Application Fast Sweeping Method with Adaptive Finite Difference Scheme Eikonal Solver

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ABSTRACT

Many seismic applications require fast and accurate Eikonal equation solvers, such as Kirchhoff migration and tomography. One of the most stable and consistent methods is the Fast Sweeping Method, in which the Eikonal equation is numerically solved by the upwind finite difference scheme. Whenever finite difference is used, a plane wavefront is implicitly assumed. When waves propagate in inhomogeneous media, the assumption of plane wave front may be invalid. The errors generated in the spots on the bending wave front will propagate to the entire calculation domain, resulting in inaccurate propagation times. In this report, an adaptive finite difference scheme is introduced to improve the accuracy of the Eikonal solver in the Fast Sweeping Method.

INTRODUCTION

Many seismic applications require fast and accurate Eikonal equation solvers to generate travel time fields, such as Kirchhoff migration and tomography, and fast and accurate solvers for Eikonal equations are very attractive. Among these techniques, one of the most stable and consistent methods is the fast Sweeping method (Zhao 2004, Huang et al. 2019). It uses the upwind finite difference scheme and Gauss-Seidel iteration method to solve the discretized Eikonal equation. This method has high computational efficiency. One of the advantages of this method is that when the sweep direction is properly controlled, it can track the wavefront of a downward or turning wave, which can provide more insights into wave propagation. The use of finite difference to solve the Eikonal equation implicitly assumes that the wavefront is a plane wave or at least a local plane wave. In fact, when waves propagate in an inhomogeneous medium, the verification of this hypothesis may be invalid, so numerical errors will be inevitable. Finite difference is also used in dynamic ray tracing, where the ray tracing system is derived from Eikonal equation. In the ray tracing equation system, a flow coordinate is used, where one of the coordinate axes is chosen to coincide with the characteristic direction. In such way, the ray path in the inhomogeneous medium can be accurately predicted. The same idea can also be introduced into finite difference Eikonal equation solver for travel time calculation. In fact, as will be explained later, accurate travel time can be obtained in a special coordinate system. However, unlike ray tracing, arbitrary orthogonality may not be suitable for pre-parameterized model grids, and we don't want to add too much computational cost to affect efficiency. In this report, the finite difference Eikonal equation solver uses a simple two finite difference schemes, which can significantly improve the accuracy of the fast sweeping method shown in examples.

FAST SWEEPING METHOD

The Fast Sweeping Method (FSM) is an iterative algorithm which computes the travel times by successively sweeping the whole grid following a specific order. Gauss-Seidel iterations is performed in alternating directions, i.e. North-East, North-West, South-

East and South-West (Combinations of traversing x and y dimensions forwards and backwards), as shown in Figure 1, so that all the possible characteristic curves of the solution to the Eikonal are obtained.

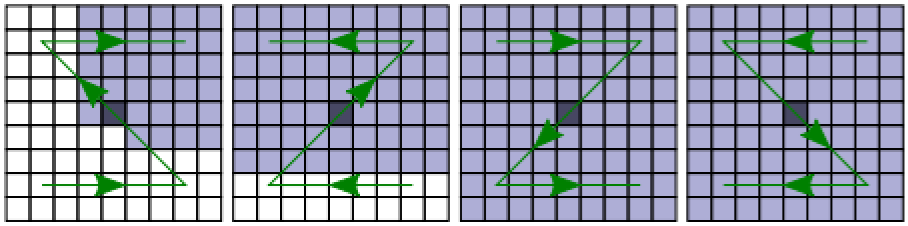


Figure 1. Fast Sweeping direction in 2D represented with arrows. The darkest cell is the initial point and the shaded cells are those analyzed by the current weep.

Sweeping may continue until no value is improved. In each sweep, the Eikonal equation is solved for every cell. In each cell with index i and j, Eikonal equation is approximated by first-order upwind-difference scheme, i.e. if let

$$T = T_{i,j}$$
$$T_x = \min\,(T_{i-1,j}, T_{i+1,j})$$
$$T_y = \min\,(T_{i,j-1}, T_{i,j+1})$$

If we are assuming that the speed of the front is positive (F > 0), T must be greater than T_x and T_y whenever the front wave has not already passed over the coordinates (i, j), we can rewrite the Eikonal Equation, for a discrete 2D space as:

$$max\left(\frac{T-T_x}{\Delta}\right)^2 + max\left(\frac{T-T_y}{\Delta}\right)^2 = \frac{1}{F_{i,j}^2} \qquad (1)$$

Equation (1) must be solved under the condition that wavefront propagation follows causality, that is, in order to reach a point with a higher arrival time, it should first propagate through the neighbors of that point with a smaller value. Equation (1) cannot guarantee causality, so before accepting the solution, the causality must be checked.

MODIFIED FAST SWEEPING METHOD

The Eikonal equation is solved numerically by the finite difference method, which implicitly assumes the plane wave front. However, when a wave propagates in an inhomogeneous medium, the assumption of the plane wave front may be invalid. For example: Figure 1 is a rectangular domain with a grid spacing of 1 km and a wave constant slowness of 1 s/km. O is the source point of $T_o = 0$ s. Using circular wave front expansion, the propagation time of grid points A and B can be accurately calculated $T_A = T_B = 1\ s$. The propagation time of point C should be equal to the time that the wave propagates from point O to C along the characteristic line, that is $T_C = \sqrt{2}\ s$. However, T_C calculated by upwind finite difference scheme is $1 + \sqrt{2}/2$ s, the percentage error of the numerical propagation time of grid point C relative to the analytical solution can reach up to 20.7%. This error always exists throughout the calculation. As the seismic wave front evolves, it will pollute the entire computational domain and make the propagation time field inaccurate.

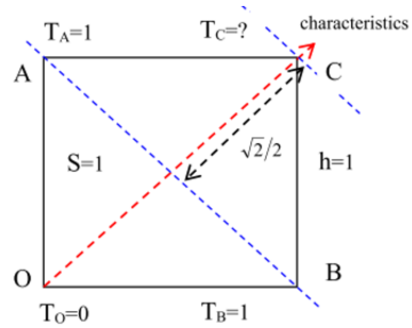


Figure 2. A Cartesian coordinate grid configuration. In this rectangular domain, wave slowness is 1 s/km and grid space is 1km. Traveltime of grid point o is set to 0 s. Traveltimes of point A and B are initialized to 1 s as wavefront pass to them from point o. Traveltime of point C need to be calculated and the wavefront reach to it along the characteristic line.

If you study this example carefully, it can be shown that when the characteristic line coincides with one of the coordinate axes, an accurate travel time can be obtained, i.e. T_A and T_B. In fact, the Eikonal equation is independent of choice of coordinate. Therefore, if the coordinates are rotated by 45 degrees with T_o as the origin of the coordinates, T_C can be accurately calculated by the central finite difference scheme

$$\left(\frac{T_C - T_O}{\sqrt{2}\Delta}\right)^2 + \left(\frac{T_A - T_B}{\sqrt{2}\Delta}\right)^2 = \frac{1}{F^2} \qquad (2)$$

In equation (2), if $T_o < \min\,(T_A, T_B)$, it is always ensured that the solution T_C of equation (2) is causal. Generally, the wavefront may not be circular as in the above example, but if the characteristic line direction of the wavefront is close to the axis of rotated coordinate, formula (2) can still give a good approximate the travel time. With this as an additional constraint, we arrive at an improved fast sweeping method (MFSM) by adaptive finite difference scheme Eikonal solver.

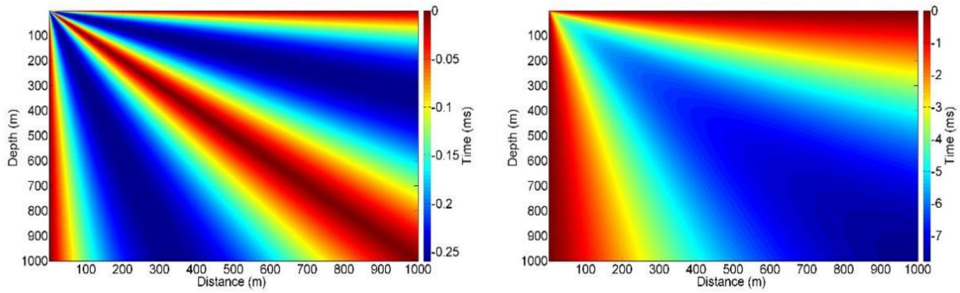
Modified Fast Sweeping Method (MFSM)

```
While (sweeping)
 Choosing sweep direction: yStart,yIncrement,yEnd and xStart, xIncrement,  xEnd
   do iy = yStart:yIncrement:yEnd
       do ix = xStart: xIncrement:xEnd
           T1 <- upwind scheme: equation (1)
                         iky=iy- yIncrement;
                         ikx=ix- xIncrement;
                         select three points T(iky,jkx), T(iky,jx),T(iy,jkx)
                        T2 <- central finite scheme:  equation (2)
                        T(iy,ix)=min(T1,t2)

 End
```

EXAMPLES

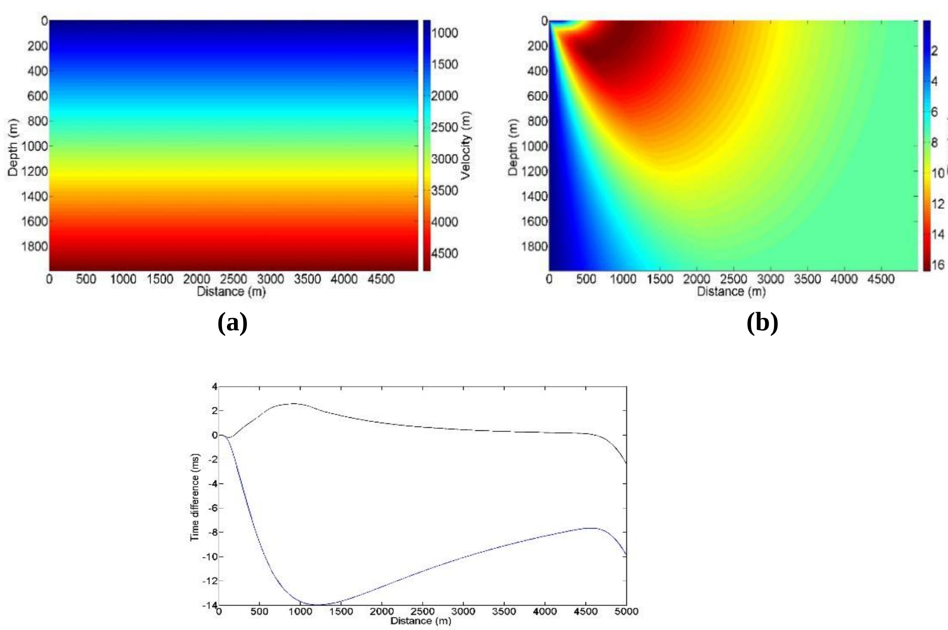
Example 1: A constant velocity model with a 100 x 100 grid; the grid space is 10 m, and the wave speed is 3000 km/s. Since the real travel time of the model can be easily calculated, only the difference between the real travel time and the calculated travel time is drawn. In Figure 3: (a): The time difference between the true traveltime and the traveltime calculated by MFSM , And (b): thetime difference between the true traveltime and traveltime calculated by FSM.



(a) **(b)**

Figure 3. Traveltime difference between true and calculated by MFSM (a); and calculated by FSM (b).

Example 2: Vertically increasing velocity model; The grid space is 10 m, and the vertical velocity is $V_z = 800 + 2Z$, where Z increases from zero to the maximum depth, as shown in Figure 4(a). Figure (b) shows the travel time difference between FSM and MFSM. On the surface of the model, where the travel time can be analytically solved. Figure 4(c) shows the time difference between analytical solution and FSM (blue), and MFSM (black) at model surface. If the calculated travel time is used to fit the first break as normally do in seismic Tomo, FSM traveltime may produce larger errors than MFSM traveltime does.

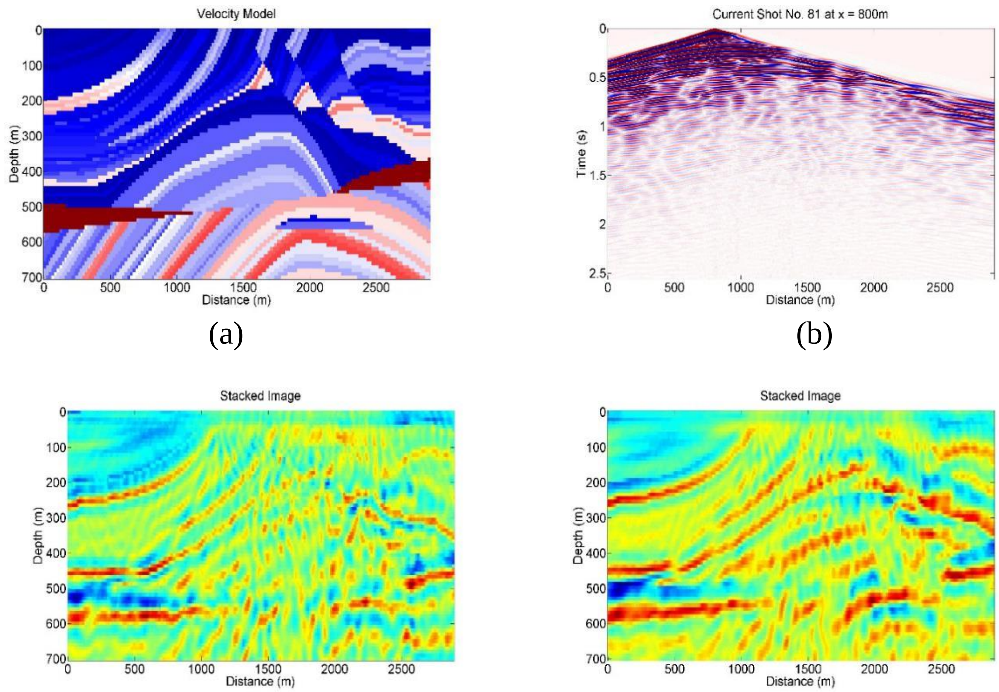


(a) **(b)**

(c)

Figure 4. (a) velocity mode; (b) traveltime difference between FSM and MFSM; (c)) traveltime difference between analytical solution and MFSM (black), FSM (blue) at model surface.

Example 3: Applying FSM and MFSM traveltime to Kirchhoff migration. In Figure 5: (a) is the velocity mode and (b) is a single shot gather generated by finite difference modelling; (c) is result of stacking all migrated single shot with FSM traveltime; (d) is same as (c) but with MFSM traveltime.



(a) (b)

(c) (d)

Figure 5: (a) velocity mode and (b) One shot gather that generated by finite difference modelling; (c) Stacked of all single shot migrated results with FSM traveltime and (d) is same as (c) but with MFSM traveltime.

CONCLUSIONS

Numerical results show that the application of the adaptive finite difference scheme Eikonal solver to the fast sweeping method can significantly improve the accuracy of the travel time. It is expected that comparing to FSM, MFSM will produce better results in seismic applications, such as migration and traveltime tomography.

ACKNOWLEDGEMENTS

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