Split-step phase-shift time stepping methods for wavefield propagation

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Overview

- Finite differences & the acoustic wave equation
- Pseudospectral methods
- Higher-order pseudospectral methods from PSTS
- The phase-shift time stepping (PSTS) Equation
- Split-step PSTS
- One-way in time split-step PSTS
- Examples

RTM Basics

- Forward model a shot field
  - two-way wave equation
    - Finite Difference
    - Pseudo Spectral Methods
    - <u>Phase-Shift Time Stepping</u>
- Back propagate a receiver field
  - two-way wave equation
- Apply an imaging Condition
  - Zero lag cross correlation

* Filter low frequency Artifacts

Finite Difference Advantages

- Accurate boundary conditions can be implemented
- Domain decomposition is flexible
- Easily adapted to more general cases
- Can use the exact velocity model

Finite difference disadvantages

- Finite difference operators are dispersive and must oversampled compared to Fourier methods.

Pseudospectral

$$\Delta = \frac{\partial^2}{\partial x^2} + \ldots + \frac{\partial^2}{\partial z^2}$$

$$\frac{\partial^2 U}{\partial t^2}(t,\vec{x}) = c^2 \Delta U(t,\vec{x})$$

- Calculated Laplacian in Fourier Domain

$$\frac{\partial^2 U}{\partial t^2}(t,\vec{x}) = -c^2 \mathrm{FT}_{\vec{k}}^{-1}\left\{ |\vec{k}|^2 \ \mathrm{FT}_{\vec{x}}\{U(t,\vec{x})\}\right\}$$

- Centered finite difference

$$U(\Delta t + t,\vec{x}) = -U(-\Delta t + t,\vec{x}) + 2U(t,\vec{x})$$
$$-(c\Delta t)^2 \mathrm{FT}_{\vec{k}}^{-1}\left\{ |\vec{k}|^2 \ \mathrm{FT}_{\vec{x}}\{U(t,\vec{x})\}\right\}$$

Higher-order pseudospectral

- Taylor series

$$U(\delta t+t,x)=U(t,x)+\frac{\partial U}{\partial t}(t,x)\delta t+\frac{1}{2}\frac{\partial^2 U}{\partial t^2}(t,x)\delta t^2+.$$

$$U(-\delta t+t,x)=U(t,x)-\frac{\partial U}{\partial t}(t,x)\delta t+\frac{1}{2}\frac{\partial^2 U}{\partial t^2}(t,x)\delta t^2+.$$

- Adding up

$$U(\delta t+t,x)+U(\delta t-t,x)=2U(t,x)+\frac{\partial^2 U}{\partial t^2}(t,x)\delta t^2+\frac{2}{4!}\frac{\partial^4 U}{\partial t^4}(t,x)\delta t^4+.$$

Higher-Order Pseudospectral

$$\frac{\partial^4 U}{\partial t^4}(t,x) = \frac{\partial^2}{\partial t^2}\frac{\partial^2 U}{\partial t^2} = \frac{\partial^2}{\partial t^2}\left(c^2 \Delta U\right) = c^2 \Delta \frac{\partial^2}{\partial t^2} U$$

$$= c^2 \Delta\left(c^2 \Delta U\right)$$

$$= c^4 \Delta^2 U + c^2 \Delta(c^2)\Delta U + c^2 \nabla(c^2) \bullet \nabla(\Delta U)$$

$$\approx c^4 \Delta^2 U$$

- Insert time derivative approximation into wave equation

$$U(\delta t + t) = -U(-\delta t + t) + 2U(t) - (c\delta t)^2 \mathrm{FT}_{\vec{k}}^{-1}\left\{|\vec{k}|^2 \ \mathrm{FT}_{\vec{x}}\{U(t)\}\right\} + \frac{2}{4!}\frac{\partial^4 U}{\partial t^4}(t)\delta t^4$$

$$U(\delta t + t) = -U(-\delta t + t) + 2U(t) - (c\delta t)^2 \mathrm{FT}_{\vec{k}}^{-1}\left\{|\vec{k}|^2 \ \mathrm{FT}_{\vec{x}}\{U(t)\}\right\}$$

$$+\frac{2}{4!}(c\delta t)^4 \mathrm{FT}_{\vec{k}}^{-1}\left\{|\vec{k}|^4 \ \mathrm{FT}_{\vec{x}}\{U(0,\vec{x})\}\right\}$$

Modified eq approach

Cohen(2002)

Pseudospectral

- Very fast
- Memory efficient
- Free surface boundary conditions difficult to implement
- Parallel processing tricky
- Difficult to adapt to more general case

Exact solution of constant velocity wave equation

$$U(\Delta t + t, \vec{x}) = -U(-\Delta t + t, \vec{x})$$
$$+2\mathrm{FT}_{\vec{k}}^{-1}\left\{\cos(2\pi c\,|\,\vec{k}\,|\,\Delta t)\mathrm{FT}_{\vec{x}}\{U(t, \vec{x})\}\right\}$$

Constant velocity to variable velocity

$$U(\Delta t, \vec{x}) = -U(-\Delta t, \vec{x}) +$$
$$2\mathrm{FT}_{\vec{k}}^{-1}\left\{\cos(2\pi c(\vec{x})\,|\,\vec{k}\,|\,\Delta t)\mathrm{FT}_{\vec{x}}\{U(0, \vec{x})\}\right\}$$

Variant on peusdospectral method

- Take the higher order method pseudo spectral method.
- Optimize the coefficients of the Taylor series expansion for the ranges of values v(x) (Soubaras and Zhang,2008).

Another variant on pseudospectral

$$U(\Delta t + t, \vec{x}) = -U(-\Delta t + t, \vec{x})$$
$$+2w_{\min}(\mathrm{c}(\vec{x}))\mathrm{FT}_{\vec{k}}^{-1}\left\{\cos(2\pi c_{\min}|\vec{k}|\Delta t)\mathrm{FT}_{\vec{x}}\{U(t,\vec{x})\}\right\}$$
$$+2w_{\max}(\mathrm{c}(\vec{x}))\mathrm{FT}_{\vec{k}}^{-1}\left\{\cos(2\pi c_{\max}|\vec{k}|\Delta t)\mathrm{FT}_{\vec{x}}\{U(t,\vec{x})\}\right\}$$

$$w_{\min}(\mathrm{c}(\vec{x})) = \frac{c_{\max}^2 - \mathrm{c}(\vec{x})^2}{c_{\max}^2 - c_{\min}^2}, w_{\max}(\mathrm{c}(\vec{x})) = \frac{\mathrm{c}(\vec{x})^2 - c_{\min}^2}{c_{\max}^2 - c_{\min}^2}$$

- Calculate two reference velocities and interpolate (Etgen, 2009)
- Power series expansion of cosines reveals that correct to 2^{nd} order pseudospectral

PSTS and Pseudospectral

$$\cos(2\pi c\,|\vec{k}|\,\Delta t) \approx 1 - \frac{(2\pi c\,|\vec{k}|\,\Delta t)^2}{2}$$

$$U(\Delta t, \vec{x}) =$$

$$-U(-\Delta t, \vec{x}) + 2\mathrm{FT}_{\vec{k}}^{-1}\left\{\left(1 - \frac{(2\pi c\,|\vec{k}|\,\Delta t)^2}{2}\right)\mathrm{FT}_{\vec{x}}\{U(0,\vec{x})\}\right\}$$

Pseudospectral:

$$U(\Delta t, \vec{x}) = -U(-\Delta t, \vec{x}) + 2U(0, \vec{x})$$

$$-(2\pi c\Delta t)^2\,\mathrm{FT}_{\vec{k}}^{-1}\left\{|\vec{k}|^2\,\mathrm{FT}_{\vec{x}}\{U(0,\vec{x})\}\right\}$$

One-way in time propagation

- Approximate the wave equation by the one-way in time wave equation

$$\frac{\partial U}{\partial t}(t,\vec{x}) = \pm c\sqrt{-\Delta}U(t,\vec{x})$$

- Or with the Fourier transform

$$\frac{\partial \hat{U}}{\partial t}(t,\vec{k}) = \pm 2\pi ci\,|\,\vec{k}\,|\,\hat{U}(t,\vec{k})\}$$

- Which has the solution

$$U(t+\delta t,\vec{x}) = \mathrm{FT}_{\vec{k}}^{-1}\left\{e^{\pm 2\pi ic\delta t|\vec{k}|}\mathrm{FT}_{\vec{x}}\{U(t,\vec{x})\}\right\}$$

-Zhang & Zhang (2009)

- One-way depth step Extrapolation:

$$U(z+\Delta z,k_x,\omega)=e^{i\Delta z\sqrt{\frac{\omega^2}{v^2}-k_x^2}}U(z,k_x,\omega)$$

- Two-way phase-shift timestepping extrapolator:

$$U(t+\Delta t,k_x,k_z)$$
$$=\left(e^{+i\Delta tv\sqrt{k_x^2+k_z^2}}+e^{-i\Delta tv\sqrt{k_x^2+k_z^2}}\right)U(t,k_x,k_z)-U(t-\Delta t,k_x,k_z)$$

- One-way phase-shift timestepping extrapolator:

$$U(t+\Delta t,k_x,k_z)=\left(e^{\pm i\Delta tv\sqrt{k_x^2+k_z^2}}\right)U(t,k_x,k_z)$$

Variable Velocity Model

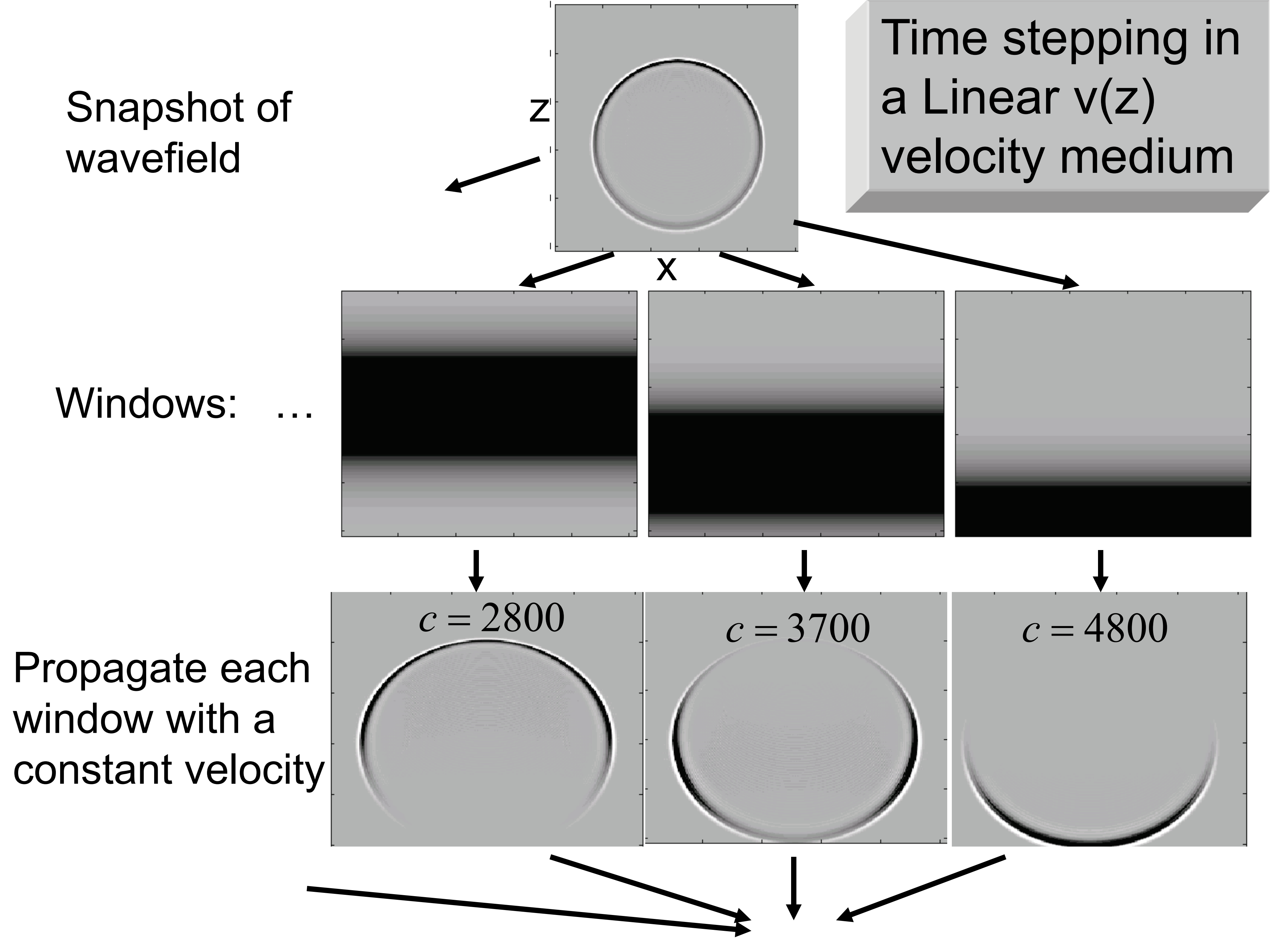
- Set of Windowing functions which sum to 1

$$\sum_i \Omega_i(\vec{x}) = 1$$

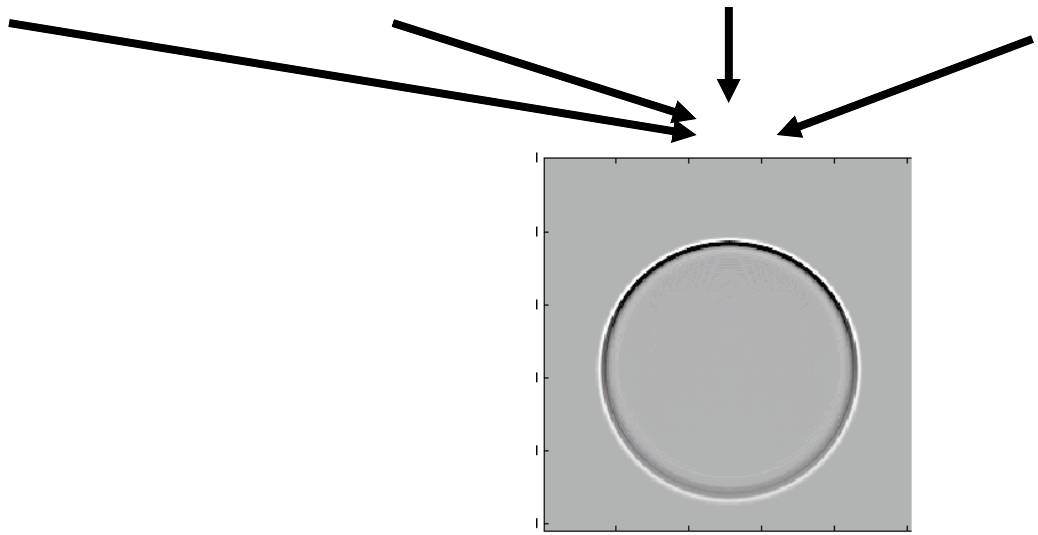
- Approximate the velocity model

$$\sum_i c_i \Omega_i(\vec{x}) \approx c(\vec{x})$$

$$U(\Delta t, \vec{x}) = -U(-\Delta t, \vec{x})$$
$$+\mathrm{FT}_{\vec{k}}^{-1}\{\sum_i 2\cos(2\pi v_i \,|\vec{k}|\,\Delta t)\mathrm{FT}_{\vec{x}}\{\Omega_i(\vec{x})U(0,\vec{x})\}\}$$

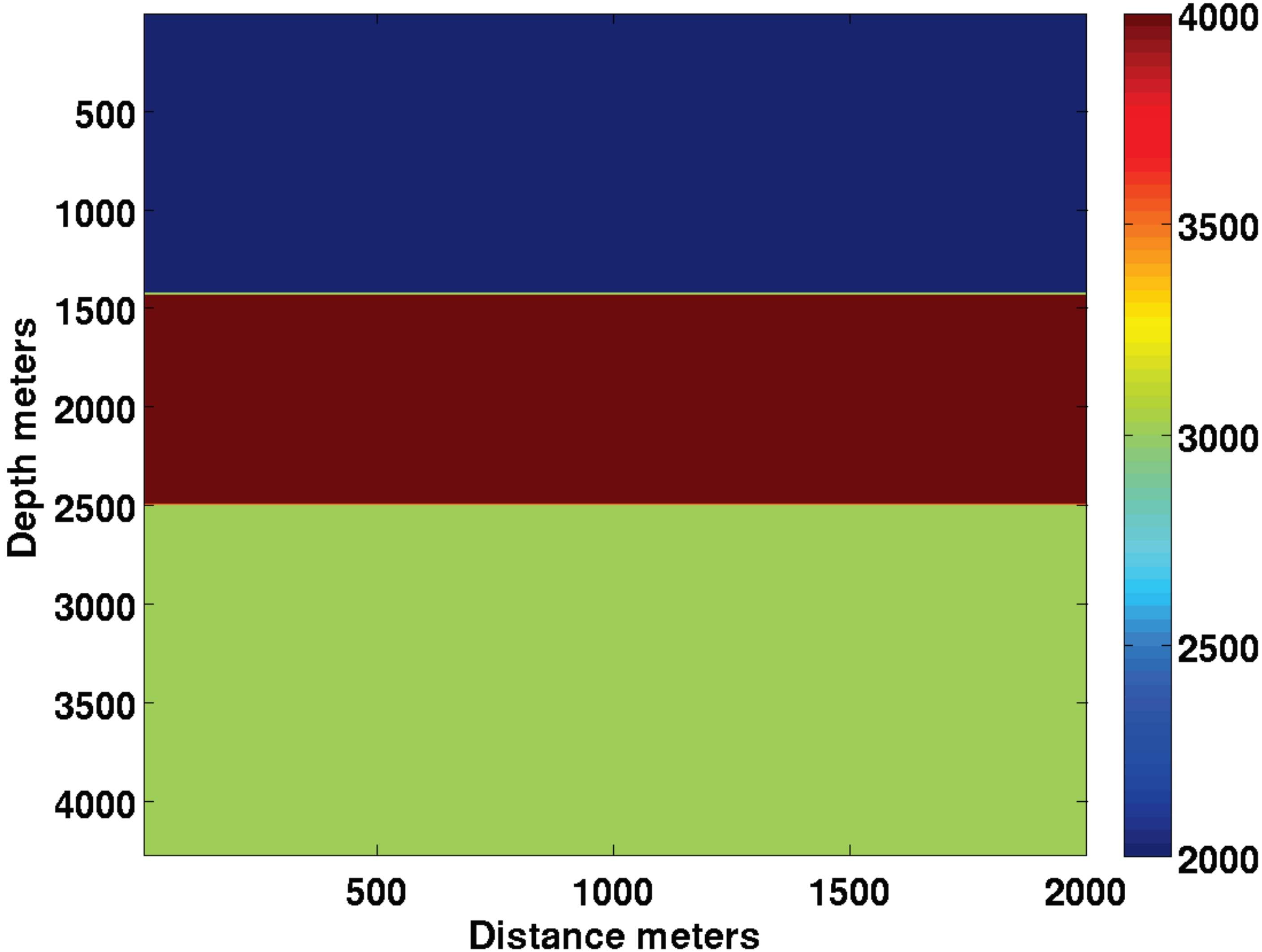


Time stepping in a Linear v(z) velocity medium
Snapshot of wavefield
z
x
Windows: …
Propagate each window with a constant velocity
c = 2800
c = 3700
c = 4800

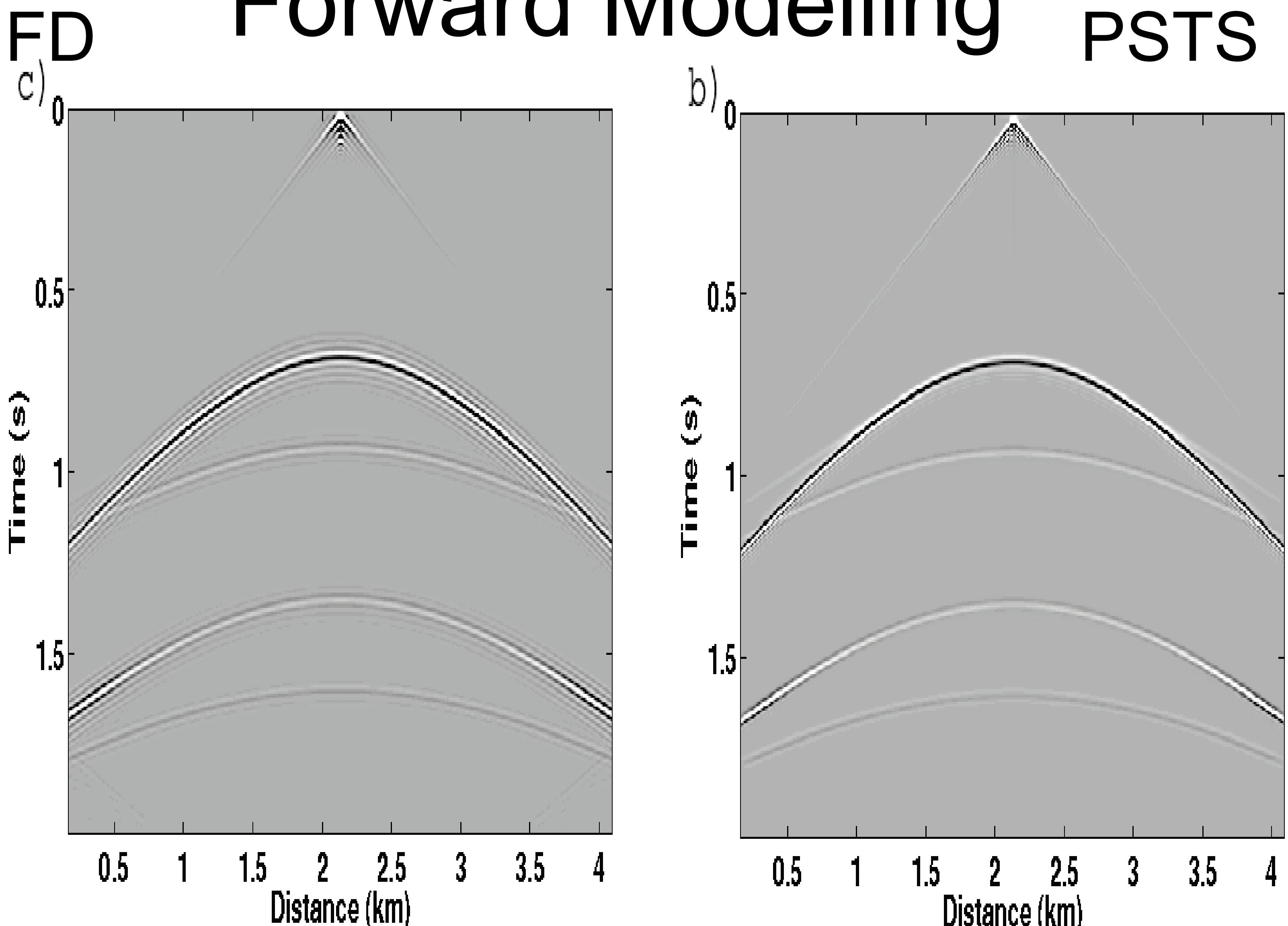


Add up each windowed snapshot of the wavefield that was propagated with a constant velocity.

Simple Velocity for Modelling



Forward Modelling



Review of Splitstep Depth stepping

- Corrects Gazdag phase-shift for velocity variations.
- Variants pseudo screen, phase screen, finite difference correctors make the split-step approximation more accurate for wider angles of propagation.

Split-Step

$$v(x) = v_{ref} + \delta v(x)$$

- Taylor series about v_{ref}

$$\cos(c(\vec{x})2\pi\,|\vec{k}|\,\Delta t) = \cos\left((v_{ref} + \delta v)2\pi\,|\vec{k}|\,\Delta t\right)$$

$$= \cos\left(v_{ref}\,2\pi\,|\vec{k}|\,\Delta t\right) + \delta v 2\pi\,|\vec{k}|\,\Delta t \sin\left(v_{ref}\,2\pi\,|\vec{k}|\,\Delta t\right)$$

$$-\frac{1}{2}\left(\delta v 2\pi\,|\vec{k}|\,\Delta t\right)^2 \cos\left(v_{ref}\,2\pi\,|\vec{k}|\,\Delta t\right) + \ldots$$

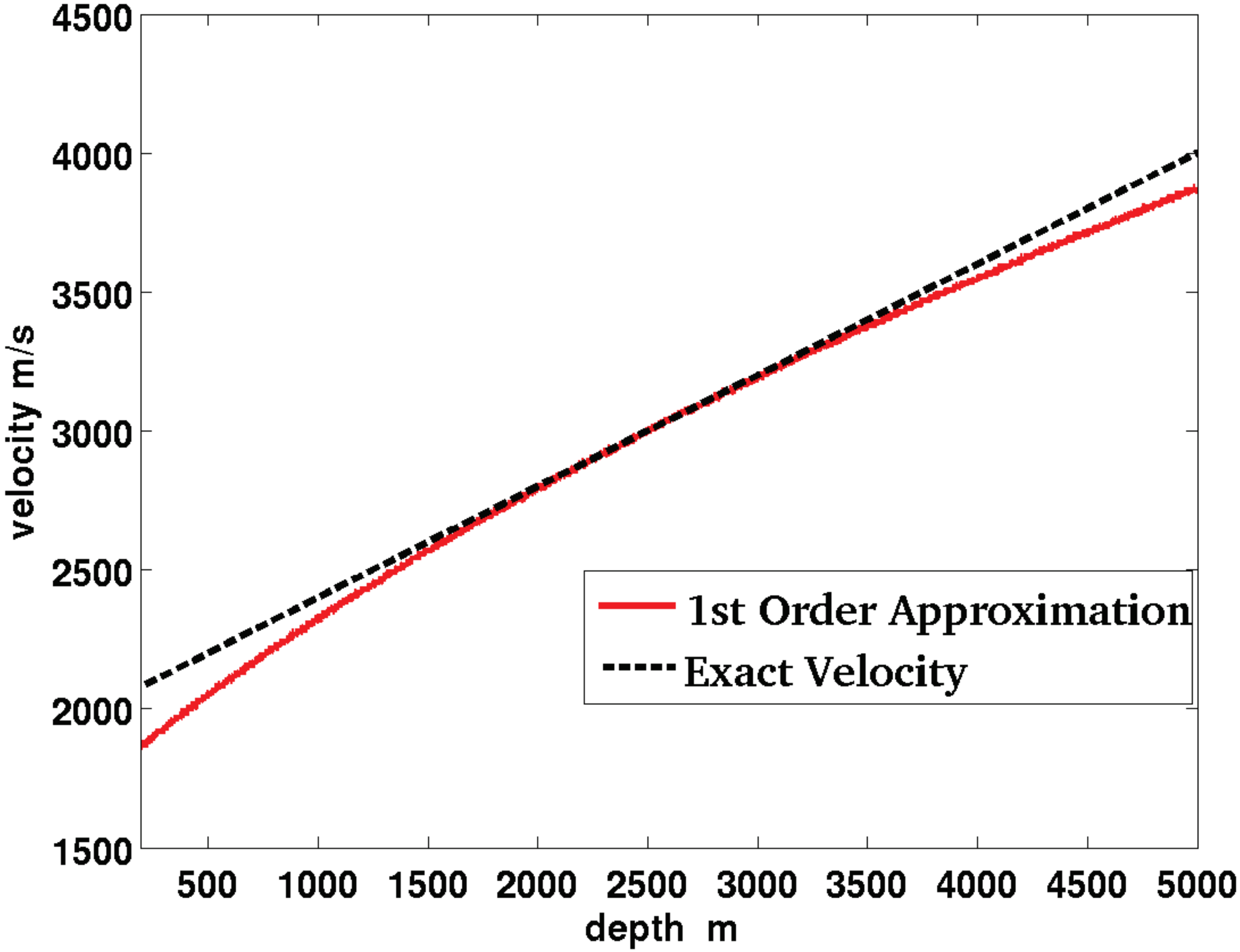
- Split-step variable velocity time stepper

$$U(\Delta t, \vec{x}) = -U(-\Delta t, \vec{x}) +$$

$$2\mathrm{FT}_{\vec{k}}^{-1}\left\{\cos(2\pi v_{ref}\,|\vec{k}|\,\Delta t)\mathrm{FT}_{\vec{x}}\{U(0,\vec{x})\}\right\}$$

$$-2\pi\delta v(x)\Delta t\,\mathrm{FT}_{\vec{k}}^{-1}\left\{|\vec{k}|\sin(2\pi v_{ref}\,|\vec{k}|\,\Delta t)\mathrm{FT}_{\vec{x}}\{U(0,\vec{x})\}\right\} + \ldots$$

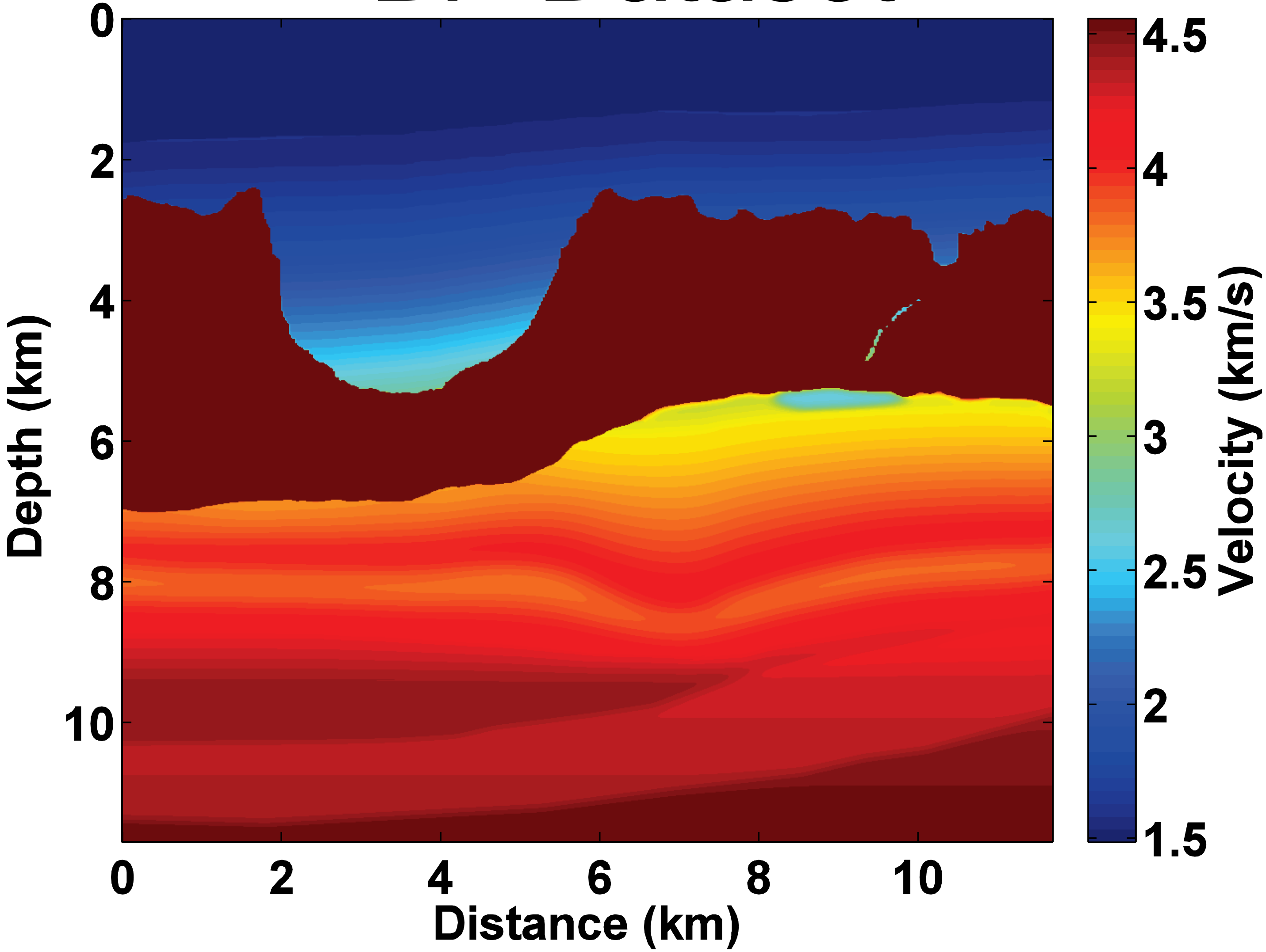
Split-Step Approximation



Review

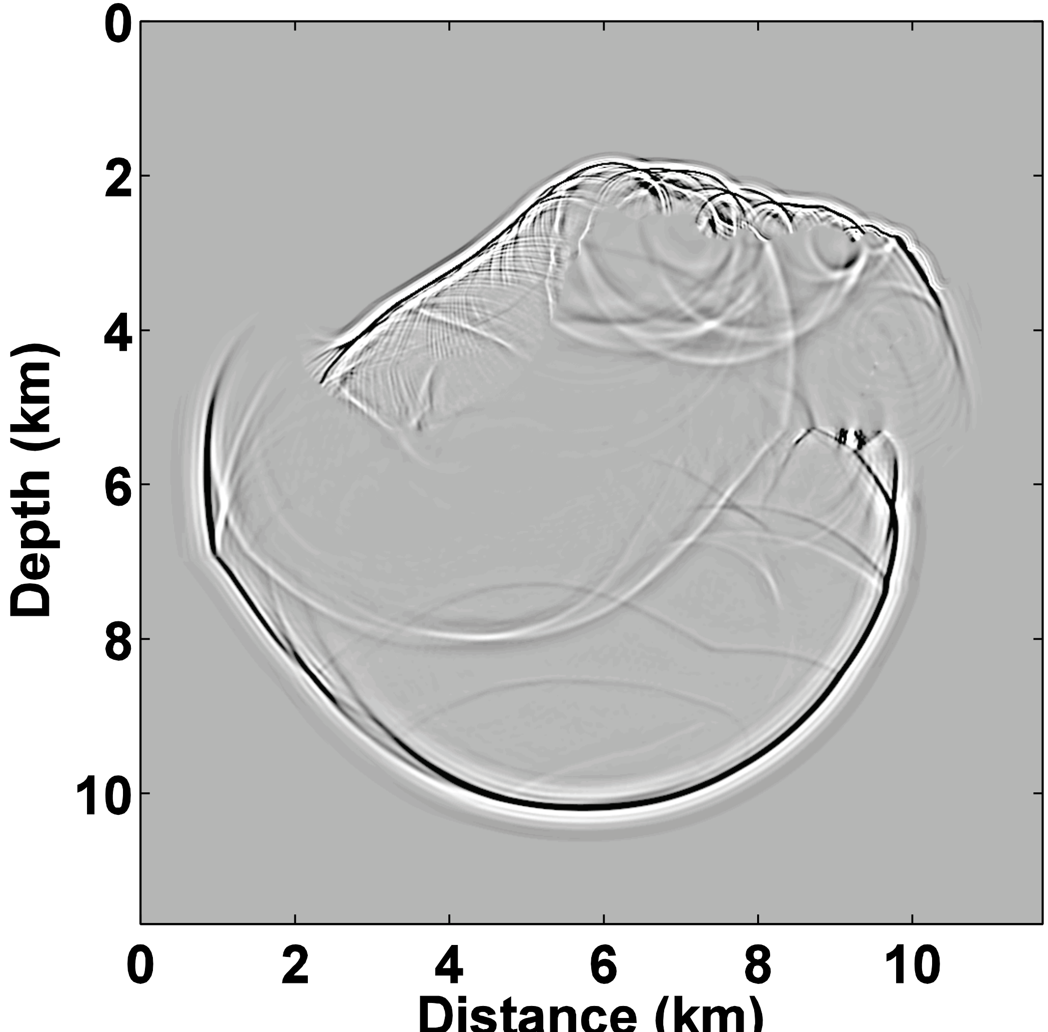
- Taylor series expand cos(c(x)|k|dt) about v=0 to get pseudospectral
- Higher order Taylor series means higher order pseudospectral
- Use constant reference velocities and corresponding windows you get PSTS
- If you Taylor series expand about c=vref you splitstep PSTS

BP Dataset



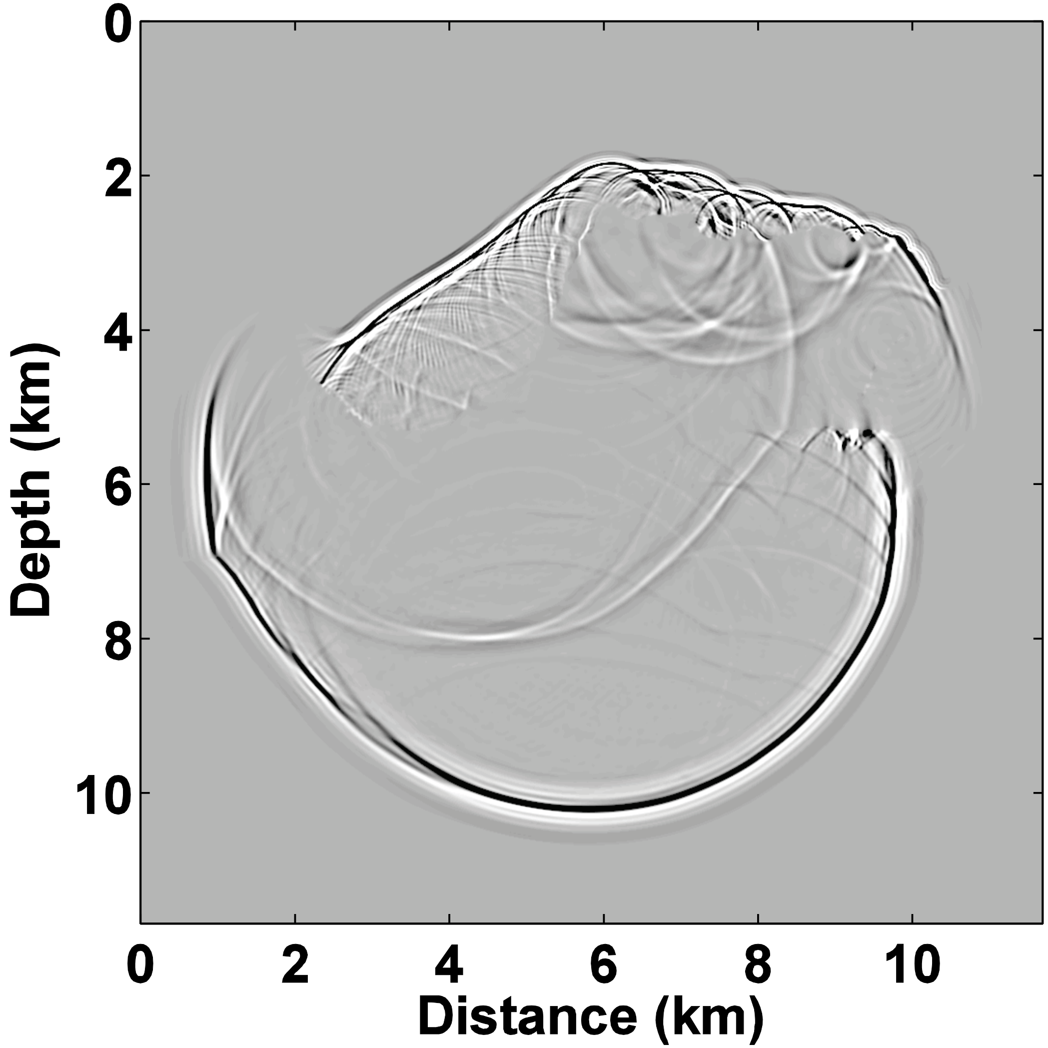
Depth (km)
0
2
4
6
8
10
0
2
4
6
8
10
Distance (km)
Velocity (km/s)
4.5
4
3.5
3
2.5
2
1.5

PSTS 10 reference velocities



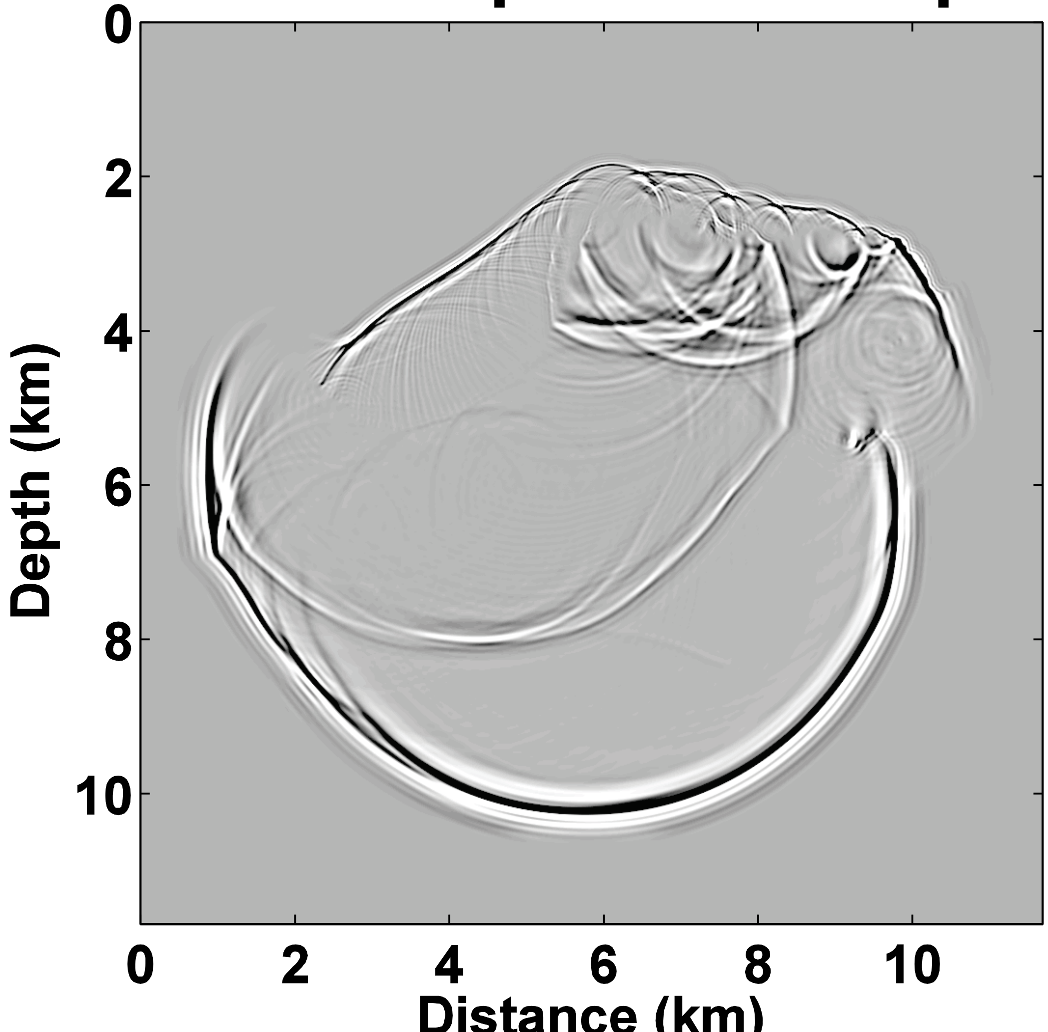
$$dt = 1.5ms, dx = 12.5m$$

PSTS 20 reference velocities



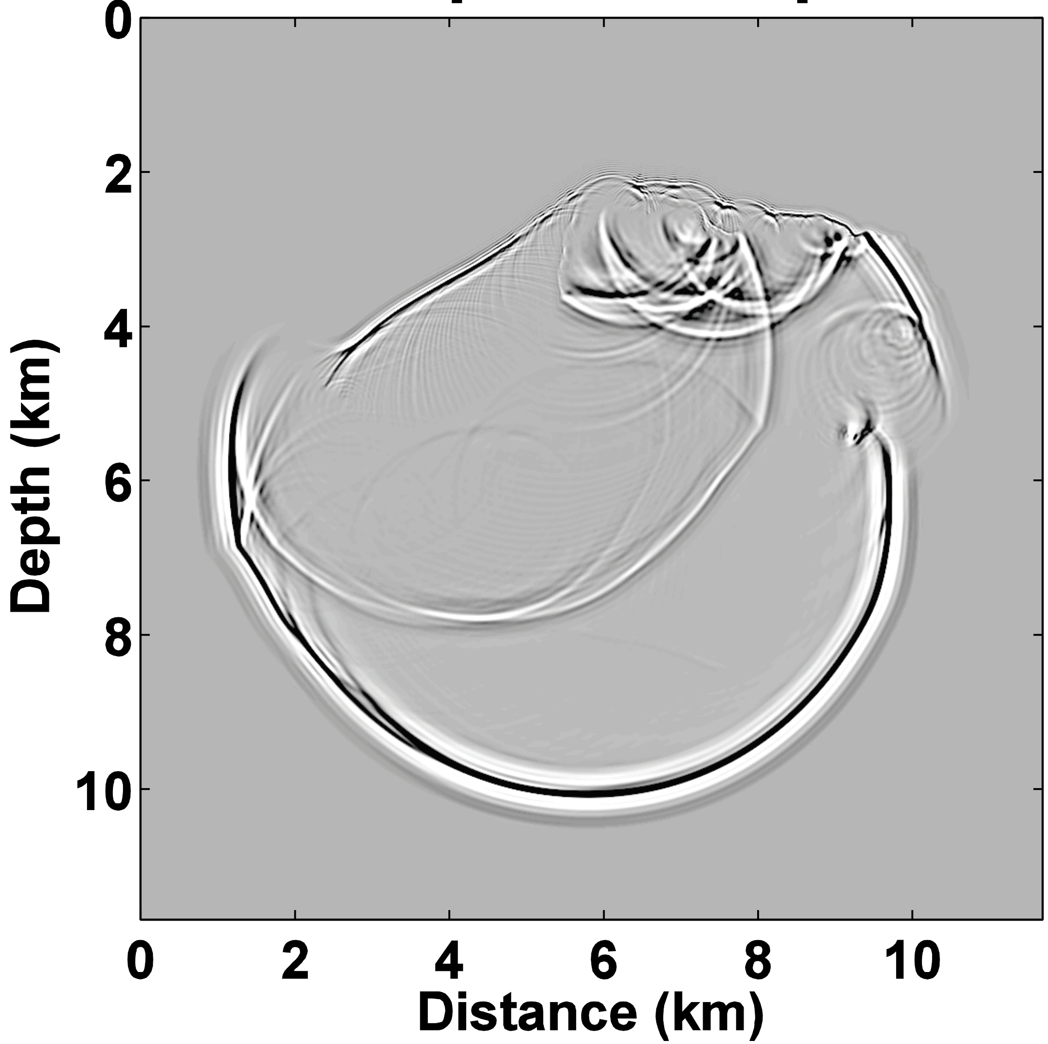
$$dt = 1.5ms, dx = 12.5m$$

second-order pseudospectral



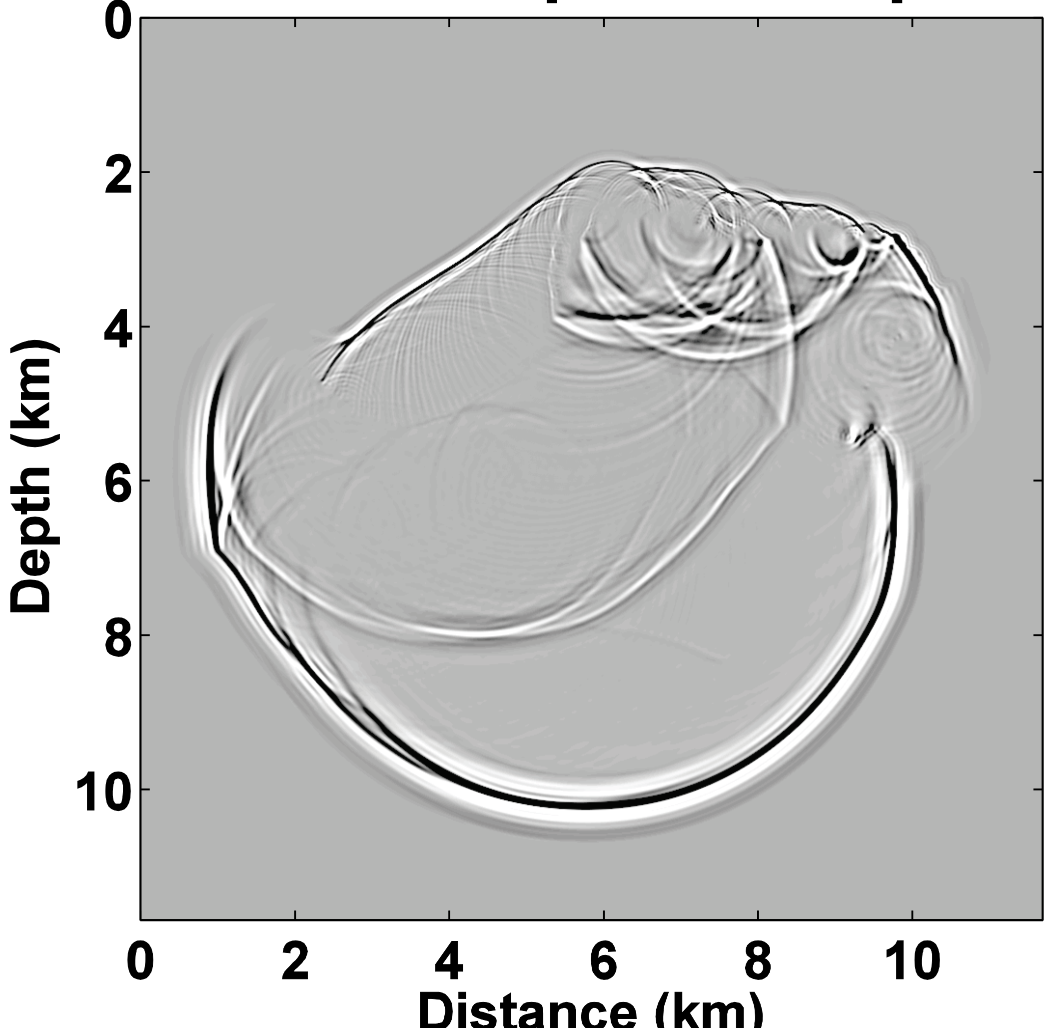
$$dt = 1.25ms, dx = 12.5m$$

first-order split-step PSTS



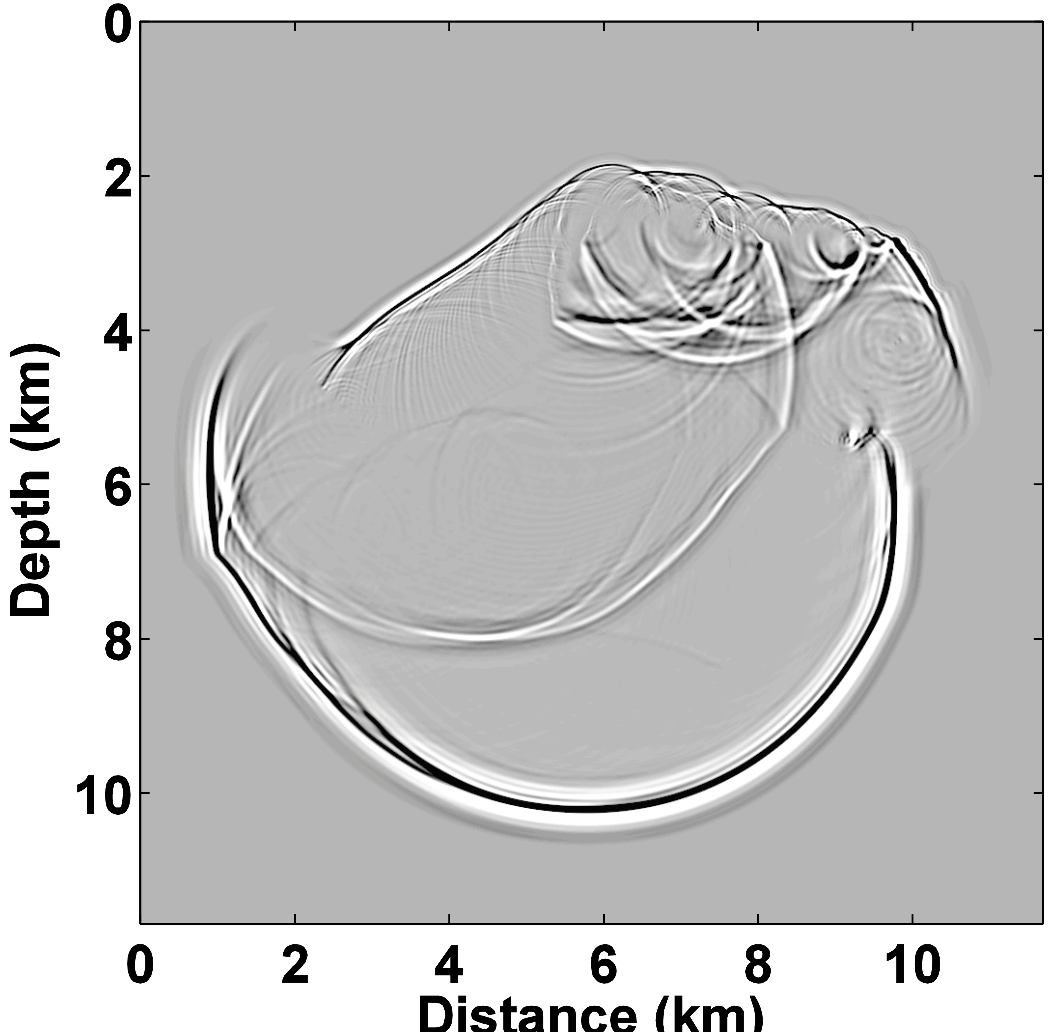
$$dt = 1.5ms, dx = 12.5m$$

second-order split-step PSTS



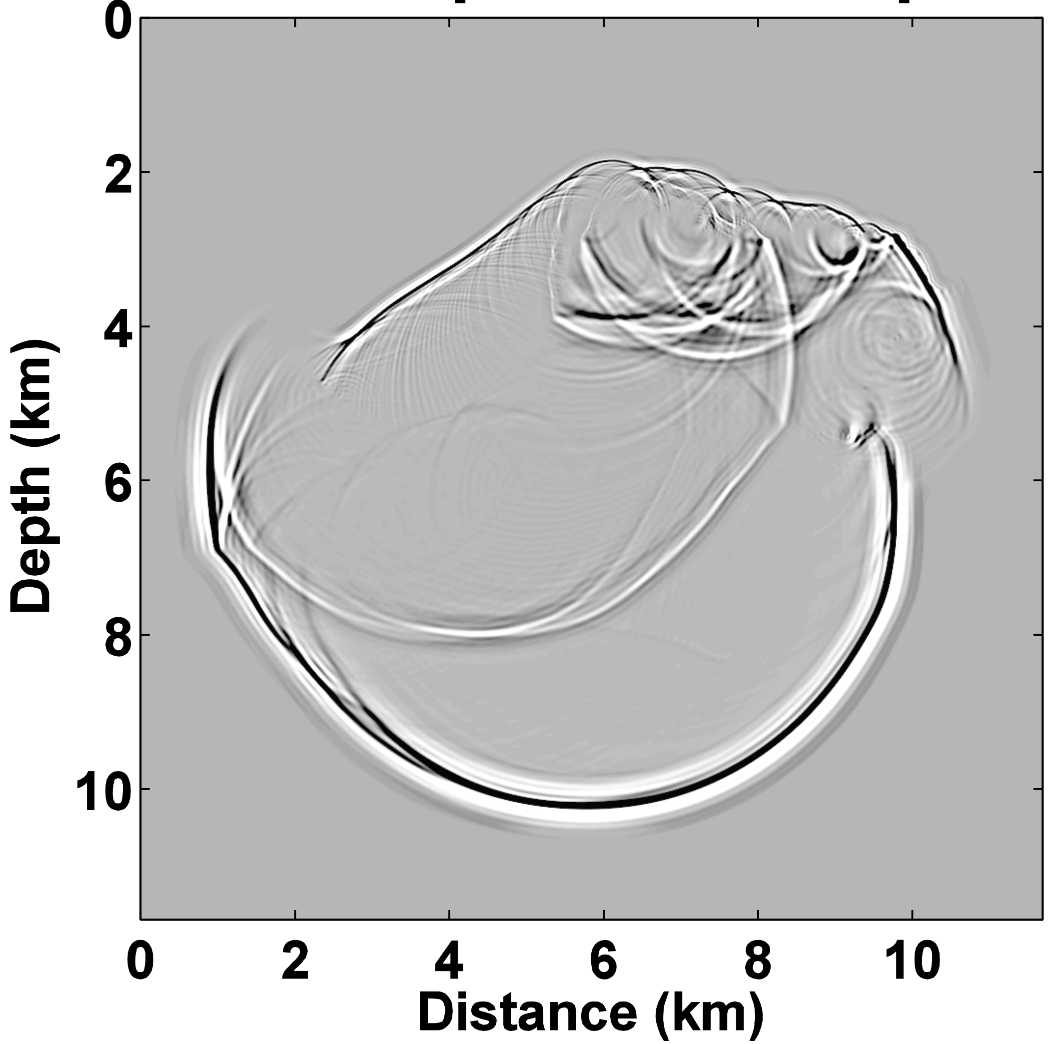
$$dt = 1.5ms, dx = 12.5m$$

cosine interpolation(v^2 linear)



$$dt = 1.0ms, dx = 12.5m$$

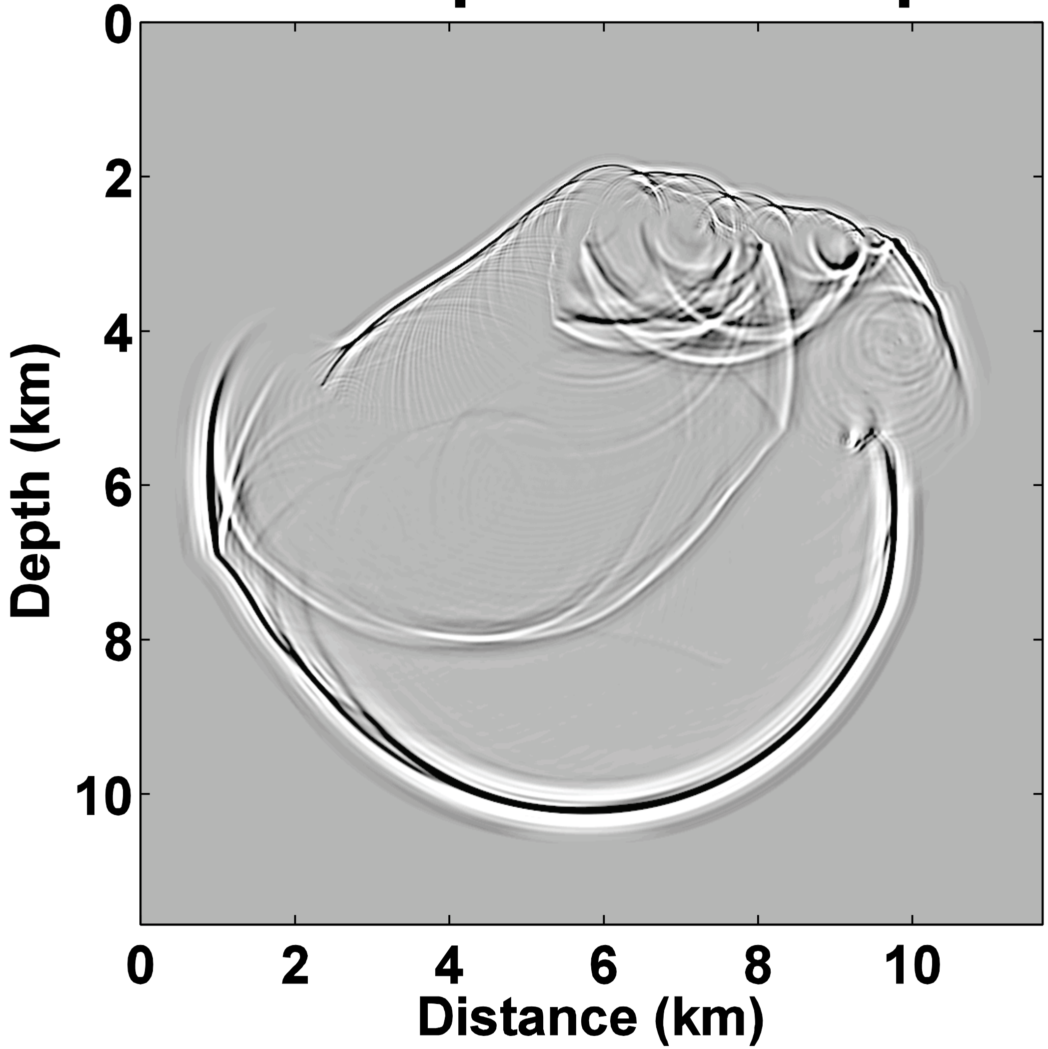
fourth-order pseudospectral



$$dt = 1.5ms, dx = 12.5m$$

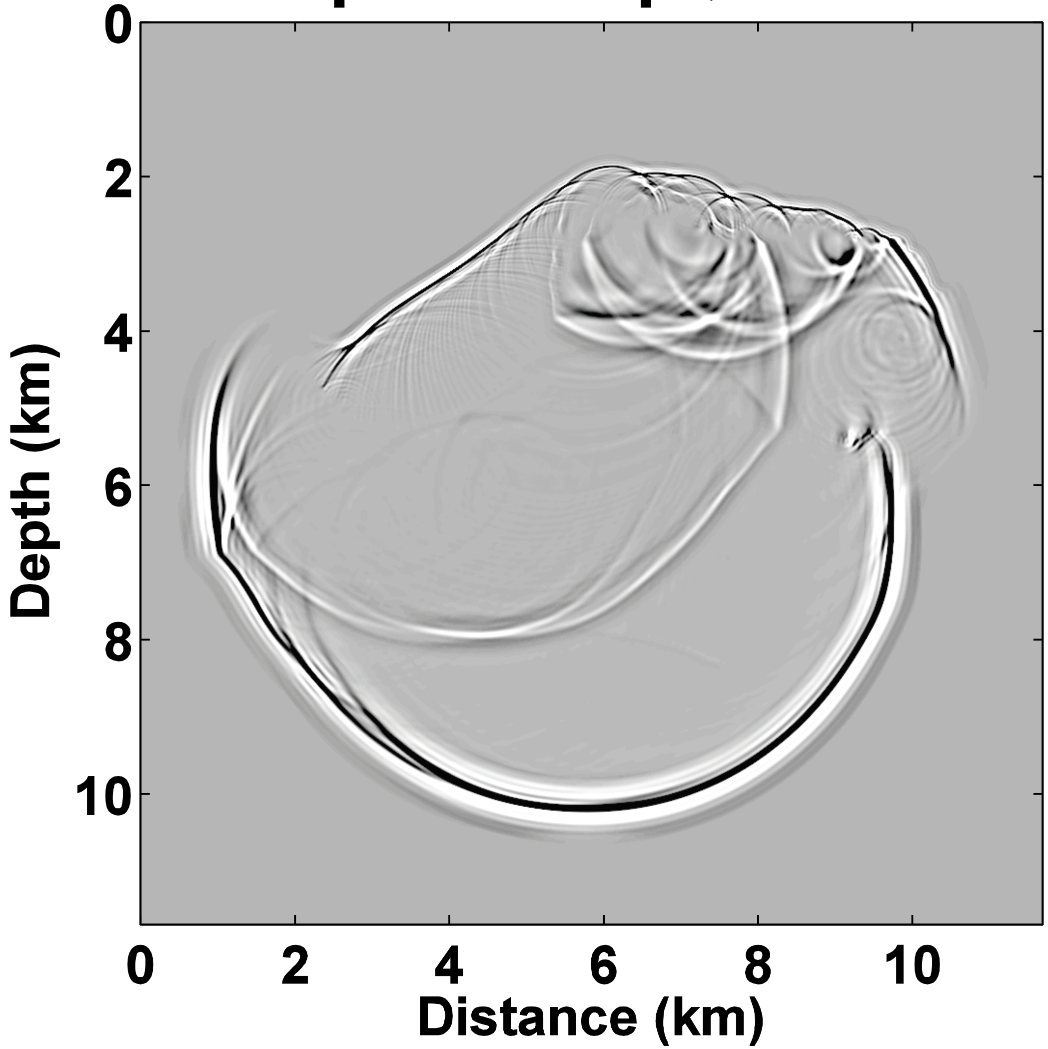
$$+c^2\Delta(c^2)\Delta U$$

fourth-order pseudospectral



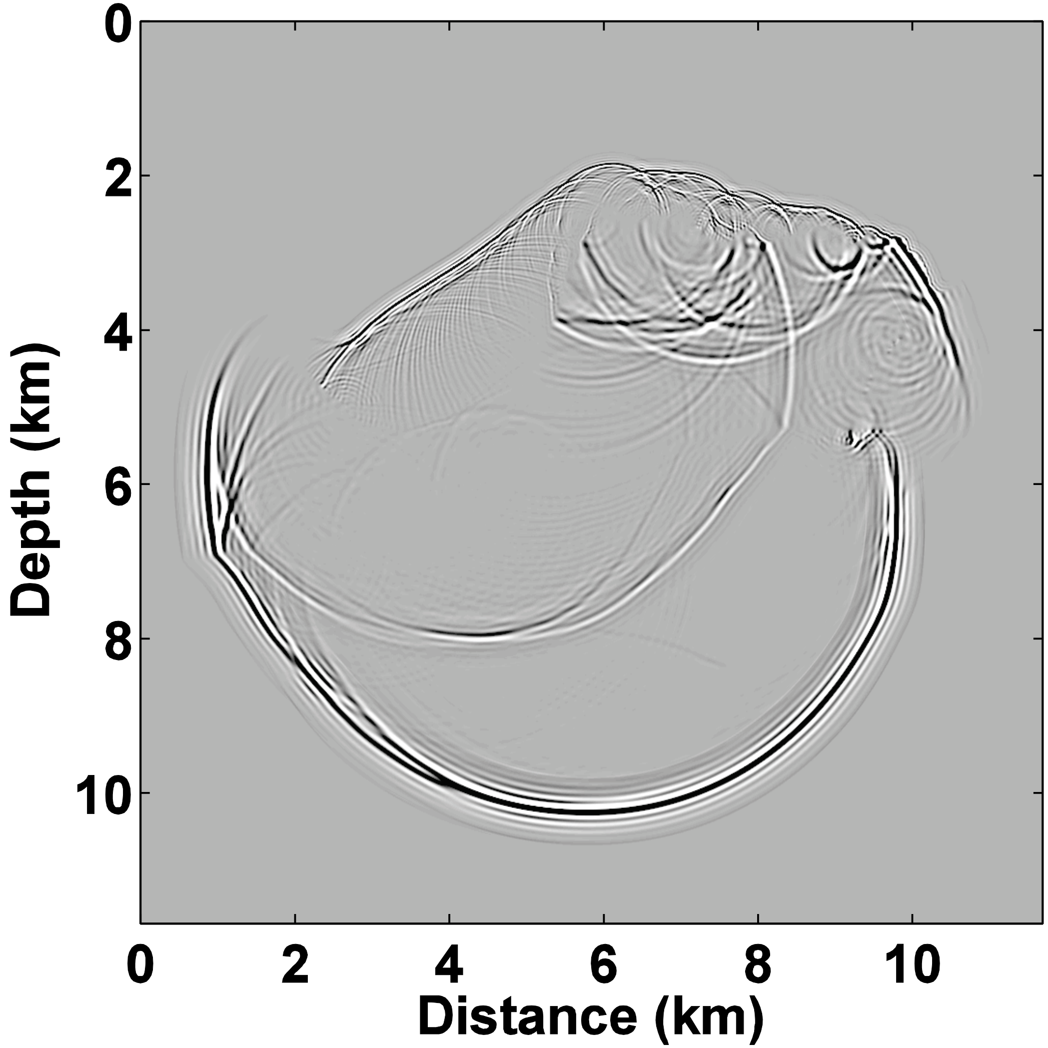
$$dt = 1.5ms, dx = 12.5m$$

first-order splitstep, 3 windows



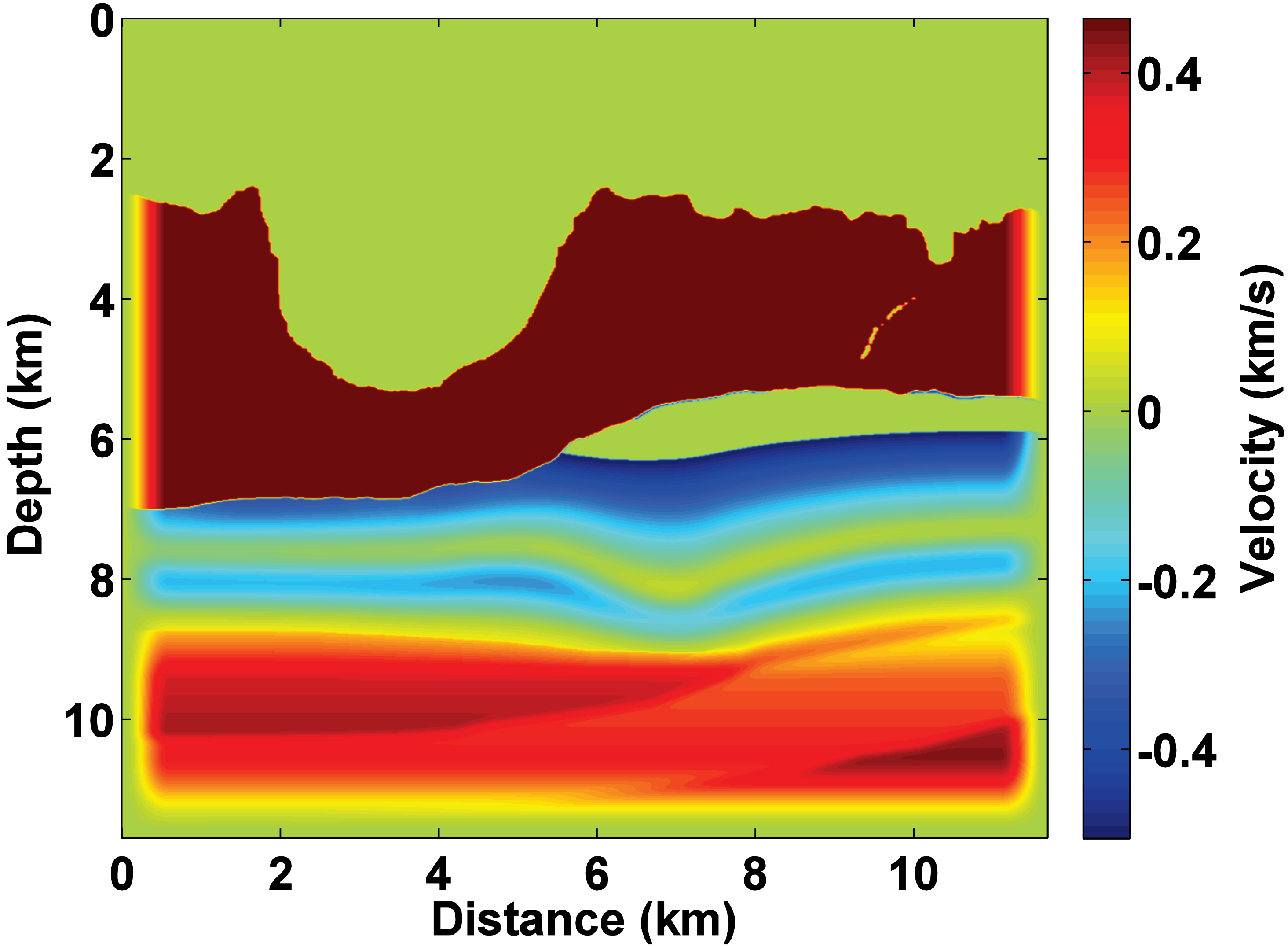
$$dt = 1.5ms, dx = 12.5m$$

Exp 3nd-order Splitstep 1 window



$$dt = 1.0ms, dx = 12.5m$$

Velocity variation for window 3 $\delta v(x)$



Algorithm	Time (s)	dt(ms)	dx(m)	#FT per time step	
PSTS 10 refs	201	1.5	12.5	11	
Pseudo 2nd order	30	1.2	12.5	2	
Pseudo 4th order	41	1.5	12.5	3	
$+\Delta\left(c^2\right)\Delta U$	41	1.5	12.5	3	
Splitstep 1nd order	43	1.5	12.5	3	
Splitstep 2th order	51	1.5	12.5	4	
Splitstep 1nd,3 wins	110	1.5	12.5	7	
Onewaysplitstep,1ndO,7win	380	1.0	12.5	15	
Onewaysplitstep,2ndO,3win	210	1.5	12.5	10	
Onewaysplitstep,2ndO,3win	136	3	12.5	10	
Onewaysplitstep,3rd, 1 win	100	1.5	12.5	5	

Windowing before propagation

$$U(\Delta t, \vec{x}) = -U(-\Delta t, \vec{x})$$
$$+\mathrm{FT}^{-1}_{\vec{k}}\{\sum_i 2\cos(2\pi v_i \,|\, \vec{k} \,|\, \Delta t)\mathrm{FT}_{\vec{x}}\{\Omega_i(\vec{x})U(0, \vec{x})\}\}$$

Windowing before and after propagation

$$U(\Delta t, \vec{x}) = -U(-\Delta t, \vec{x})$$
$$+\sqrt{\Omega_i(\vec{x})}\sum_i \mathrm{FT}^{-1}_{\vec{k}}\{2\cos(2\pi v_i \,|\, \vec{k} \,|\, \Delta t)\mathrm{FT}_{\vec{x}}\{\sqrt{\Omega_i(\vec{x})}U(0, \vec{x})\}\}$$

Sampling issues

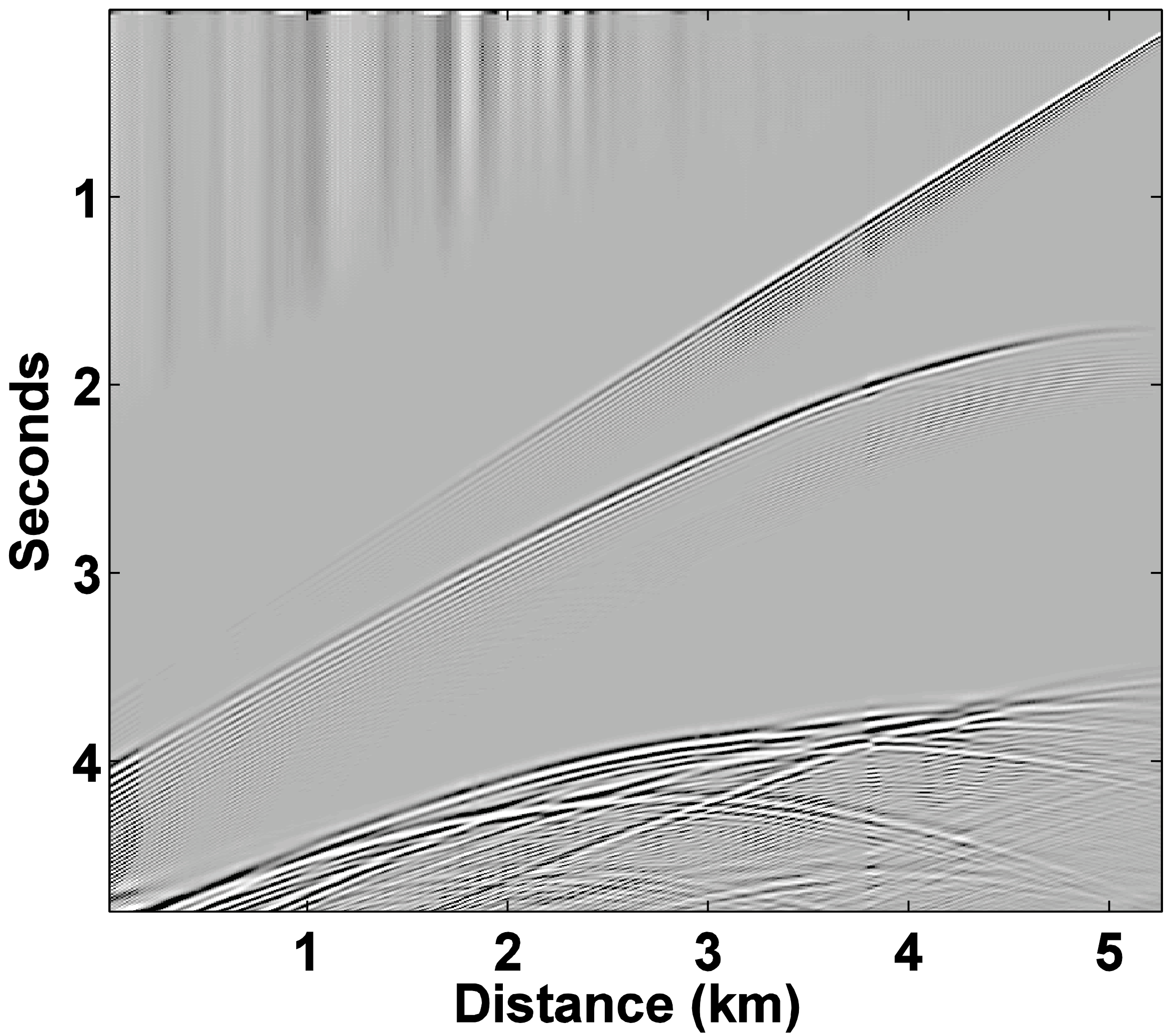
Two way algorithms

$$\frac{c\Delta t}{\Delta x} < \frac{1}{\sqrt{2}}$$

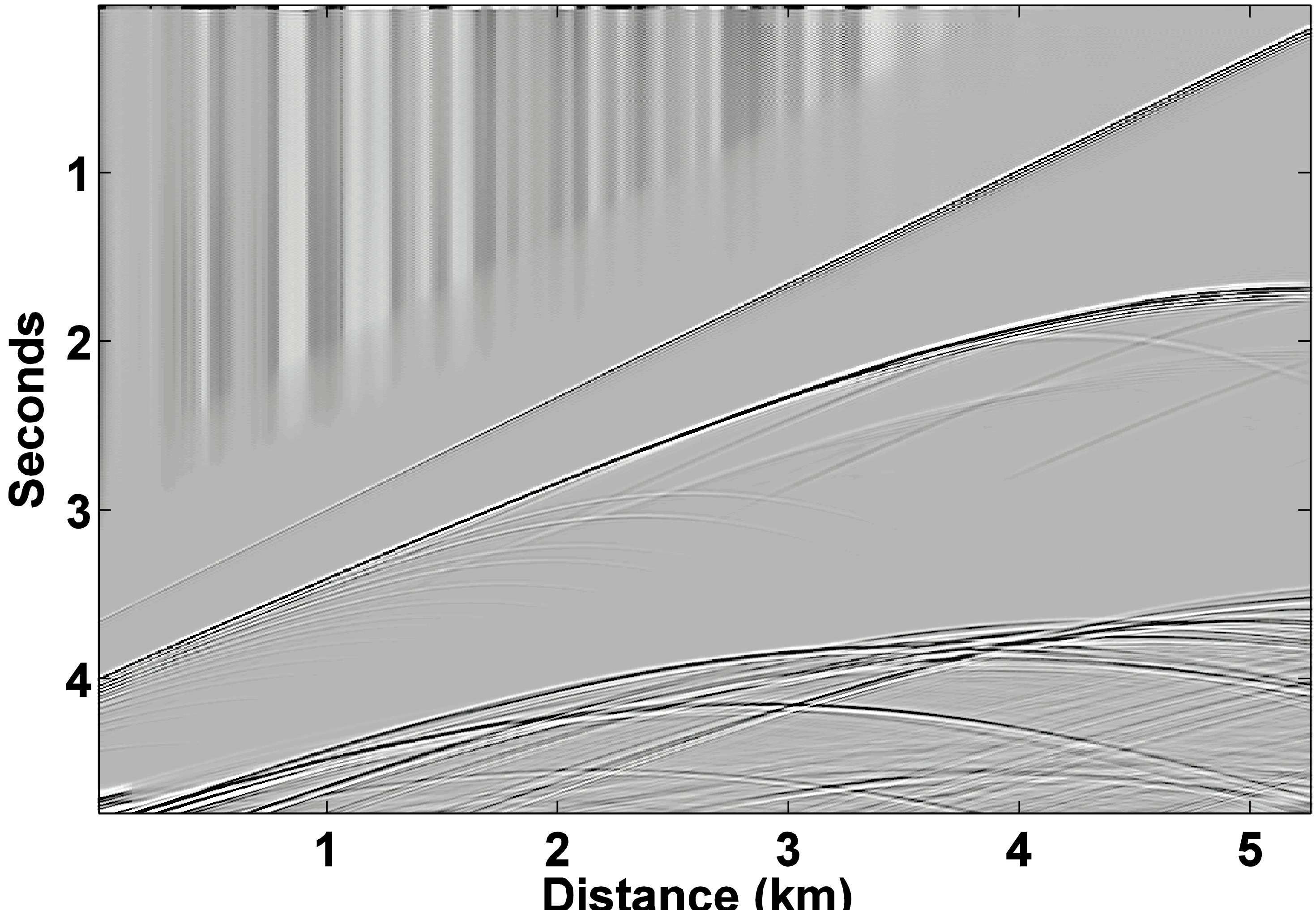
One-way algorithms

$$\Delta t < \frac{1}{2 f_{\max}}$$

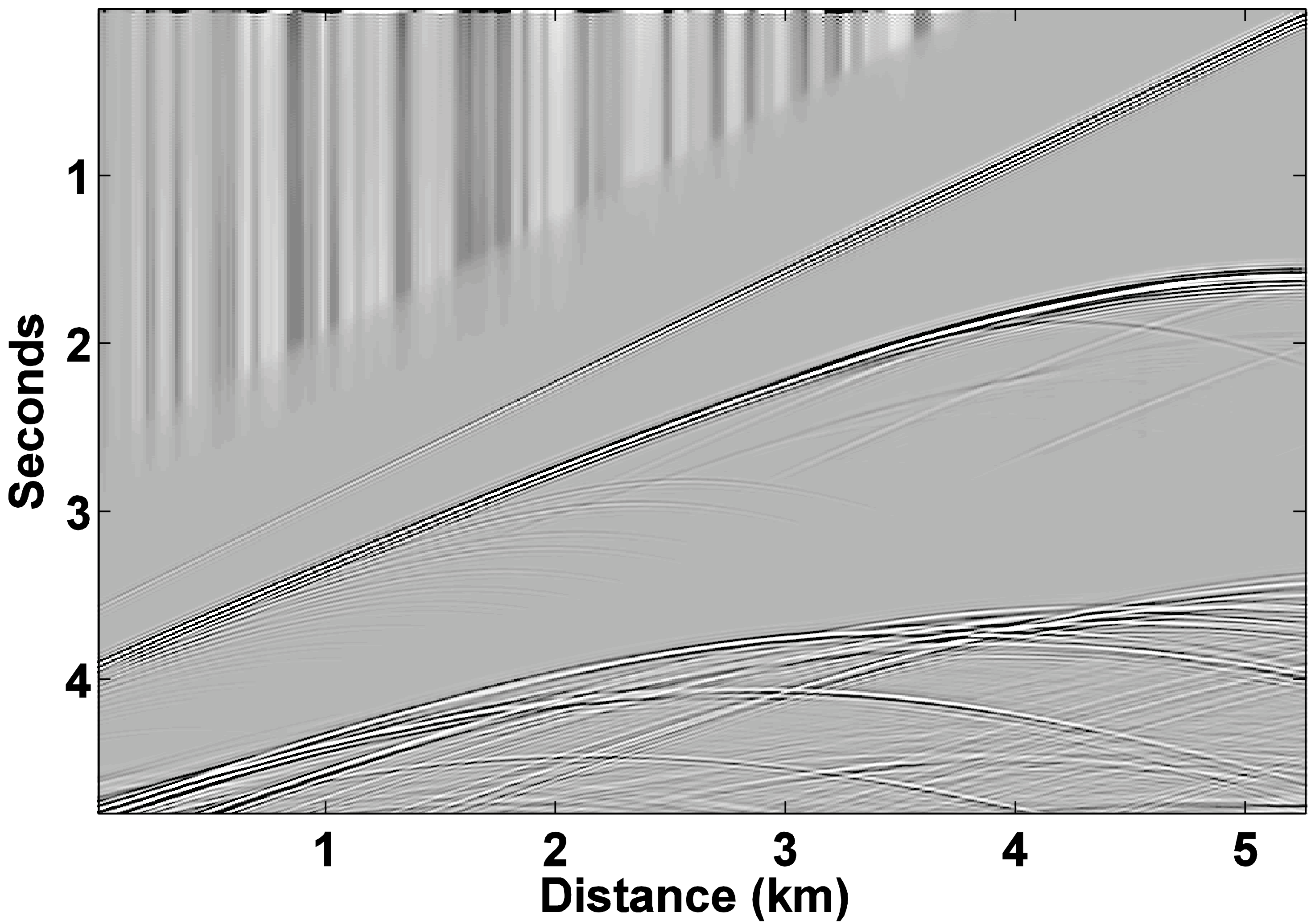
second order splitstep cosine



fourth order Splitstep cosine



fourth order pseudospectral



Conclusions

- A higher order method is preferred to a lower order method with windowing

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