Vectors and tensors in curved space

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ABSTRACT

Elsewhere in this report we develop the idea of navigating in model space (e.g., to construct the best estimate of an Earth model) not as usual, by being guided by an external objective function and its derivatives, but rather, by following the simplest possible paths in a space that is curved by the objective function. This requires ideas of Riemannian curvature, parallel displacement, and geodesics to be reviewed. This review can be considered as an addendum to the review on tensor analysis in non-Cartesian coordinate systems in the 2020 report.

INTRODUCTION

The 2020 CREWES report included a review (Innanen, 2020) of indicial notation for vector and tensor analysis in non-Cartesian coordinate systems. This document adds to that review some results involving changes to vector and tensor quantities when the space is curved. These results are then used in a companion paper in this year's report. As in 2020, the ideas here are expanded and embellished versions of results found in many texts. Dirac's short text introducing general relativity (Dirac, 1975) is my personal favourite. It is a little short on discourse, figures, etc., but the author manages to get a lot across with an extreme economy of words. Especially the tricky derivation of parallel displacement comes mostly from that resource.

NOTATION

Tensors are denoted and classified by Roman letters with upper, lower, or combined indices which label components. For instance, $t^{\mu}_{\nu\lambda}$ is a third rank tensor with one contravariant index and two covariant indices. In this report we will allow the indices to be Roman or Greek letters, so, s^n is a tensor of rank 1 (or vector), and constructions like r^{ν}_{m} are also permitted. This is a useful device for keeping things clear when different indices range over different numbers of dimensions.

If a vector x^{μ} is perturbed to produce $x^{\mu} + dx^{\mu}$, the length of the displacement is the scalar ds, where

$$ds^2 = dx^{\mu} dx_{\mu} = g_{\mu\nu} dx^{\mu} dx^{\nu}. \tag{1}$$

Here $g_{\mu\nu}$ is the metric tensor, a quantity that contains all relevant information about obliquity and scaling relationships between the coordinate axes. If the space is Cartesian, $g_{\mu\nu} = \delta_{\mu\nu}$, i.e., it is the Kronecker delta. The metric tensor $g_{\mu\nu}$ acts as an operator for raising or lowering indices:

$$a_{\mu} = g_{\mu\nu} a^{\nu},\ b^{\mu} = g^{\mu\nu} b_{\nu}. \tag{2}$$

The contravariant and covariant components of the metric tensor itself are related by

$$g^{\mu\nu} = (g_{\mu\nu})^{-1},\ g_{\mu\nu} = (g^{\mu\nu})^{-1},\ g^{\nu}_{\mu} = \delta^{\nu}_{\mu}. \tag{3}$$

Derivatives are expressed in a number of ways depending on the situation. In general

$$\frac{\partial a^\mu}{\partial x^\nu} = \partial_\nu a^\mu = a^\mu_{,\nu} \tag{4}$$

are all equivalent. The process of contraction, or identification of two indices, in these cases produce divergence, or $\nabla\cdot$ type operations, e.g., $a^\mu_{,\mu} = \nabla \cdot \mathbf{a}$.

THE METRIC TENSOR IN CURVED SPACE

In a flat space, the metric tensor is everywhere constant, i.e., it can be written such that it has no space-dependence; in a curved space, the components of the tensor depend on position. Diagnosing the presence of curvature is important in many applications. The main obstacle to proper diagnosis is the possibility of apparent curvature, produced not by the space itself but by the choice of coordinates.

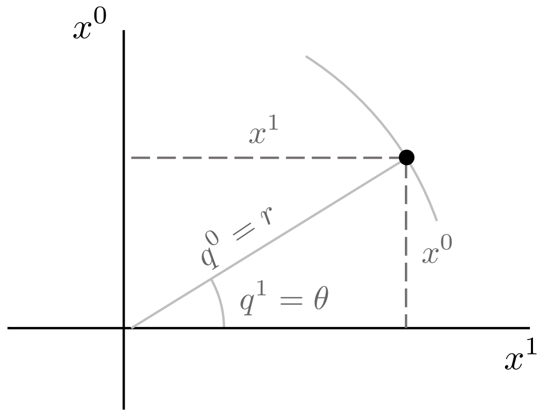

FIG. 1. 2D plane and both Cartesian and polar coordinates. The metric tensor in the polar coordinates is spatially-varying, whereas its counterpart in the Cartesian system is constant.

In Figure 1 is an example system in which we can observe apparent curvature. It comprises a Cartesian system (x^0, x^1) with x^0 representing the vertical axis and x^1 the horizontal axis. Onto it we overlay a polar coordinate system (q^0, q^1) where $q^0 = r$ and $q^1 = \theta$, the angle being measured counterclockwise from the $+x^1$ axis. The x / q coordinates are related by

$$x^0(q^0, q^1) = q^0 \sin q^1, \;\; x^1(q^0, q^1) = q^0 \cos q^1, \tag{5}$$

and

$$q^0(x^0, x^1) = \sqrt{(x^0)^2 + (x^1)^2}, \;\; q^1(x^0, x^1) = \tan^{-1}\left(\frac{x^0}{x^1}\right). \tag{6}$$

A small change in the q coordinates goes over into a small change in the x coordinates via

$$\begin{aligned} dx^0 &= x^0_{,0} dq^0 + x^0_{,1} dq^1 = \sin q^1 dq^0 + q^0 \cos q^1 dq^1 \\ dx^1 &= x^1_{,0} dq^0 + x^1_{,1} dq^1 = \cos q^1 dq^0 - q^0 \sin q^1 dq^1, \end{aligned} \tag{7}$$

or

$$\begin{bmatrix} dx^0 \\ dx^1 \end{bmatrix} = \begin{bmatrix} \sin q^1 & q^0 \cos q^1 \\ \cos q^1 & -q^0 \sin q^1 \end{bmatrix} \begin{bmatrix} dq^0 \\ dq^1 \end{bmatrix}. \tag{8}$$

To keep $ds^2 = dx^\mu dx_\mu$ in the x system invariant under the transformation to polar coordinates q requires that

$$ds^2 = [dq^0 \; dq^1] \begin{bmatrix} \sin q^1 & \cos q^1 \\ q^0 \cos q^1 & -q^0 \sin q^1 \end{bmatrix} \begin{bmatrix} \sin q^1 & q^0 \cos q^1 \\ \cos q^1 & -q^0 \sin q^1 \end{bmatrix} \begin{bmatrix} dq^0 \\ dq^1 \end{bmatrix}, \tag{9}$$

which, upon expanding, gives the 2D polar coordinate metric tensor

$$g_{\mu\nu} = \begin{bmatrix} 1 & 0 \\ 0 & (q^0)^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & r^2 \end{bmatrix}. \tag{10}$$

Unlike in the oblique/rectilinear case, this metric tensor depends on spatial position – specifically, radial distance from the origin. If we were not previously certain that the 2D plane was flat, the spatial dependence in (10) would be suggestive of curvature. In fact it is the existence of the transform in (8), to the Cartesian system, that confirms that the curvature is merely apparent. A flat space is one for which a transformation to a coordinate system in which the metric tensor is constant can be found. In general it is difficult to know if such a transform is available, so other more direct (though more complex) techniques exist.

QUANTIFYING CURVATURE

To discuss curved spaces and flat spaces, it can be helpful to imagine beings who inhabit those spaces, and measure positions and lengths within them. This artifice is particularly useful when describing curved spaces embedded in flat spaces of higher dimension, so we will use language of this kind regularly.

Let us start our analysis by constructing an example of a space that is curved. We start with a 2D space, within which its inhabitants describe their positions with polar coordinates x^μ, $\mu = (0, 1)$, where

$$x^0 = r, \quad x^1 = \theta, \tag{11}$$

as illustrated in Figure 2a. We saw previously that such a coordinate system gives an apparent curvature, but let us this time give it an actual curvature. Consider that this space in actuality forms a curved paraboloid surface embedded in a flat 3D space y, with Cartesian coordinates y^0, y^1, y^2. The 2D space is described in the y system by the equation

$$y^2 = y^2(y^0, y^1) = \frac{1}{2}(y^0 - a)^2 + \frac{1}{2}(y^1 - b)^2, \tag{12}$$

as illustrated in Figure 2b. What does this mean for vectors and tensors defined in x? To answer this, consider first the metric tensor of the 3D non-curved y space. This tensor, which we will call h_{mn}, with m, n ranging over $0, 1, 2$, has components:

$$h_{mn} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & h_{11} & 0 \\ 0 & 0 & h_{22} \end{bmatrix}, \tag{13}$$

where h_{11} and h_{22} have been initially set unequal to 1 to leave open the possibility that the y coordinates have different mutual length scales. In contrast, the metric tensor of the curved x space, say $g_{\mu\nu}$ with μ, ν ranging over 0, 1, will not be constant, but rather it will be seen to be a function of either x^0 or x^1 or both. Question: given h_{mn} on the y space and the equations governing the paraboloid x space, what is $g_{\mu\nu}(x)$?

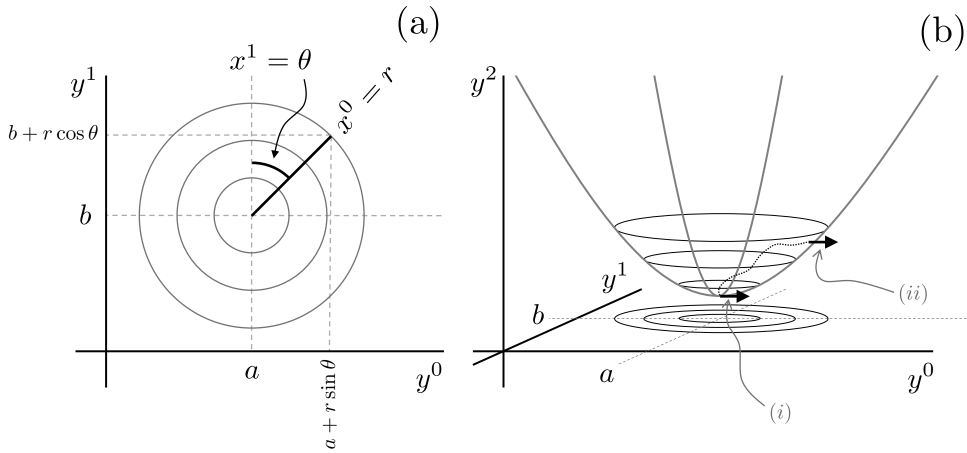

FIG. 2. (a) A 2D space x overlain with a polar coordinate system. (b) The 2D space as a curved surface embedded in a 3D flat space y.

Let us first collect up several derivatives. Mapping from the x system to the y system, making use of Figure 2a, we have

$$y^0 = a + x^0 \sin x^1,\ y^1 = b + x^0 \cos x^1,\ \text{and}\ y^2 = \frac{1}{2}(x^0)^2, \tag{14}$$

the last equation being obtained by substituting the previous two into (51) and simplifying. From these relations we obtain

$$y^0_{,0} = \sin x^1,\ y^0_{,1} = x^0 \cos x^1,\ y^1_{,0} = \cos x^1,\ y^1_{,1} = -x^0 \sin x^1,\ y^2_{,0} = x^0,\ y^2_{,1} = 0. \tag{15}$$

Next we determine the metric tensor relations by considering a small displacement dy^n in the 3D space. If this displacement is constrained to lie in the embedded paraboloid, then it is completely specified by a pair of x coordinates and we may write it as

$$dy^n = \left(\frac{\partial y^n}{\partial x^0}\right) dx^0 + \left(\frac{\partial y^n}{\partial x^1}\right) dx^1 = y^n_{,\mu} dx^\mu. \tag{16}$$

The squared length of this displacement is then

$$ds^2 = dy^n dy_n = h_{mn} dy^m dy^n = h_{mn}\left(y^m_{,\mu} dx^\mu\right)\left(y^n_{,\nu} dx^\nu\right), \tag{17}$$

where in the last step we used equation (16). Since the $g_{\mu\nu}$ we are seeking satisfies

$$ds^2 = dx^\mu dx_\mu = g_{\mu\nu} dx^\mu dx^\nu, \tag{18}$$

by comparison with (17) it must be

$$g_{\mu\nu} = h_{mn} y^m_{,\mu} y^n_{,\nu} = y^n_{,\nu} y_{n,\mu} = y^n_{,\mu} y_{n,\nu}. \tag{19}$$

This can be written in algebraic notation as the product of three matrices. Using equations (13) and (15), we have

$$g_{\mu\nu} = \begin{bmatrix} \sin x^1 & \cos x^1 & x^0 \\ x^0 \cos x^1 & -x^0 \sin x^1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & h_{11} & 0 \\ 0 & 0 & h_{22} \end{bmatrix} \begin{bmatrix} \sin x^1 & x^0 \cos x^1 \\ \cos x^1 & -x^0 \sin x^1 \\ x^0 & 0 \end{bmatrix}, \tag{20}$$

or, evaluating the products,

$$\begin{aligned} g_{\mu\nu}(x^0, x^1) &= \begin{bmatrix} \sin^2 x^1 + h_{11} \cos^2 x^1 + h_{22} (x^0)^2 & x^0 \cos x^1 \sin x^1 (1 - h_{11}) \\ x^0 \cos x^1 \sin x^1 (1 - h_{11}) & (x^0)^2 \big(\cos^2 x^1 + h_{11} \sin^2 x^1 \big) \end{bmatrix} \\ &= \begin{bmatrix} \sin^2 \theta + h_{11} \cos^2 \theta + h_{22} r^2 & r \cos\theta \sin\theta (1 - h_{11}) \\ r \cos\theta \sin\theta (1 - h_{11}) & r^2 \big(\cos^2 \theta + h_{11} \sin^2 \theta \big) \end{bmatrix}. \end{aligned} \tag{21}$$

Notice if the y system was chosen to be orthonormal/Cartesian, then $h_{11} = h_{22} = 1$, causing both the off-diagonal terms and the θ-dependence to vanish, such that the curvature becomes a function only of radial distance from (a, b):

$$g_{\mu\nu}(x^0) = \begin{bmatrix} 1 + (x^0)^2 & 0 \\ 0 & (x^0)^2 \end{bmatrix} = \begin{bmatrix} 1 + r^2 & 0 \\ 0 & r^2 \end{bmatrix}. \tag{22}$$

Indices of vectors in x can be lowered with this matrix, or raised with its inverse:

$$g^{\mu\nu}(x^0) = \begin{bmatrix} \big(1 + (x^0)^2\big)^{-1} & 0 \\ 0 & (x^0)^{-2} \end{bmatrix} = \begin{bmatrix} \big(1 + r^2\big)^{-1} & 0 \\ 0 & r^{-2} \end{bmatrix}. \tag{23}$$

So, two different 2D spaces produce two different metric tensors

$$g^{(1)}_{\mu\nu} = \begin{bmatrix} 1 + r^2 & 0 \\ 0 & r^2 \end{bmatrix}, \quad g^{(2)}_{\mu\nu} = \begin{bmatrix} 1 & 0 \\ 0 & r^2 \end{bmatrix}, \tag{24}$$

with the first corresponding to an intrinsically curved space and the other to a flat space with coordinates that produce apparent curvature.

PARALLEL DISPLACEMENT

The general procedure for characterizing curvature, and for distinguishing between intrinsically flat and curved spaces is based on parallel displacement. To help visualize this process, let us carry out a simple operation on the small vector in the x space at the position marked (i) in Figure 2b*. To observers in the 2D curved space x, this vector exists and can be analyzed. Like all vectors in x, this vector is also perfectly well defined in the 3D y

*The fact that this vector lies in the x space is reflected in the diagram by the fact that it appears as a tangent to the paraboloid surface. Any vector with a component perpendicular to the paraboloid lies in y, but not in x.

system. To observers in y, whose metric tensor h_{nm} has constant components, this vector also exists and can be analyzed. Such observers might notice that the vector is special in that it is constrained to lie in the paraboloid, but otherwise they would see nothing remarkable about it. Now, suppose the y observers undertook to shift the vector along a path in the paraboloid, starting at (i) and ending at the position (ii), keeping its components (as they see them) constant. Afterwards, comparing the initial and final vectors, the y observers would describe them as trivially parallel, since the components were constrained never to change during the move. This is parallel displacement — quite a trivial operation in a space with a constant metric (like y). To observers in x, however, the same process is not only nontrivial, it does not even make sense. The vector at the end of the shift (see Figure 2b) cannot be compared to the initial vector, since the final vector does not lie in x, and hence from the x-observer point of view, it does not exist, and cannot be analyzed. So, what was a trivial parallel displacement for y-observers does not exist for x-observers. Question: is there any comparable process available to x beings which at least approximates parallel displacement?

The answer is yes — but the process requires some subtle re-definition. Let us describe it in words first:

1. We define a differential step in x, dx^μ, away from (i);
2. We move the vector through this step, keeping its y components constant;
3. The result no longer lies in x, but we throw away the component orthogonal to x, producing an approximation to the original vector.

Any desired path within x, such as the one from (i) to (ii) in Figure 2b, can then be built up as an integral over these steps. This is parallel displacement proper. The components of the vector at (ii) will not be the same as those at (i), in fact they will depend on the path from (i) to (ii). If the process follows a closed path the vector upon its return will tend to be different from the one that left. An x-observer would in fact be hard-pressed to argue that the two were parallel. Nonetheless, all of the differences between the vectors at the start and end of the process are the result of the curvature of the space through which the path passes, and not of any fundamental change to the vector itself.

Mathematically, the process unfolds as follows. Consider an arbitrary vector $\mathbf{q}$ which lies in x (the vector at (i) is an example). Like all vectors in x it also lies in y. It therefore has well-defined components in the coordinate system (x^0, x^1), which we will denote $q^\mu(x)$, or explicitly $q^0(x)$ and $q^1(x)$, and also in the coordinates (y^0, y^1, y^2), which we will denote $q^n(y)$, or explicitly $q^0(y)$, $q^1(y)$, and $q^2(y)$.

Using the rule in equation (16), which holds for the contravariant components of vectors which lie in both spaces,

$$q^n(y) = q^n\big(y(x)\big) = y^n_{,\mu}(x)q^\mu(x). \tag{25}$$

The vector on the right hand side of this equation, with contravariant components $q^\mu(x)$, and consequently with covariant components $q_\mu(x) = g_{\mu\nu}(x)q^\mu(x)$, is our starting point.

Each $y^n_{,\mu}(x)$ in equation (25) can be interpreted as a vector in y which gives the contribution to q^n of the μ^{th} direction in x.† This allows us to give a definition for a vector $\mathbf{r}$ in y which is orthogonal to x: such a vector must be perpendicular to every direction in x, hence every $y^n_{,\mu}(x)$, and so it must satisfy

$$r^n\big(y(x)\big)y_{n,\mu}(x) = 0. \tag{26}$$

Next, consider a displacement of our vector through an infinitesimal distance in x, represented abstractly by dx. After the displacement, its x components will go over from $q^\mu(x)$ to $q^\mu(x+dx)$, and the y components will go over from $q^n(y)$ to $q^n\big(y(x+dx)\big)$. While doing so, we hold all of its y components fixed, forming $q^n\big(y(x+dx)\big)$. The components labelled n do not change, so this is equal to $q^n\big(y(x)\big)$, and in fact when we run into this vector later on we will find we can simply write it as q^n. However, although its y components do not change, the vector now no longer lies in x, and so can no longer be the left-hand side of an equation like (25). The best we can do is resolve it into two vectors, one which does lie in x, and one which is orthogonal to every vector in x:

$$q^n\big(y(x+dx)\big) = p^n\big(y(x+dx)\big) + r^n\big(y(x+dx)\big). \tag{27}$$

The p vector is in x and so it *can* be written like the one in equation (25),

$$p^n\big(y(x+dx)\big) = y^n_{,\nu}(x+dx)p^\nu(x+dx), \tag{28}$$

and the orthogonal vector satisfies equation (26):

$$r^n\big(y(x+dx)\big)y_{n,\mu}(x+dx) = 0. \tag{29}$$

So, if we take the product of equation (27) with $y_{n,\mu}(x+dx)$, the orthogonal vector drops out, and we obtain

$$\begin{aligned} q^n\big(y(x+dx)\big)y_{n,\mu}(x+dx) &= p^n\big(y(x+dx)\big)y_{n,\mu}(x+dx) \\ &= p^\nu(x+dx)y^n_{,\nu}(x+dx)y_{n,\mu}(x+dx), \end{aligned} \tag{30}$$

or, using equation (19),

$$q^n\big(y(x+dx)\big)y_{n,\mu}(x+dx) = p^\nu(x+dx)g_{\mu\nu}(x+dx). \tag{31}$$

But the metric tensor simply lowers indices, and so we in fact wind up with:

$$p_\mu(x+dx) = q^n\big(y(x+dx)\big)y_{n,\mu}(x+dx) = q^n y_{n,\mu}(x+dx), \tag{32}$$

where in the second line we recognize as mentioned earlier that the components q^n do not depend on x or y.

From our discussions, we have come to expect that the motion of the vector through dx will change its x components, even if its y components are held fixed. So, we expect to be able

†This contributing vector has a free index n which can be lowered using $y_{n,\mu}(x) = h_{nm}y^m_{,\mu}(x)$.

to associate with dx a characteristic $dq_\mu = q_\mu(x + dx) - q_\mu(x)$. But, the vector after the motion through dx does not exist in x, and so we do not have the first of the terms needed to form this difference. The $p_\mu(x + dx)$ in equation (32) is the closest thing to $q^\mu(x + dx)$ have, so we will make do with this, and define:

$$dq_\mu = p_\mu(x + dx) - q_\mu = q^n y_{n,\mu}(x + dx) - q_\mu. \tag{33}$$

But, applying a Taylor's series expansion,

$$y_{n,\mu}(x + dx) = y_{n,\mu}(x) + y_{n,\mu\sigma} dx^\sigma + ..., \tag{34}$$

so to leading order in dx we have

$$dq_\mu = q^n \big(y_{n,\mu}(x) + y_{n,\mu\sigma} dx^\sigma\big) - q_\mu(x) = q^n y_{n,\mu\sigma} dx^\sigma. \tag{35}$$

This equality requires several lines of mathematics to produce; for details see Appendix A. Equation (35) gives us a viable differential change in the x components of the vector after the shift.

Our last two steps are aimed at removing all reference to the larger y space, so that we end up characterizing the properties of parallel displacement solely in terms of the curved x space. First, we replace q^n with $q^\nu y^n_{,\nu}(x)$:

$$dq_\mu(x) = q^\nu(x) y^n_{,\nu}(x) y_{n,\mu\sigma}(x) dx^\sigma. \tag{36}$$

Now, the only reference to n is in the coordinate transform quantities. Next, we analyze these. First recall from equation (19) that $g_{\mu\nu} = y^n_{,\mu} y_{n,\nu}$. Applying to this the chain rule, we obtain

$$g_{\mu\nu,\sigma}(x) = y^n_{,\nu\sigma} y_{n,\mu} + y^n_{,\nu} y_{n,\mu\sigma}, \tag{37}$$

which means, by interchanging indices, we also have

$$g_{\sigma\nu,\mu}(x) = y^n_{,\nu\mu} y_{n,\sigma} + y^n_{,\nu} y_{n,\sigma\mu} \;\text{ and }\; g_{\mu\sigma,\nu}(x) = y^n_{,\sigma\nu} y_{n,\mu} + y^n_{,\sigma} y_{n,\mu\nu}. \tag{38}$$

Now form

$$\Gamma_{\mu\nu\sigma} = \frac{1}{2}\Bigg(g_{\mu\nu,\sigma}(x) + g_{\mu\sigma,\nu}(x) - g_{\nu\sigma,\mu}(x)\Bigg), \tag{39}$$

which is found (see Appendix B) to equal

$$\Gamma_{\mu\nu\sigma}(x) = y^n_{,\nu}(x) y_{n,\mu\sigma}(x). \tag{40}$$

So, we obtain

$$dq_\mu(x) = q^\nu(x)\Gamma_{\mu\nu\sigma}(x) dx^\sigma, \tag{41}$$

as a definition of covariant parallel displacement of a contravariant vector without reference to the y space. The quantity $\Gamma_{\mu\nu\sigma}$ is called a Christoffel symbol of the first kind. It happens not to be a tensor, but nonetheless it can legitimately have its indices raised and lowered. Specifically, it is possible to construct from it a Christoffel symbol of the second kind:

$$\Gamma^\mu_{\nu\sigma}(x) = g^{\mu\lambda}(x)\Gamma_{\lambda\nu\sigma}(x), \tag{42}$$

and with this define contravariant parallel displacement of a contravariant vector as

$$dq^\mu(x) = -q^\nu(x)\Gamma^\mu_{\nu\sigma}(x) dx^\sigma. \tag{43}$$

THE CURVATURE AND RICCI TENSORS AND THE SCALAR CURVATURE

Parallel displacement teaches us that there is no absolute way of saying or implementing a parallel shift of vectors in curved space. We can do it approximately, with error at second order in the step we take when moving the vector, and so if we define a path made up of differential elements of shift, and move the vector along that path, it has in a sense been displaced parallel to itself. However, what we give up for this is that the vector at the end of the path depends on the path itself. So the situation in curved space is quite different from flat space.

This is also the key however to unambiguously determining if the space you currently inhabit is curved. If it is, then paralllel displacement along two different paths to the same point will produce different answers. This is true even for a differential path, meaning, if we take two spatial derivatives of the components of a vector, and then we do it again *in a different order*, in a curved space we will get a different answer.

The curvature tensor is an object that measures these differences. We won't go through all of the calculus, but perhaps not surprisingly given the formulas for parallel displacement and derivative-taking, it is dominated by the Christoffel symbol:

$$R^{\beta}_{\nu\rho\sigma} = \Gamma^{\beta}_{\nu\sigma,\rho} - \Gamma^{\beta}_{\nu\rho,\sigma} + \Gamma^{\alpha}_{\nu\sigma}\Gamma^{\beta}_{\alpha\rho} - \Gamma^{\alpha}_{\nu\rho}\Gamma^{\beta}_{\alpha\sigma} \tag{44}$$

The Ricci tensor is generated by contracting the curvature tensor by identifying β and σ:

$$R_{\nu\rho} = R^{\beta}_{\nu\rho\beta}, \tag{45}$$

and the scalar curvature is generated by raising one of these indices $R^{\nu}_{\rho} = g^{\nu\mu}R_{\mu\rho}$ and contracting over these:

$$R = R^{\nu}_{\nu}. \tag{46}$$

GEODESICS

We can choose an arbitrary vector, and an arbitrary path, and carry out parallel displacement of the one along the other. However, if we begin the process of parallel displacement with an element of path that is fixed to be in the same direction as the vector, the rest of the path, built up step by step, is uniquely determined. This path is called a geodesic. Again, skipping some of the calculus, we find that the Christoffel symbol is a key piece of the computation of a geodesic.

A curve is a continuous range of positions, and so it appears in the theory as a position vector that is a function of a single continuously-changing scalar parameter τ, i.e., $p^{\mu}(\tau)$. The geodesic satisfies

$$\frac{d^2p^{\mu}}{d\tau^2} + \Gamma^{\mu}_{\nu\sigma}\frac{dp^{\nu}}{d\tau}\frac{dp^{\sigma}}{d\tau} = 0. \tag{47}$$

CONCLUSIONS

We are curious to examine whether any useful possibilities for seismic inversion are introduced by taking a different view of actions taken on a point in model space when in the presence of a well-behaved objective function. This view involves considering the objective function to introduce intrinsic curvature to model space, rather than treating it as a completely independent quantity. To examine this requires we make use of some aspects of Riemannian geometry, thus we have been motivated to do this review. This review is probably best read in combination with the review of vectors and tensors in non-Cartesian space in the 2020 report.

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APPENDIX A

Starting from

$$dq_\mu = q^n\big(y_{n,\mu}(x) + y_{n,\mu\sigma}dx^\sigma\big) - q_\mu(x),$$

we find

$$\begin{aligned} dq_\mu &= q^n\big(y_{n,\mu}(x) + y_{n,\mu\sigma}dx^\sigma\big) - q_\mu(x) \\ &= q^n y_{n,\mu}(x) + q^n y_{n,\mu\sigma}dx^\sigma - q_\mu(x) \\ &= q^\nu(x) y^n_\nu(x) y_{n,\mu}(x) + q^n y_{n,\mu\sigma}dx^\sigma - q_\mu(x) \\ &= q^\nu(x) g_{\mu\nu}(x) + q^n y_{n,\mu\sigma}dx^\sigma - q_\mu(x) \\ &= q_\mu(x) + q^n y_{n,\mu\sigma}dx^\sigma - q_\mu(x) \\ &= q^n y_{n,\mu\sigma}dx^\sigma. \end{aligned} \tag{48}$$

APPENDIX B

We have

$$g_{\mu\nu,\sigma} = y^n_{,\nu\sigma} y_{n,\mu} + y^n_{,\nu} y_{n,\mu\sigma}, \tag{49}$$

and a definition of inner products such that if desired the first term can be re-written $y_{n,\nu\sigma} y^n_{,\mu}$, and the second term can if desired be re-written $y_{n,\nu} y^n_{,\mu\sigma}$. Thus the assemblage $g_{\mu\nu,\sigma} + g_{\mu\sigma,\nu} - g_{\sigma\nu,\mu}$ becomes

$$y^n_{,\nu\sigma} y_{n,\mu} + y^n_{,\nu} y_{n,\mu\sigma} + y^n_{,\mu\nu} y_{n,\sigma} + y^n_{,\mu} y_{n,\sigma\nu} - y^n_{,\nu\sigma} y_{n,\mu} - y^n_{,\sigma} y_{n,\nu\mu}. \tag{50}$$

We can re-write the last term as $y_{n,\sigma} y^n_{,\nu\mu}$, and further recognize that the order of any pair of indices before or after a comma does not matter. We find then that (1) the first and fifth terms cancel, (2) the third and sixth terms cancel, (3) the second and fourth terms are equal, and (4), in order to match to standard terminology, we can re-write $g_{\sigma\nu,\mu}$ as $g_{\nu\sigma,\mu}$. Thus

$$g_{\mu\nu,\sigma} + g_{\mu\sigma,\nu} - g_{\nu\sigma,\mu} = 2\, y^n_{,\mu} y_{n,\sigma\nu} = 2\Gamma_{\mu\nu\sigma}, \tag{51}$$

as desired.

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