An analytic approach to minimum phase signals

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- Eg: in deconvolution, where the reflectivity and the wavelet are separated from the recorded seismic data.

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- FIR, IIR filter converted to min phase by reflecting poles, zeros across unit circle.
- or via Hilbert transform on log amplitude spectrum.

Min phase vs zero phase

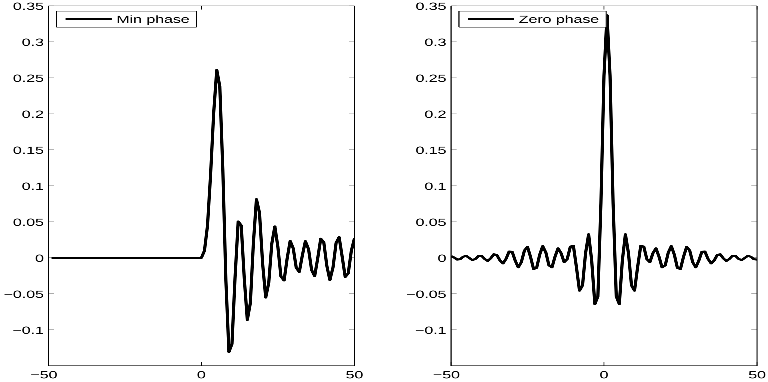

Figure: A minimum phase IIR filter response and zero phase equivalent.

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- Impossible to have a causal stable signal with causal stable inverse on $\mathbb{R}$.
- Hilbert transform not defined on arbitrary log spectrum.

Why no causal stable signal with causal stable inverse?

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- For instance, locally integrable ("stable") implies the convolution is locally integrable.

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  $$\log(\text{abs}(\text{Fourier transform}) + \epsilon)$$
- Does that work?

Analytic approach

- Given a causal, stable signal $\mathbf{f} = (f_0, f_1, f_2, \ldots .)$, define

$$F(z) = \sum_{n=0}^{\infty} f_n z^n, \qquad \text{for complex numbers } z \text{ with } |z| < 1.$$

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- $F(e^{i\omega})$ is just the usual Fourier transform. $F(z)$ is an extension of the spectrum to the disk.
- $F(z)$ is a power series, differentiable everywhere on the unit disk. An analytic function. A function in Hardy space $H^1(\mathbb{D})$.

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- An analytic function can't be zero on an interval (or curve) in the disk, unless it is zero everywhere.
- Similarly, a causal signal can't have an interval of zeros in its spectrum.
- There are no band limited, causal signals.
- There are no band limited, minimum phase signals.

No min phase, band limited spike

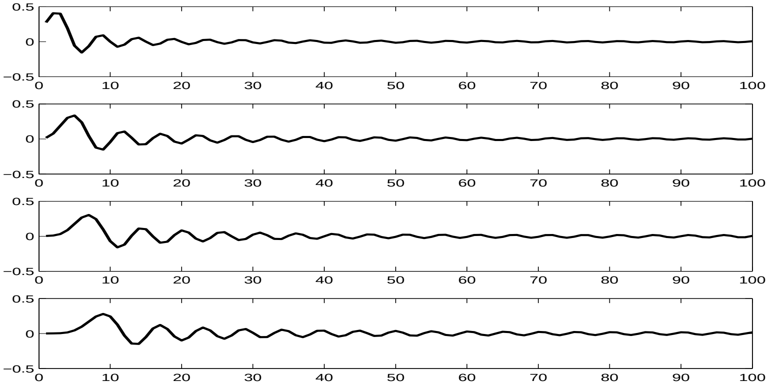

Figure: Trying to compute a min phase signal that does not exist. Stability factor ϵ goes to zero....

Existence conditions

Theorem: An amplitude spectrum $|F(e^{i\omega})|$ is the spectrum of a causal signal if and only if

- $\int |F(e^{i\omega})|\, d\omega < \infty$, and
- $\int \log |F(e^{i\omega})|\, d\omega$ is finite.

Thus, the spectrum can not have an interval of zeros, since $\log 0 = -\infty$.

Theorem: Any causal signal has a minimum phase equivalent.

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- Outer functions are like minimum phase filters.
- Inner functions are like all pass filters.
- The inner, outer definitions (complicated) apply to general signals, not just filters.

A better definition for "min phase" signals.

- **Definition:** A causal signal $\mathbf{f} = (f_0, f_1, f_2, \ldots)$ is *front-loaded* if its partial energies are maximized, relative to any other causal signal with the sample amplitude spectrum. That is,

$$\sum_{n=0}^{N} |g_n|^2 \leq \sum_{n=0}^{N} |f_n|^2, \qquad \text{for each } N = 0, 1, 2, \ldots.$$

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- **Theorem:** A discrete signal $\mathbf{f} = (f_0, f_1, f_2, \ldots)$ is front-loaded if and only if $F(z)$ is an outer function.

Compare with old definition

Old definition works:

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- A function $F(z)$ is *inner* if $|F| \equiv 1$ on the unit circle.
- Every function can be written as outer times inner.

Non singular formula for discrete, min phase signal.

From Hardy theory, we get alternate formulas for min phase signals. Signal coefficients from the outer function:

$$f_n = \frac{1}{r^n} \int_0^1 F(re^{2\pi i\phi}) e^{-2\pi i n\phi}\, d\phi, \qquad \text{any } r < 1\ .$$

Signal coefficients from the amplitude spectrum:

$$f_n = \frac{1}{r^n} \int_0^1 \exp\left(\int_0^1 \frac{e^{2\pi i\theta} + re^{2\pi i\phi}}{e^{2\pi i\theta} - re^{2\pi i\phi}} \log|F(e^{2\pi i\theta})|\, d\theta \right) e^{-2\pi i n\theta}\, d\phi.$$

Signals on the real line.

- Similarly, we have

Definition: A causal signal $\mathbf{f} : \mathbb{R}^+ \to \mathbb{R}$ is *front-loaded* if its partial energies are maximized, relative to any other causal signal with the sample amplitude spectrum. That is,

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- **Theorem:** A causal signal $f : \mathbb{R}^+ \to \mathbb{R}$ with spectrum $F(z)$ an outer function, is front-loaded.
- **Conjecture:** This is if and only if.

Computing approximately band-limited min phase signals.

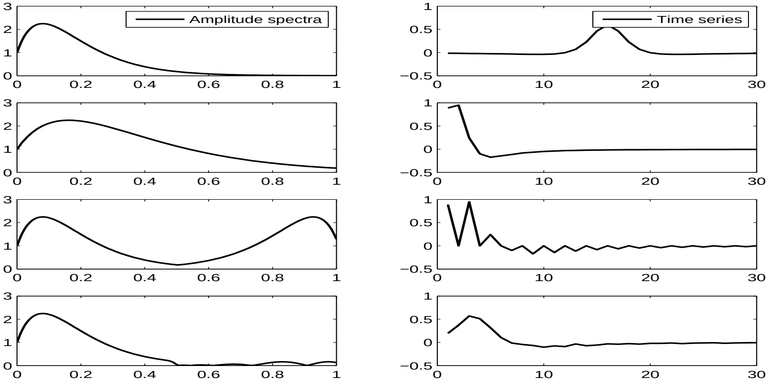

Figure: Step by step construction, using spectrum wrapping.

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- Front-loaded, energy concentration a useful alternative definition.
- Front-load signals equivalent to outer functions in Hardy space.
- No band-limited causal signals, no band-limited min phase signals.
- Hardy space theory gives useful formulations for computing min phase signals.

References

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