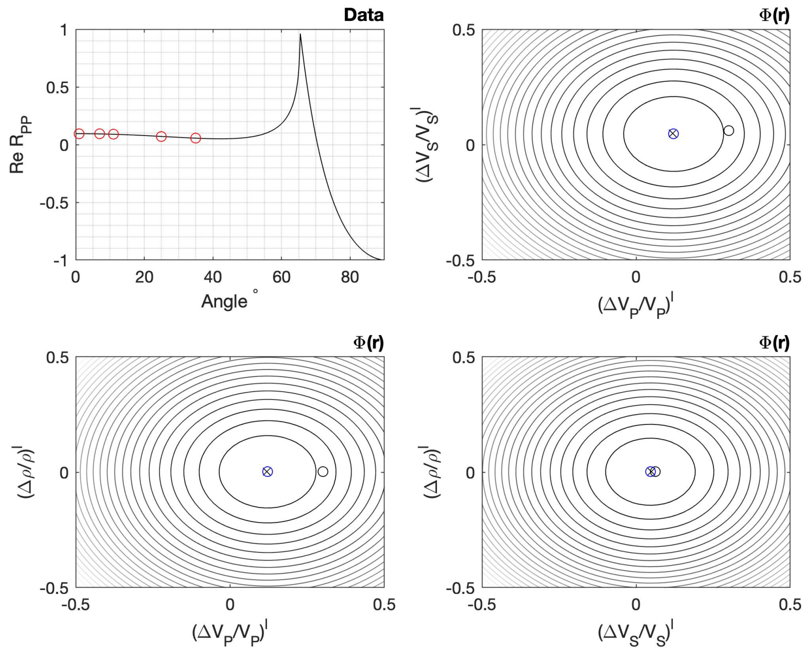

FIG. 4. Objective function in the transformed system.

The steepest-descent update in the transformed system:

$$r^{\mu}_{\min} = -g^{\mu\nu} \sum_{\theta} F^{(r)}_{\nu}(\theta) R_{\mathrm{PP}}(\theta) \tag{50}$$

is illustrated in Figure 4. We observe that the minimally weighted stack in this case gives us a result very close to the exact solution.

CONCLUSIONS

The standard re-expressions used in AVO analysis and inversion (e.g., from velocity-density to modulus-density, or from the Aki-Richards approximation to the Shuey approximation, etc.) are formally coordinate transforms between oblique-rectilinear (i.e., non-Cartesian) coordinate systems. Alternative coordinate transforms leading to favourable updating properties represent in this sense a valid suite of AVO re-expressions. In a low-dimensional model space like that of AVO (which involves dimensionalities in the low single digits), analytic forms for transformation matrices to systems in which the Hessian operator is an identity matrix can be found. These imply new AVO approximations, within which updates in AVO inversion require no 2nd order objective function information. This may have consequences both for iterative linear AVO inversion algorithms and "weighted stack" algorithms, the latter of which can be based on much simpler weights.

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