Numerical modelling of viscoelastic waves by a pseudospectral domain decomposition method

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$$f(x_{i+1}) = f(x_i) + (\Delta x)f'(x_i) + \frac{(\Delta x)^2}{2}f'(x_i) + O((\Delta x)^3),$$
$$f(x_{i-1}) = f(x_i) - (\Delta x)f'(x_i) + \frac{(\Delta x)^2}{2}f'(x_i) + O(\Delta x)^3).$$

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- If $(x_{i+1} - x_i) = (x_i - x_{i-1}) = \Delta x$ then the *finite-difference* approximations for $f'(x_i)$ are

$$f'(x_i) = \frac{f(x_{i+1}) - f(x_i)}{\Delta x} + O(\Delta x),$$
$$f'(x_i) = \frac{f(x_i) - f(x_{i-1})}{\Delta x} + O(\Delta x),$$
$$f'(x_i) = \frac{f(x_{i+1}) - f(x_{i-1})}{2\Delta x} + O(\Delta x)^2.$$

- A more general approach is to build a *Lagrange interpolating polynomial* and differentiate that.

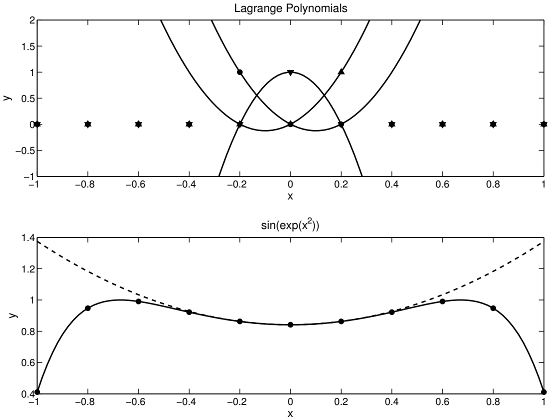

Figure: 3 node Lagrange polynomials defined on equally spaced nodes and the resulting interpolation.

- The 3 point differentiation matrix that act on the sampled values of f and returns approximately the sampled values of f' is

$$\frac{1}{2\Delta x}\begin{pmatrix} -3 & 4 & -1 & 0 & 0 & \dots & 0 \\ -1 & 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & -1 & 0 & 1 & 0 & \dots & 0 \\ \vdots & & \ddots & \ddots & \ddots & & \vdots \\ 0 & \dots & 0 & -1 & 0 & 1 & 0 \\ 0 & \dots & \dots & 0 & -1 & 0 & 1 \\ 0 & \dots & \dots & 0 & 1 & -4 & 3 \end{pmatrix}$$

- We can generalize this approach until the differentiation matrix is fully populated, but the nodes must be chosen carefully due to Runge's phenomenon.

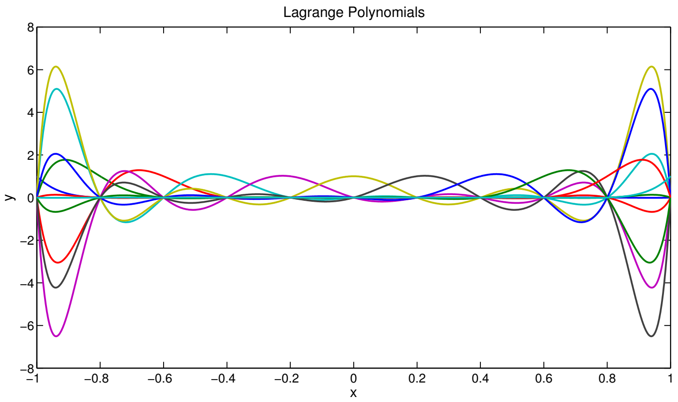

Figure: 11 Lagrange polynomials defined on equally spaced nodes.

- Two popular choices are the *Chebyshev* and *Legendre* points.

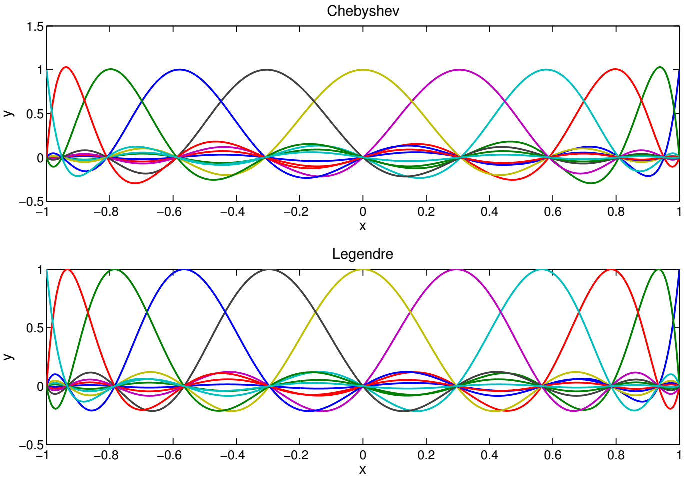

Figure: Chebyshev and Legendre polynomials.

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- And a set of *Gauss-Lobatto* integration weights $\mathbf{w}$ that are exact for polynomials of degree less than or equal $2N-1$, where N is number of points.

$$\int_{-1}^{1} f(x)g(x)dx = \sum_{i=1}^{N} f(x_i)g(x_i)w_i.$$

- $\|D_N\mathbf{f} - \mathbf{f}'\|_\infty$ for $f = x^{10}$. $\mathbf{f} = [f(x_0), ..., f(x_N)]^T$. D_N is the $(N+1) \times (N+1)$ pseudospectral differentiation matrix.

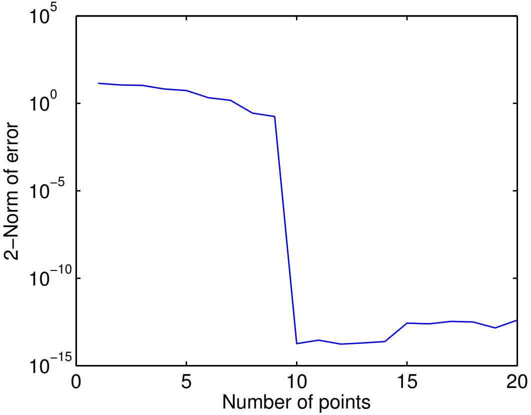

Figure: Convergence to machine-precision

- To compare accuracies, consider an ugly function

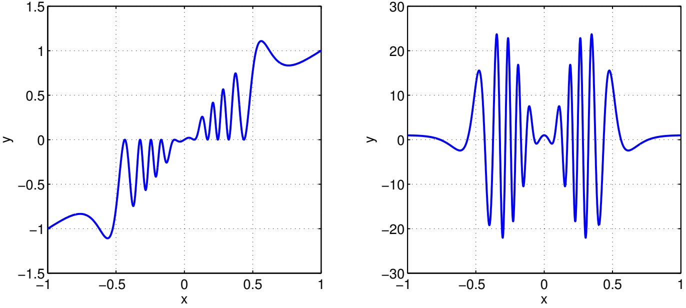

Figure: An ugly function $f(x) = x(1 + sin(10\pi exp(-10x^2)))$ and it's derivative.

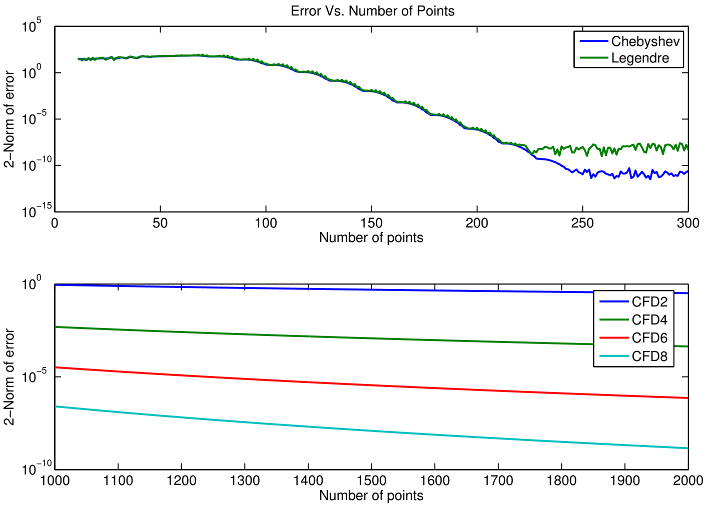

Figure: Various derivative approximations

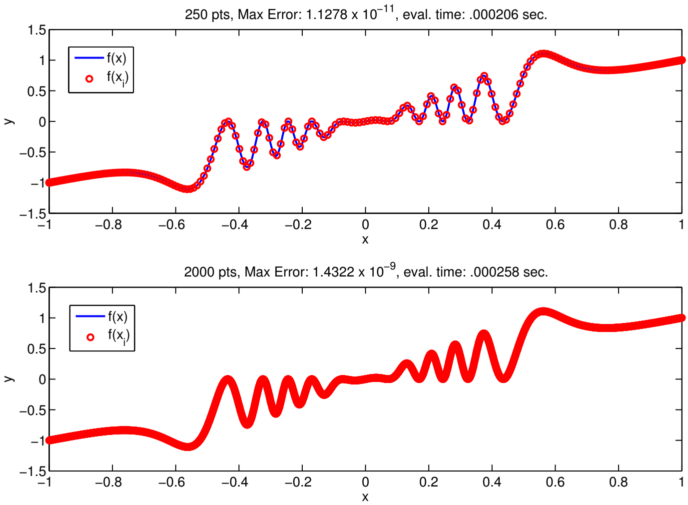

Figure: Final number of points, times and errors for Chebyshev and 8^{th} order finite differences.

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$$\rho \ddot{u}_i = \partial_j \sigma_{ij}(\mathbf{u}) + f_i, \ \mathbf{x} \in \Omega, \ t > 0$$

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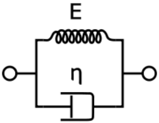
- For a Kelvin-Voigt material



$$\sigma(u) = E\epsilon + \eta\dot{\epsilon} \quad (1-D)$$

$$\sigma_{ij}(\mathbf{u}) = \lambda \nabla \cdot \mathbf{u}\delta_{ij} + 2\mu\epsilon_{ij}(\mathbf{u}) + \lambda' \nabla \cdot \mathbf{v}\delta_{ij} + 2\mu'\epsilon_{ij}(\mathbf{v}) \quad (N-D)$$

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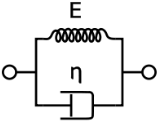

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- λ and μ are the elastic parameters, λ' and μ' are the anelastic parameters.

- Let $u_j(x,z,t) = \hat{u}_j(x,z)e^{i\omega t}$, then

$$\begin{aligned}\sigma_{ij} &= \lambda\nabla\cdot\hat{\mathbf{u}}\delta_{ij} + 2\mu\varepsilon_{ij}(\hat{\mathbf{u}}) + i\omega\left(\lambda'\nabla\cdot\hat{\mathbf{u}}\delta_{ij} + 2\mu'\varepsilon_{ij}(\hat{\mathbf{u}})\right)\\ &= \Lambda\nabla\cdot\hat{\mathbf{u}}\delta_{ij} + 2M\varepsilon_{ij}(\hat{\mathbf{u}})\end{aligned}$$

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- The frequency-dependent P and S wave quality factors

$$Q_p = \frac{\lambda + 2\mu}{\omega(\lambda' + 2\mu')}, \quad Q_s = \frac{\mu}{\omega\mu'}.$$

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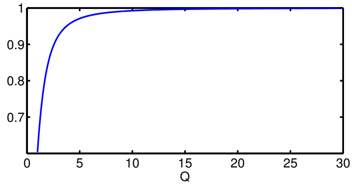
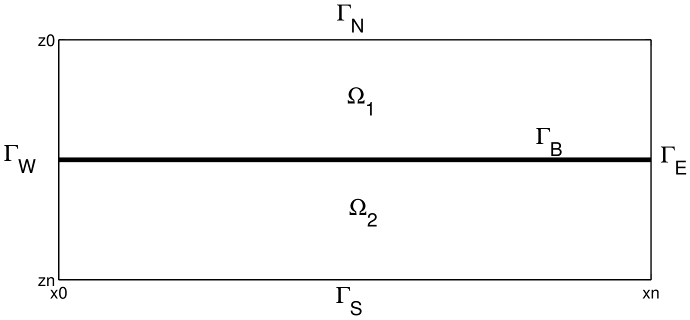
- g is obtained algebraically from the above equations as

$$g(Q) = \frac{1}{2}(1 + Q^{-2})^{-1/2}(1 + (1 + Q^{-2})^{-1/2}).$$



- Consider a 2-layer, 2-D model.



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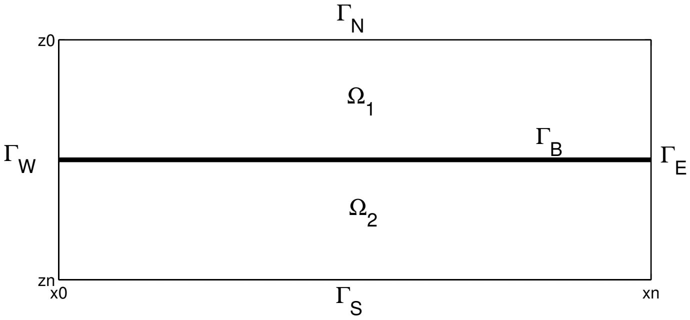

- The idea is to solve the wave equation in each subdomain and connect them using interface conditions.

- Let's ignore the force term for now leaving

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- Now we split the integral over the two regions

$$\sum_{k=1}^{2} \int_{\Omega^k} \rho \ddot{u}_i^k \phi d\Omega^k = \sum_{k=1}^{2} \int_{\Omega^k} \partial_j \sigma_{ij}(\mathbf{u}^k) \phi d\Omega^k$$

Where $\mathbf{u}^k$ are the displacements in the region Ω^k.

- Integrating the right hand side by parts produces

$$\sum_{k=1}^{2} \int_{\Omega^k} \partial_j \sigma_{ij}(\mathbf{u}^k) \phi d\Omega^k$$
$$= \sum_{k=1}^{2} \left\{ \oint_{\partial\Omega_k} \sigma_{ij}(\mathbf{u}^k) \phi n_j^k dS - \int_{\Omega^k} \sigma_{ij}(\mathbf{u}^k) \partial_j \phi d\Omega^k \right\}$$

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- We now have

$$\sum_{k=1}^{2} \int_{\Omega^k} \left\{ \rho \ddot{u}_i^k \phi + \sigma_{ij}(\mathbf{u}^k)\partial_j\phi \right\} d\Omega^k = \sum_{k=1}^{2} \oint_{\partial\Omega_k} \sigma_{ij}(\mathbf{u}^k)\phi n_j^k dS$$

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- These conditions determine the reflection and transmission coefficients at the interface.

- Continuity is enforced by construction of the basis elements.

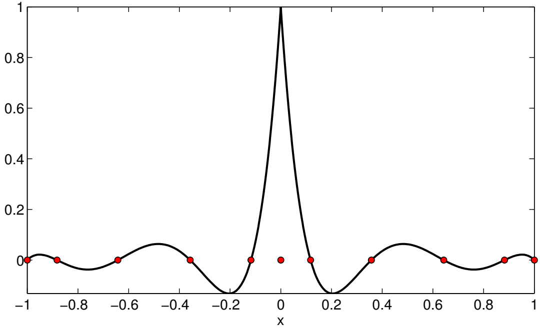


Figure: Interface function in 1-D.

- Higher-dimensional constructions use product-bases.

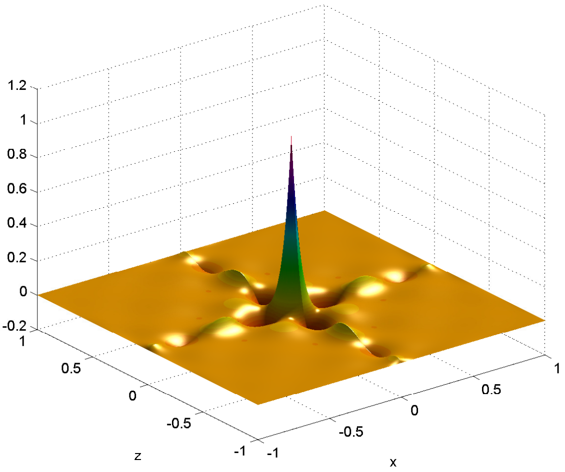
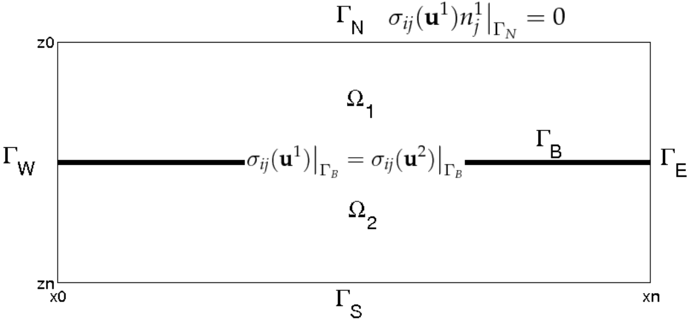


Figure: Interface function in 2-D.

- Terms involving stresses are enforced by modifying the surface integrals

$$\oint_{\partial\Omega_1} \sigma_{ij}(\mathbf{u}^1)n_j^1\phi dS + \oint_{\partial\Omega_2} \sigma_{ij}(\mathbf{u}^2)n_j^2\phi dS$$



- The discretization results in a system of equations for the k^{th} element

$$M^k \ddot{\mathbf{u}}_i^k(t) + \sum_j \hat{K}_{ij}^k \dot{\mathbf{u}}_i^k(t) + \sum_j K_{ij}^k \mathbf{u}_j^k(t) = M^k \mathbf{f}_i^k(t).$$

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- Absorbing boundaries are enforced by replacing interior derivatives with one-way wave equations.

$$\partial_z u_1 \leftarrow -\frac{1}{V_s} v_1 + \frac{V_s - V_p}{V_s} \partial_x u_2, \; x \in \Gamma_S$$

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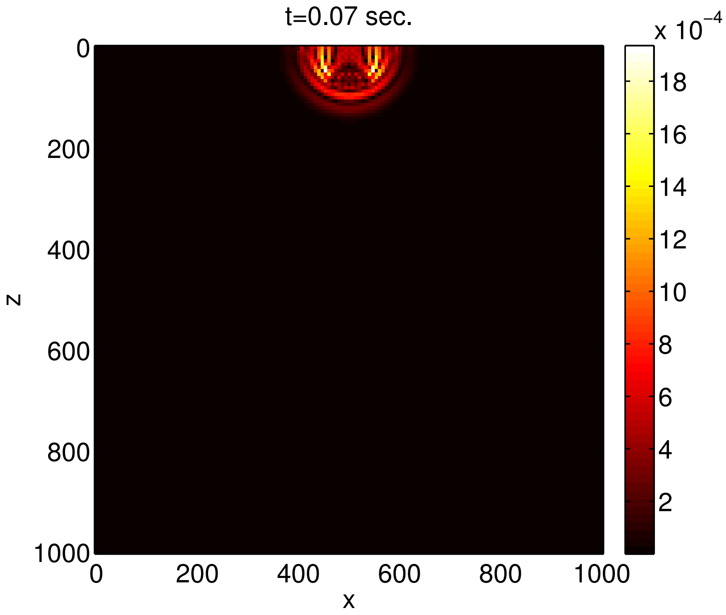
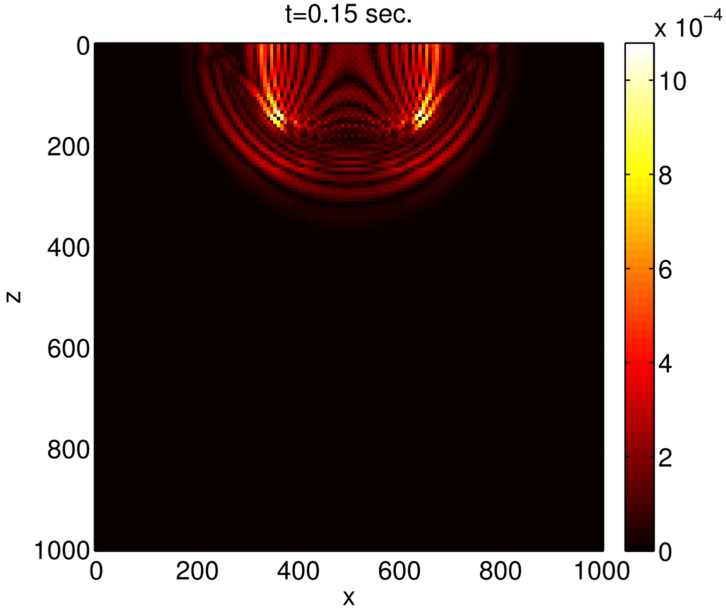
- The system is written in block form

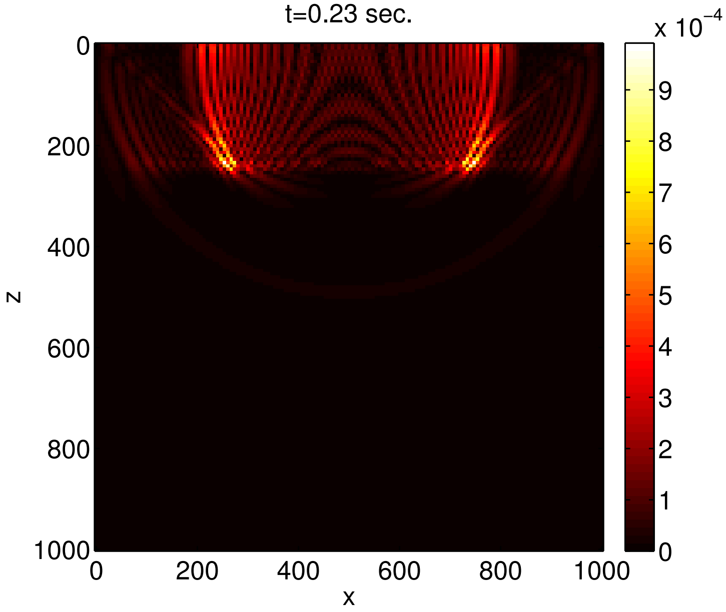
$$\begin{pmatrix} M & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} \dot{\mathbf{V}} \\ \dot{\mathbf{U}} \end{pmatrix} + \begin{pmatrix} \hat{K} & K \\ I & 0 \end{pmatrix} \begin{pmatrix} \mathbf{V} \\ \mathbf{U} \end{pmatrix} = \begin{pmatrix} \mathbf{F} \\ 0 \end{pmatrix}$$

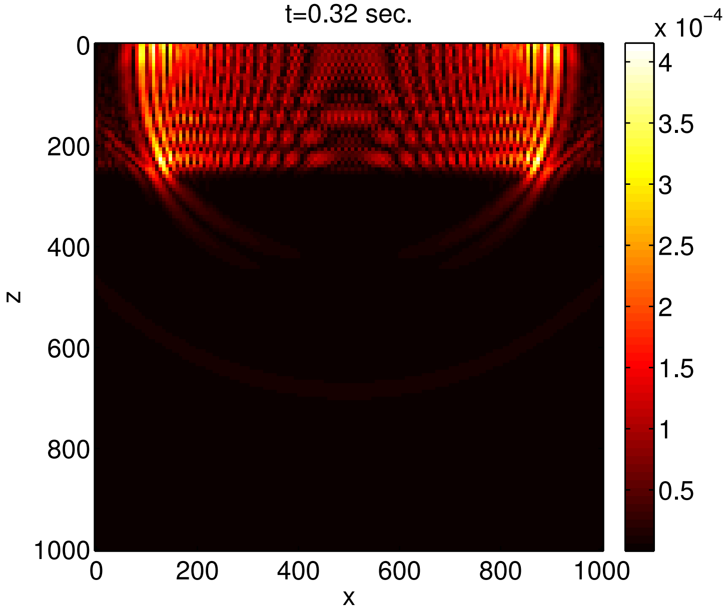
- To show the high-frequency damping present in the anelastic part of the model we purposefully choose a grid too coarse to represent the source wavelet (30 Hz Ricker). The boundary is at $z = 250m$. The model is time-stepped using a 4^{th} low-storage explicit Runge-Kutta (LSERK) method.

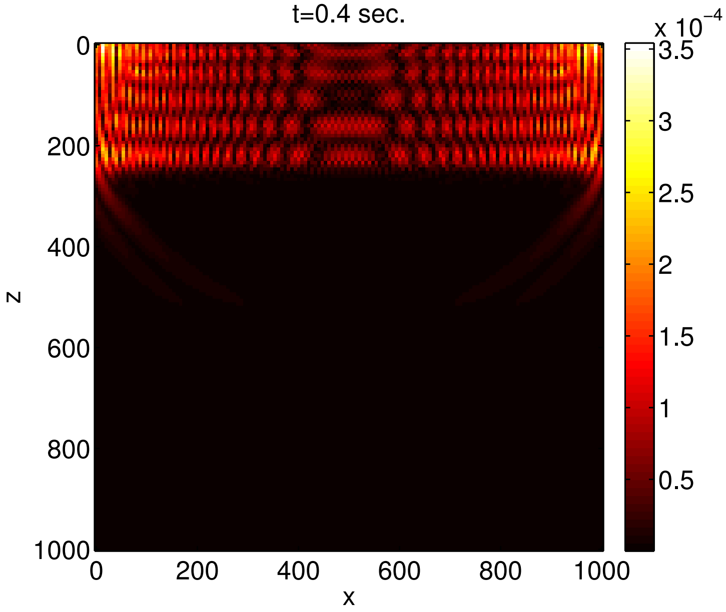
	ρ	V_p	V_s	Q_p	Q_s
Ω_1	2.06	2400	1500	∞	∞
Ω_2	2.06	2400	1500	10	10

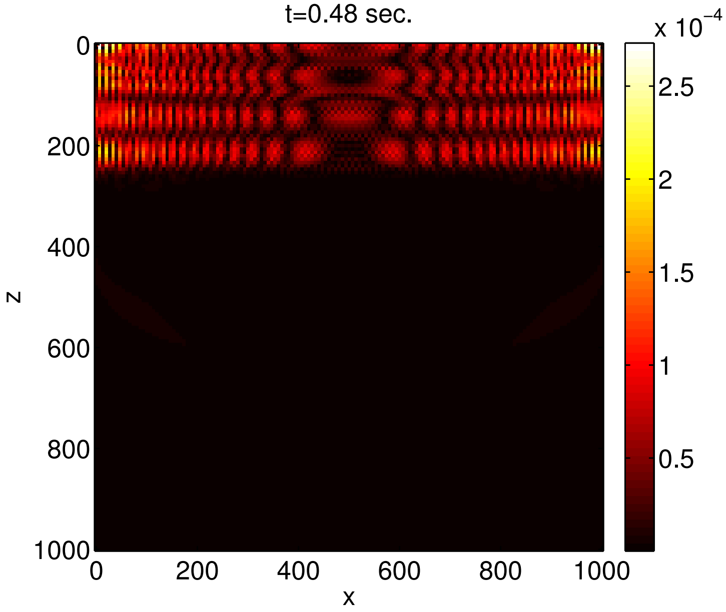

t=0.07 sec.
x 10^-4
z
x
0
200
400
600
800
1000
18
16
14
12
10
8
6
4
2


t=0.15 sec.
x 10^-4
0
200
400
600
800
1000
z
x
10
8
6
4
2
0


t=0.23 sec.
z
x
0
200
400
600
800
1000
x 10^-4
0
1
2
3
4
5
6
7
8
9

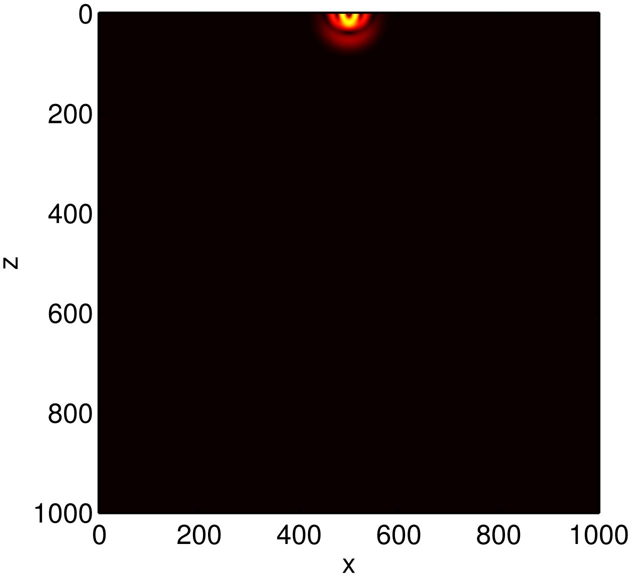

t=0.32 sec.
x 10^-4
4
3.5
3
2.5
2
1.5
1
0.5
0
200
400
600
800
1000
z
0
200
400
600
800
1000
x

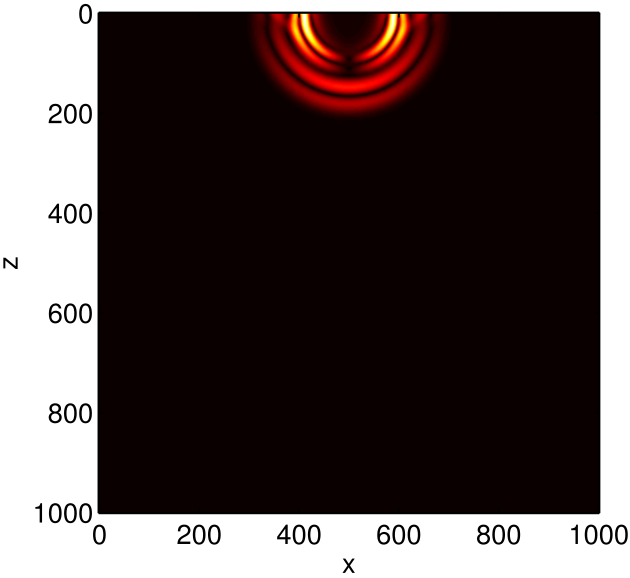

t=0.4 sec.
x 10^-4
3.5
3
2.5
2
1.5
1
0.5
0
200
400
600
800
1000
z
0
200
400
600
800
1000
x


t=0.48 sec.
x 10^{-4}
2.5
2
1.5
1
0.5
z
0
200
400
600
800
1000
x
0
200
400
600
800
1000

- Consider the case of a reflection strictly from a difference in Q_p and Q_s. The boundary is at $z = 500m$ and is again time-stepped using 4^{th} order LSERK.

	ρ	V_p	V_s	Q_p	Q_s
Ω_1	2.06	2400	1500	∞	∞
Ω_2	2.06	2400	1500	20	30


0
200
400
600
800
1000
z
0
200
400
600
800
1000
x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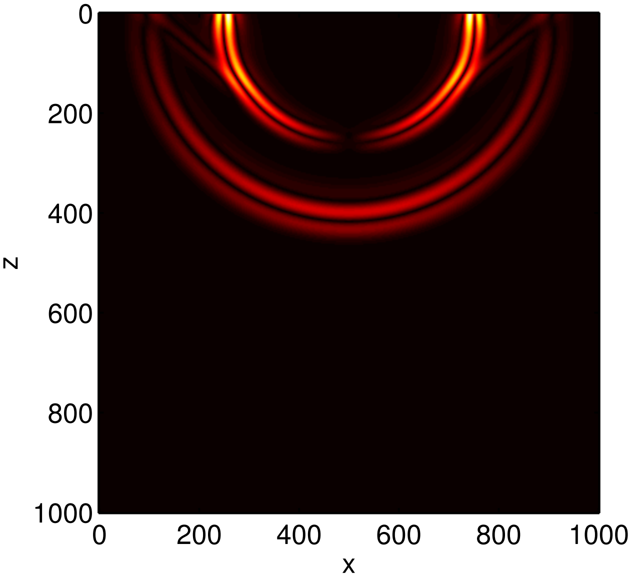
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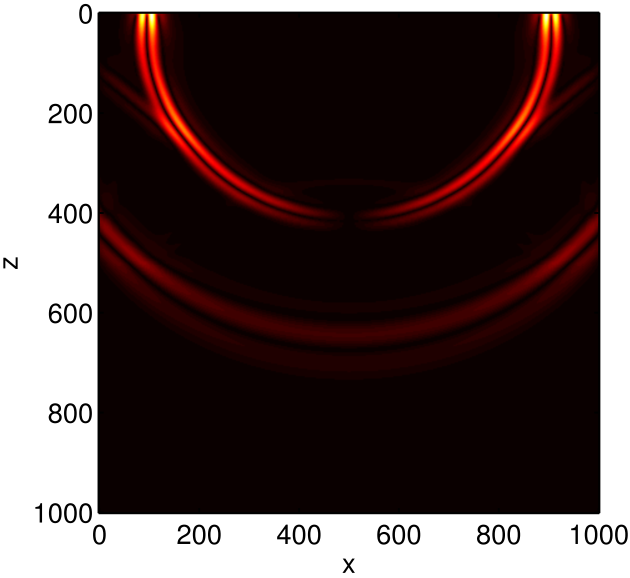
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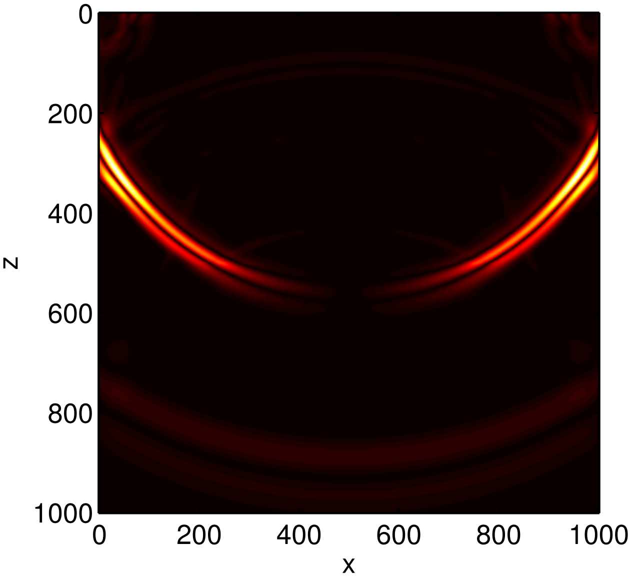
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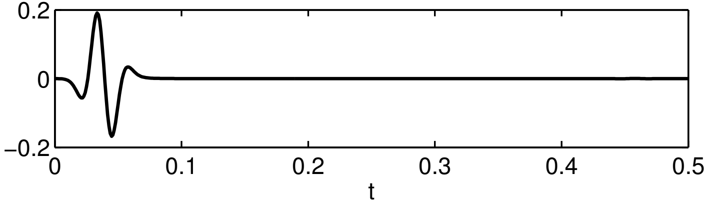
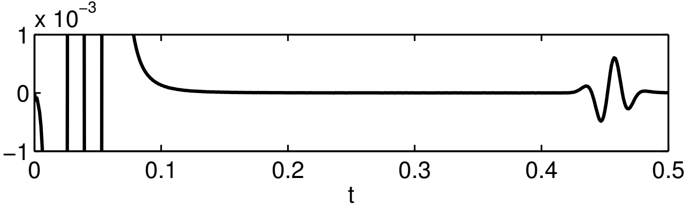


(a) Original trace.



(b) Clipped trace.

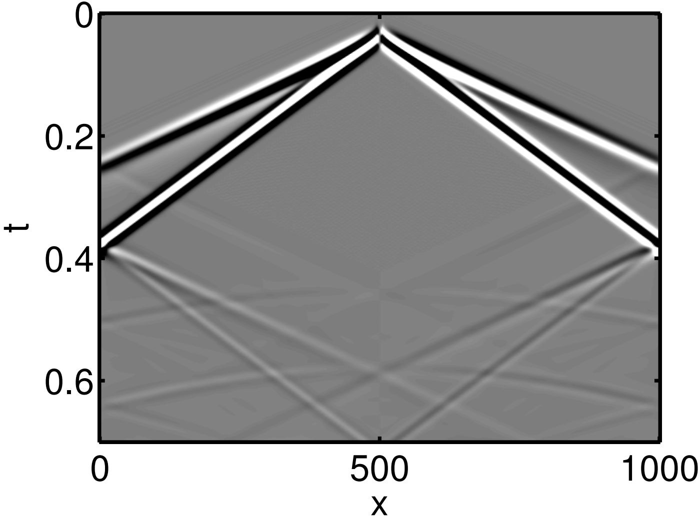


Figure: Horizontal displacement.

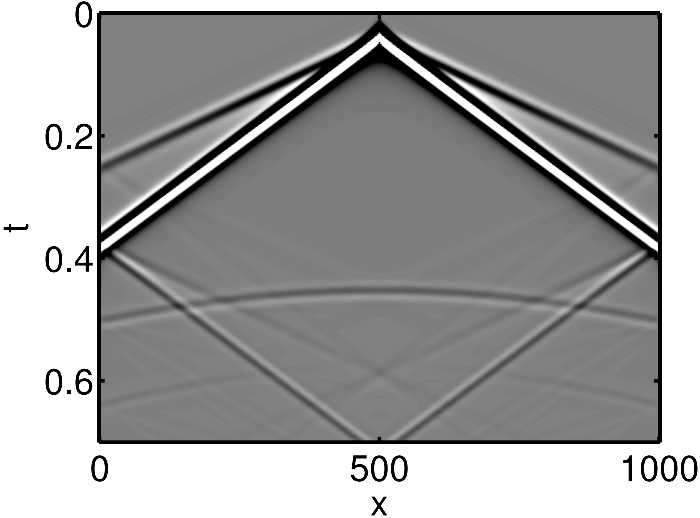


Figure: Vertical Displacement.

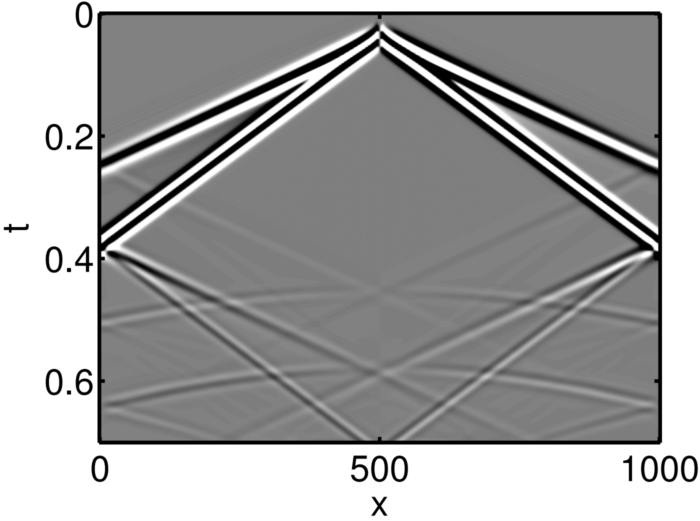


Figure: Horizontal velocity.

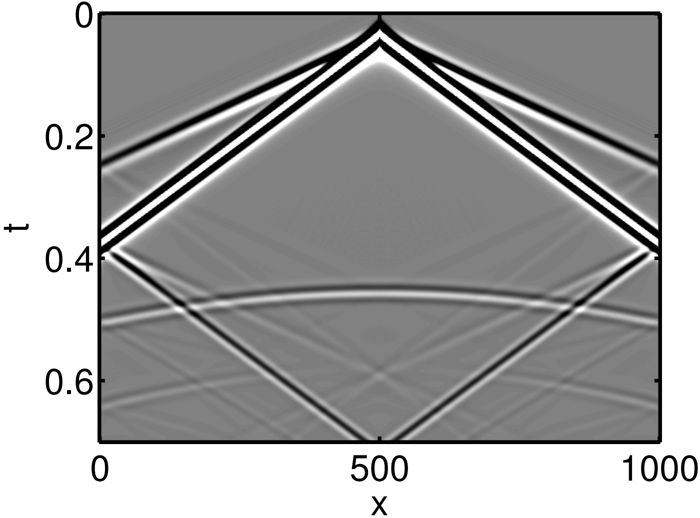


Figure: Vertical velocity.

Thank you!

- Chris Bird
- Michael Lamoureux
- Crewes
- Potsi
- mprime
- Pims
- Nserc