Instantaneous frequency computation: theory and practice

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Why study the instantaneous frequency?

Since the paper of Taner et al (1979) this attribute has been used to interpret seismic data and relate the interpretation of meaningful geological signatures.

Phase of the rectified trace parameter as an aid to seismic interpretation (B. Stebens, R. Parsons, D. Terral, R.T. Baumel and M. Yedlin) 6 patents granted to Conoco Inc. in: Australia, Belgium-France-Netherlands, Canada, West Germany, Great Britain, U.S.A [Feb. 1988].

Outline of Presentation:

1. Classical Instantaneous Frequency
2. Fomel's Improvement
3. Gabor and Cohen and Stockwell
4. Data examples
5. Conclusions

1. Classical Instantaneous Frequency

Start with

$$e^{i\omega t} = \cos(\omega t) + i\sin(\omega t)$$

Interpretation?

Inspiration for the analytic signal:

$$A(t) = f(t) + ig(t)$$

where $g(t)$ is the Hilbert transform of $f(t)$:

$$g(t) = \int_{-\infty}^{\infty} f(\tau) \frac{1}{\pi(t-\tau)} d\tau.$$

Interpretation?

Now define (by analogy):

Envelope: Env(t)= $\sqrt{f(t)^2 + g(t)^2}$

Phase: $\Phi(t) = \tan^{-1}\left[\dfrac{g(t)}{f(t)}\right]$ and

Frequency: $\Omega(t) = \dfrac{d\Phi(t)}{dt}$

$$= \frac{f(t)\dfrac{dg(t)}{dt} - g(t)\dfrac{df(t)}{dt}}{f(t)^2 + g(t)^2}$$

$$\Omega(t) = \frac{n(t)}{d(t)} = \frac{f(t)\frac{dg(t)}{dt} - g(t)\frac{df(t)}{dt}}{f(t)^2 + g(t)^2}$$

has serious problems!

These are:

1. Numerator problems – non-physical results
2. Denominator problems – can lead to instability

Need to fix these!

2. Fomel's Improvement (2007)

Write $f_{inst}(t) = \dfrac{\Omega(t)}{2\pi} = \dfrac{n(t)}{2\pi d(t)}$ in discrete

matrix-vector form: $\mathbf{f} = \mathbf{D}^{-1}\mathbf{n}$

Instability is obvious!

Stabilize this using Tikhonov regularization:

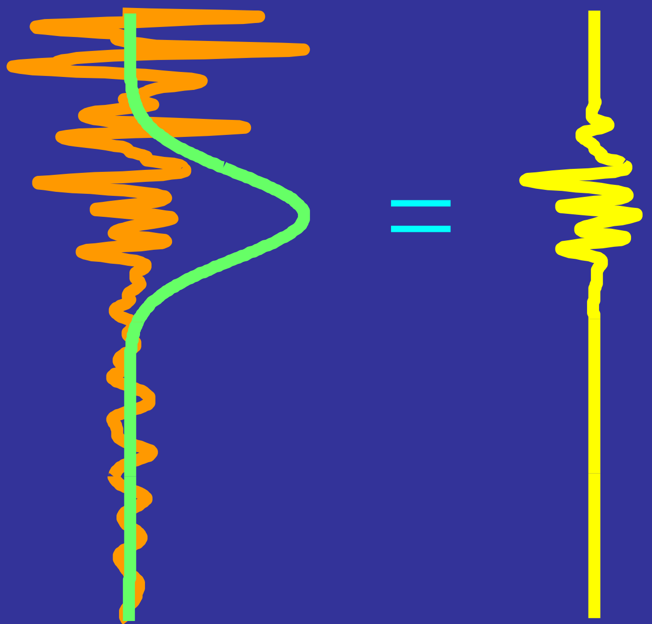
$\mathbf{f} = \mathbf{D}^{-1}\mathbf{n}$ becomes

$\mathbf{f}_{loc} = \left(\mathbf{D} + \varepsilon^2\mathbf{R}\right)^{-1}\mathbf{n}$, where $\mathbf{R}$ is a regularization operator

3. Gabor, Cohen and Stockwell

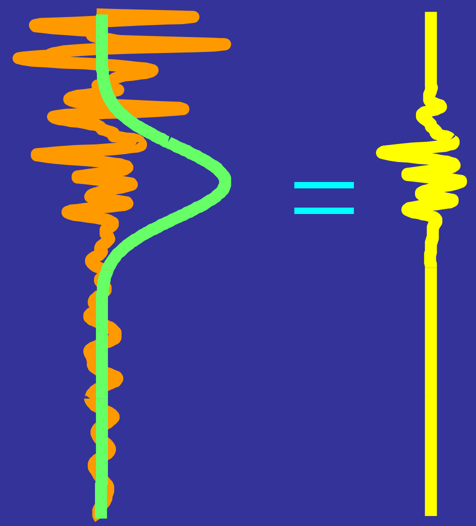
Dennis Gabor proposed (1946) proposed the expansion of a wave in terms of “Gaussian wave packets”. The mathematics for the continuous Gabor transform emerged quickly.

The discrete Gabor transform emerged in the 1980s due to Bastiaans.

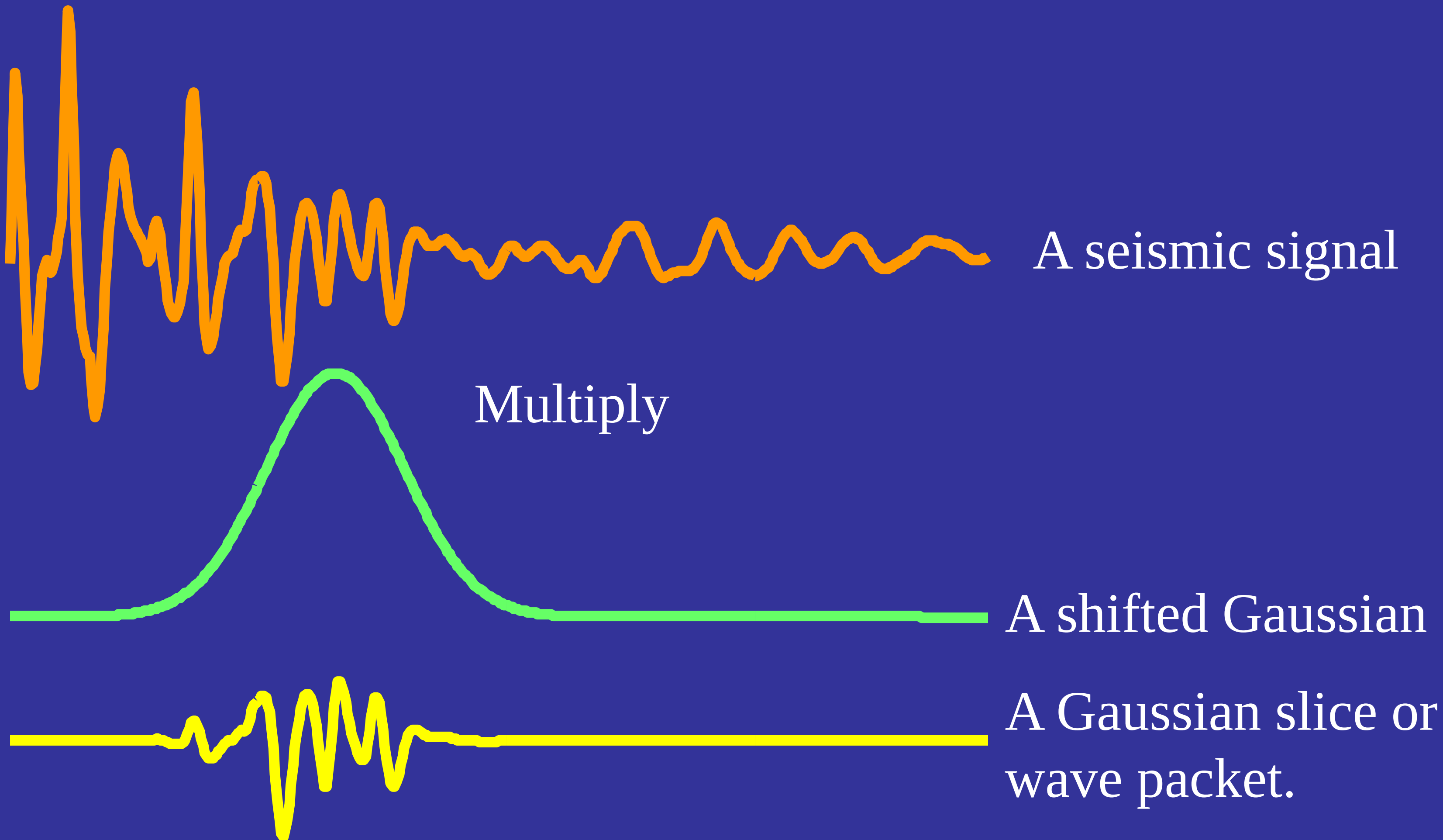


The Gabor Idea

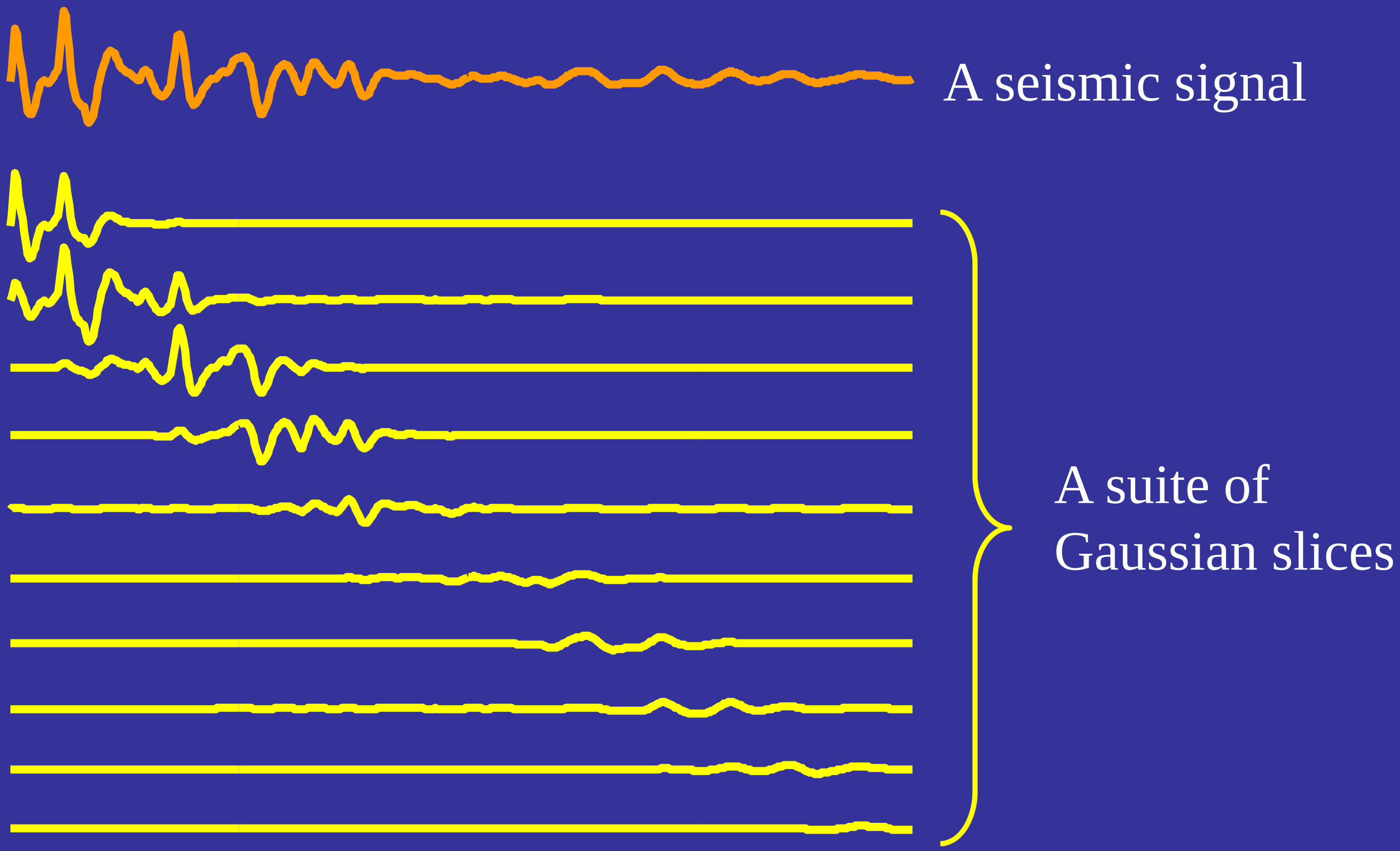
- Dennis Gabor proposed (1946) proposed the expansion of a wave in terms of “Gaussian wave packets”. This is effectively a “local” Fourier transform.



The Gabor Idea



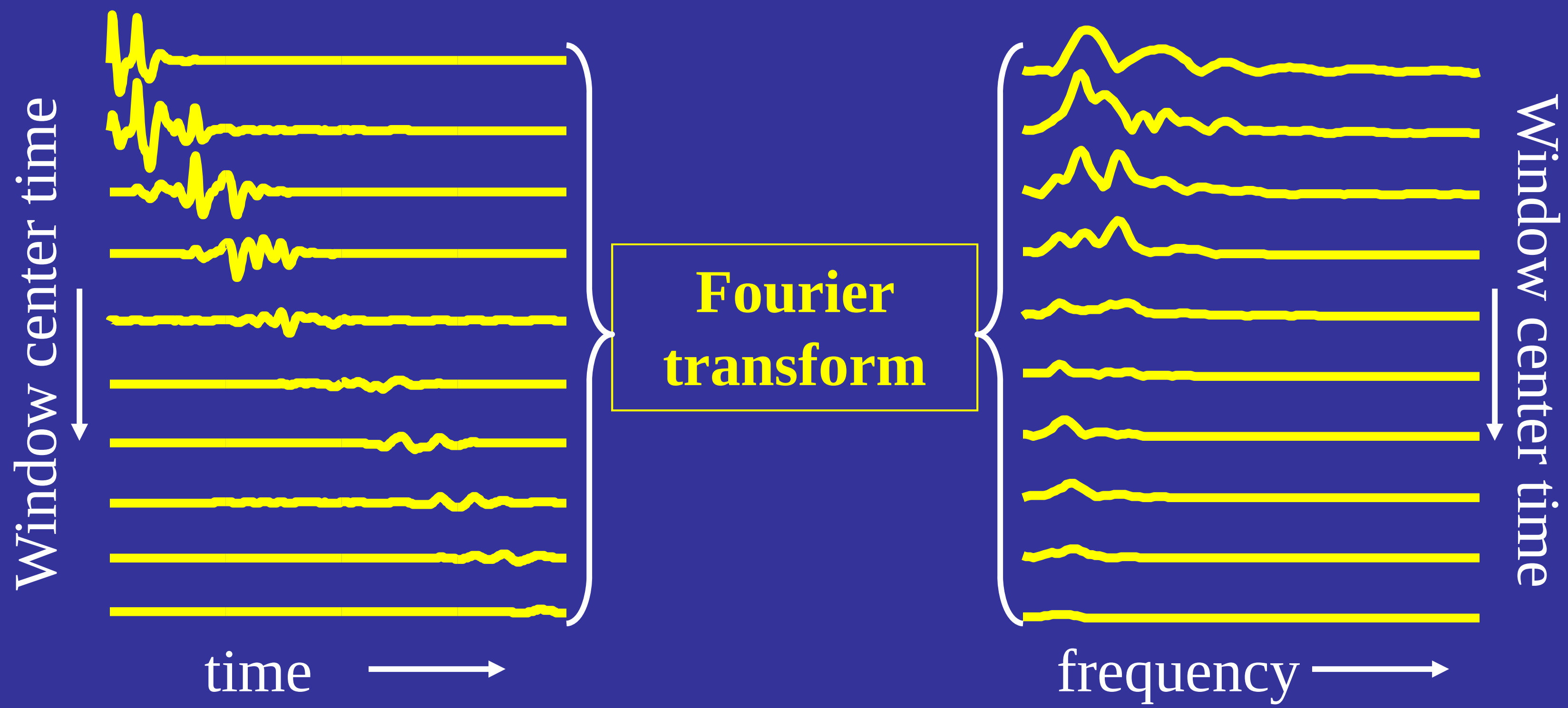
The Gabor Idea



Remarkably, the suite of Gaussian slices can be designed such that they sum to recreate the original signal with high fidelity.

The Gabor Idea

The Gabor transform

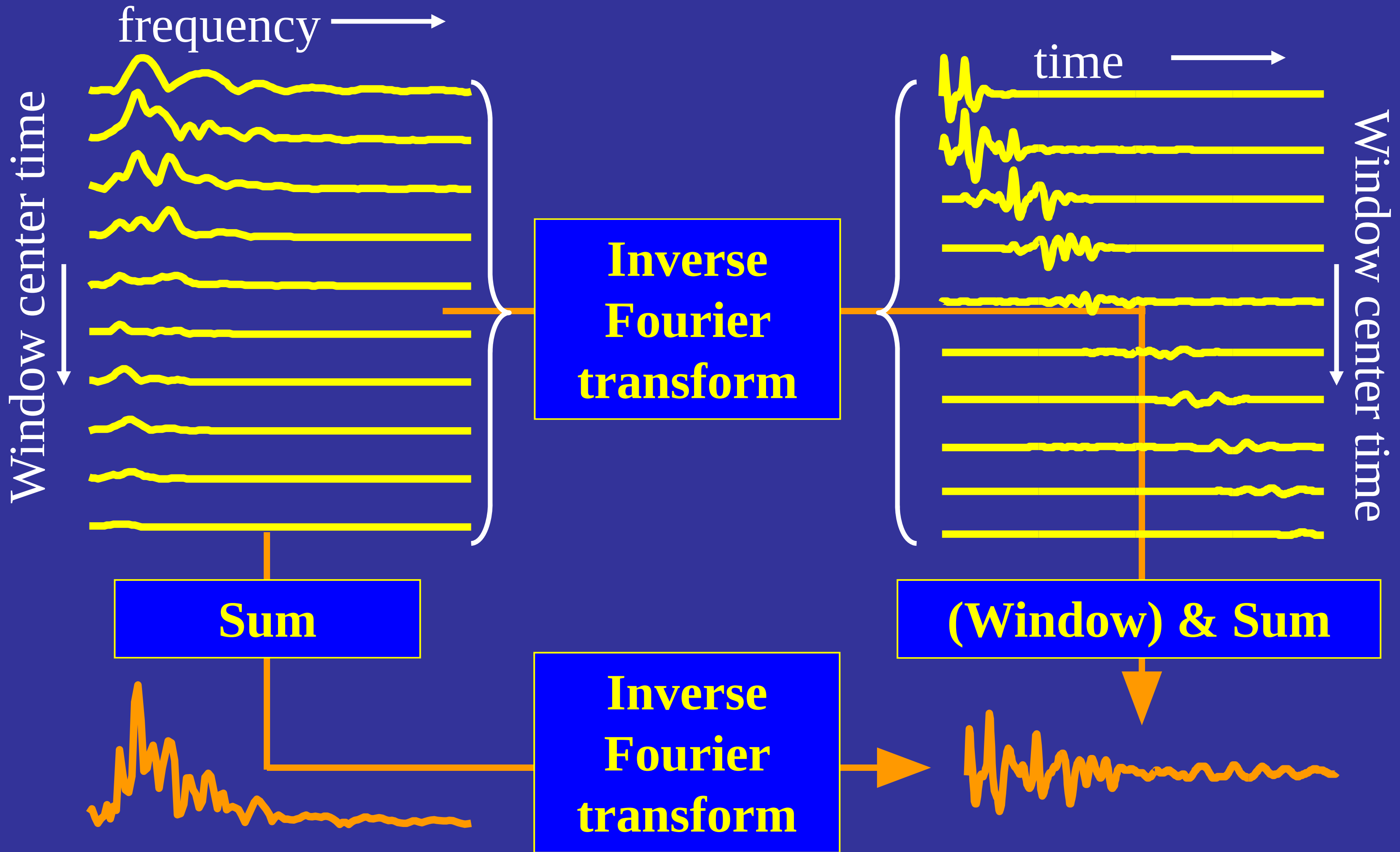


Suite of Gabor slices

Gabor spectrum or Gabor transform

The Gabor Idea

The inverse Gabor transform done two ways



The Continuous Gabor Transform

$$S_{Gabor}(t, f) = \int_{-\infty}^{\infty} g(\tau)\, e^{-(\tau - t)^2/2\sigma^2} e^{-i\pi 2 f \tau}\, d\tau$$

where σ is fixed. $report\ typo - eq.16 - -no\ |f|$

Following Cohen's theorem (1995) eqns (7.52-7.54) [first moment or power centroid]:

$$f_{loc}(t) = \frac{\int_0^{f_{NYQ}} f \left|S_{Gabor}(t, f)\right|^2 df}{\int_0^{f_{NYQ}} \left|S_{Gabor}(t, f)\right|^2 df} \rightarrow f_{inst}(t) \;\; as\ \sigma \rightarrow 0$$

But the variance of this mean tends to infinity!
There is clearly a problem with the concept of instantaneous frequency!

We can compute the local frequency by using the Stockwell transform instead of the Gabor transform in the centroid calculation.

The Stockwell transform is given by

$$S_{Stockwell}(t, f) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(\tau) \, | f | \, e^{-(\tau - t)^2 / 2 f^2} e^{-i\pi 2 f \tau} d\tau$$

4. Data Examples

We will now apply the foregoing theory to four data examples:

1. Chirp: 10 – 100 Hz;
2. Two sine waves: 40 Hz + 60 Hz;
3. Nonstationary seismic trace;
4. Quarry blast.

Figure 1.

Chirp Signal

Instantaneous Frequency

f_{loc} Fomel

f_{loc} Gabor

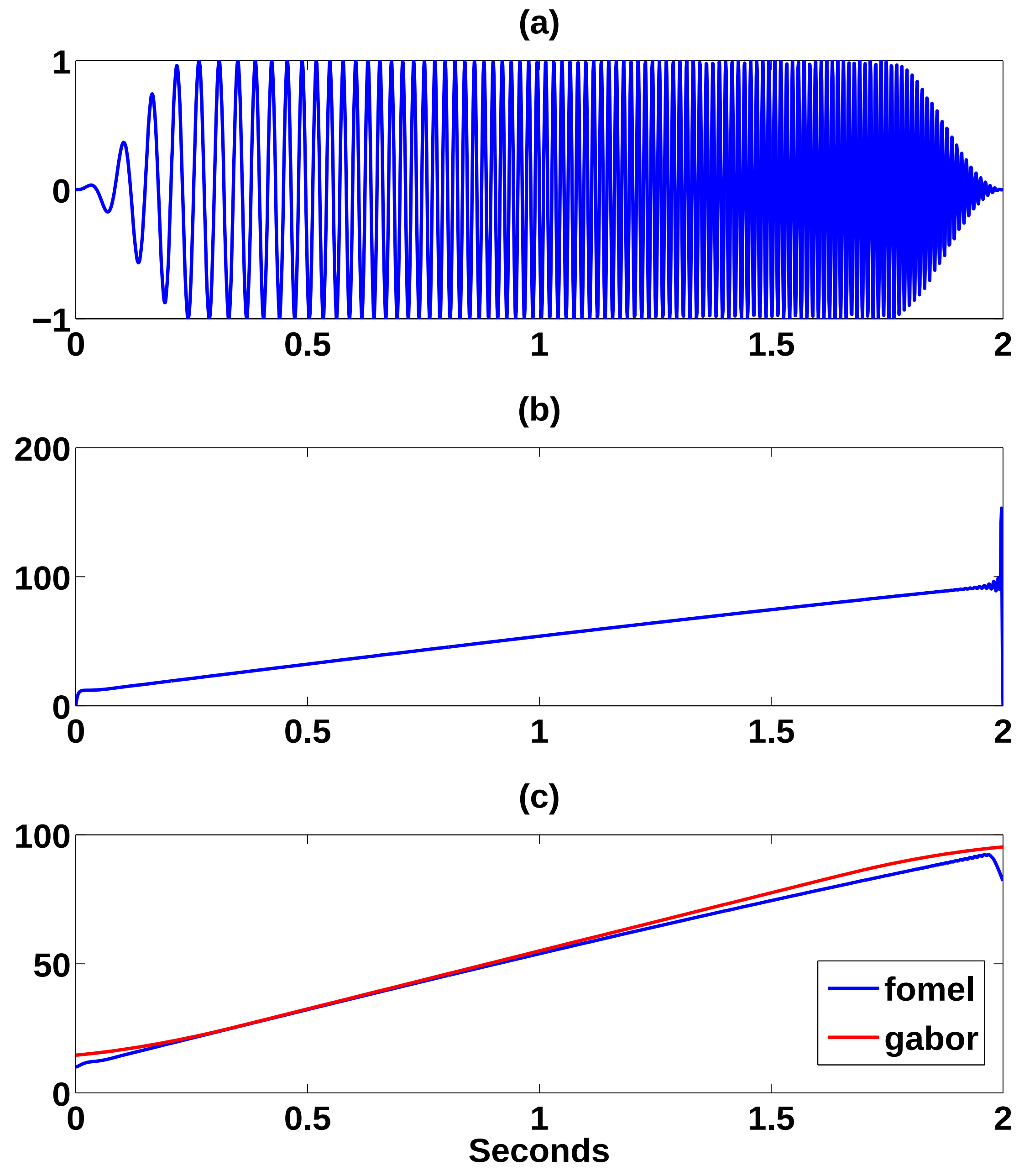


Figure 2.

Two sine waves
40 and 60 Hz

Instantaneous
Frequency

f_{loc} Fomel
f_{loc} Gabor

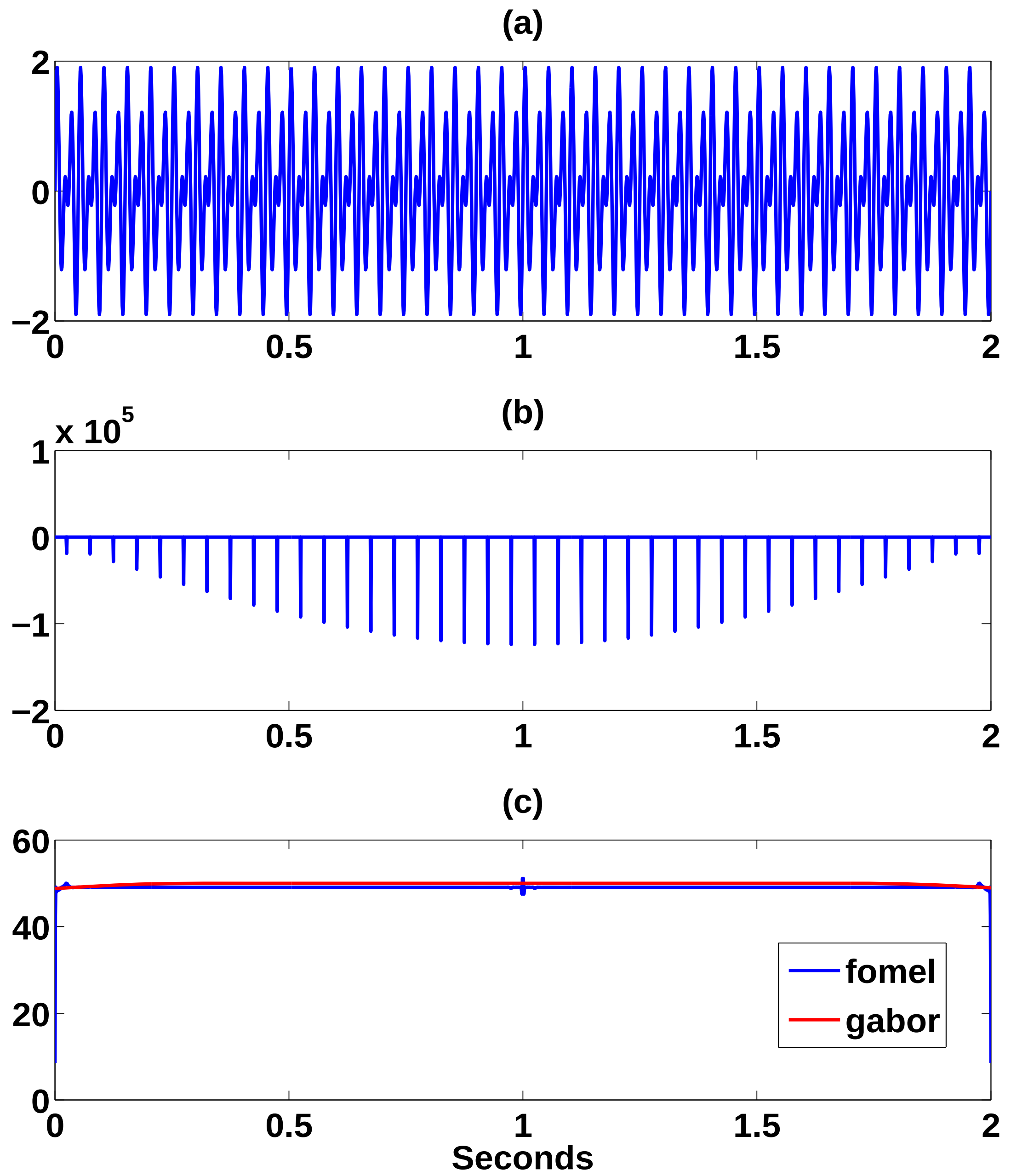
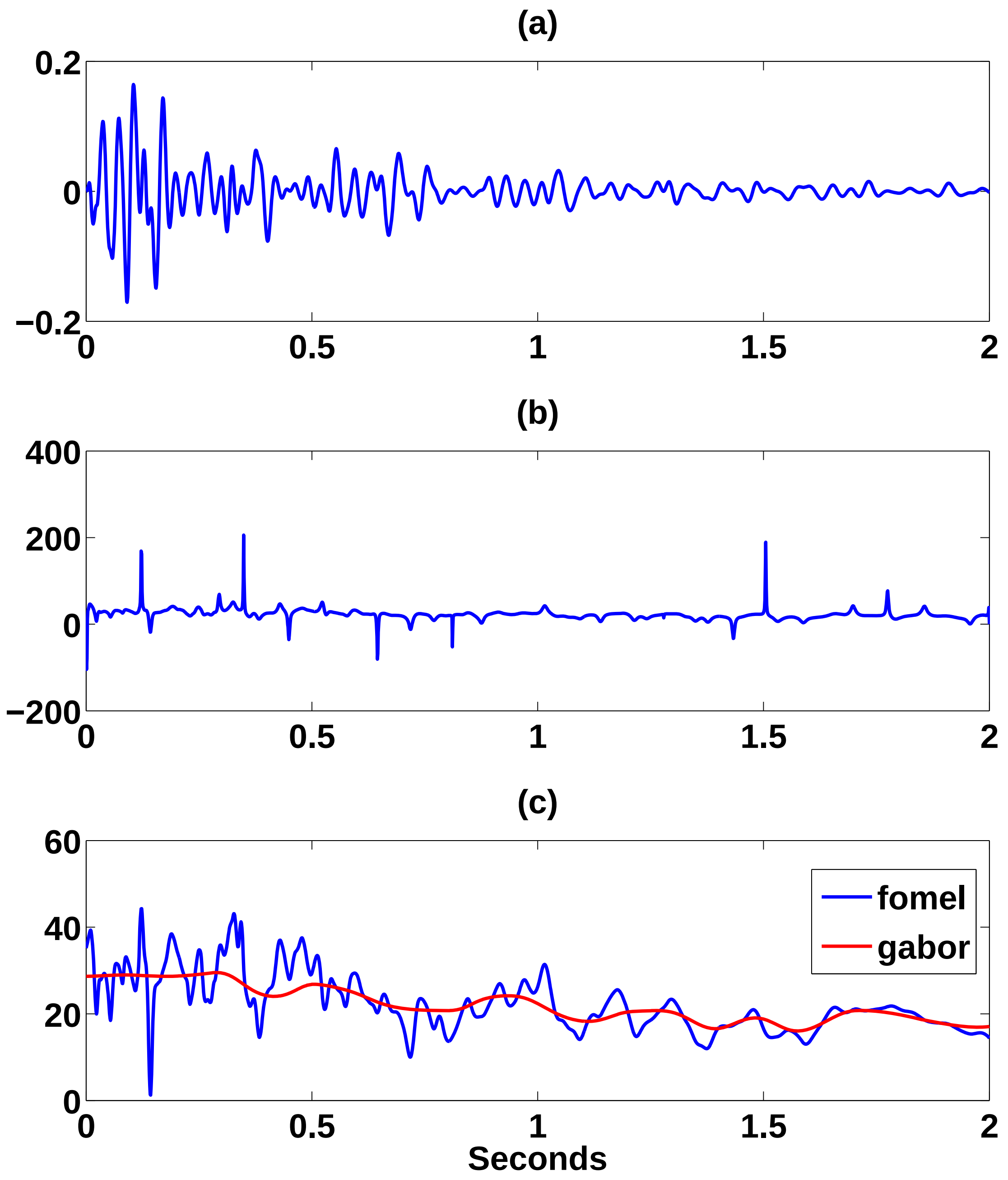


Figure 3.

Non-stationary seismic trace

Instantaneous Frequency



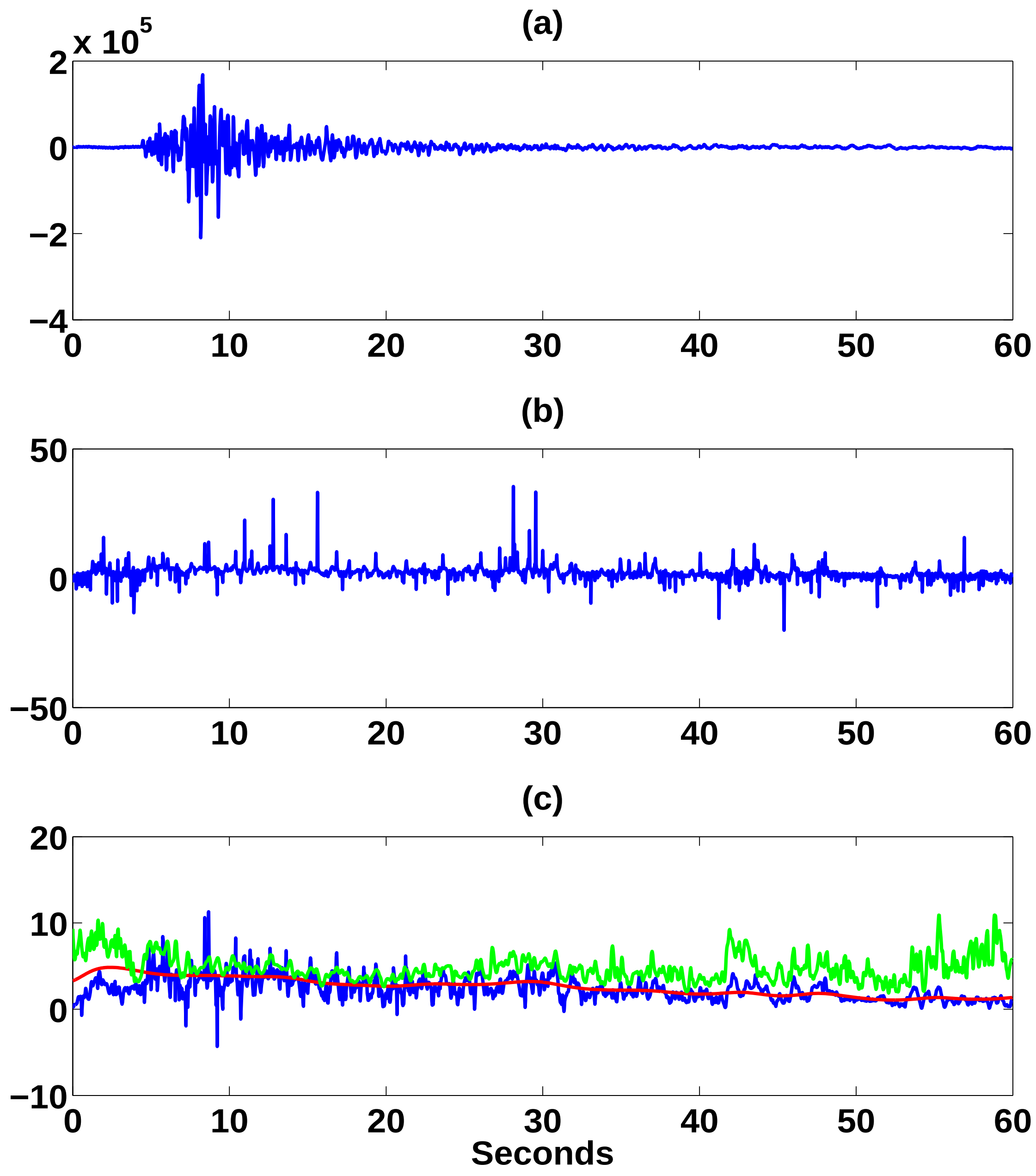
f_{loc} Fomel

f_{loc} Gabor

Figure 4.

Quarry Blast

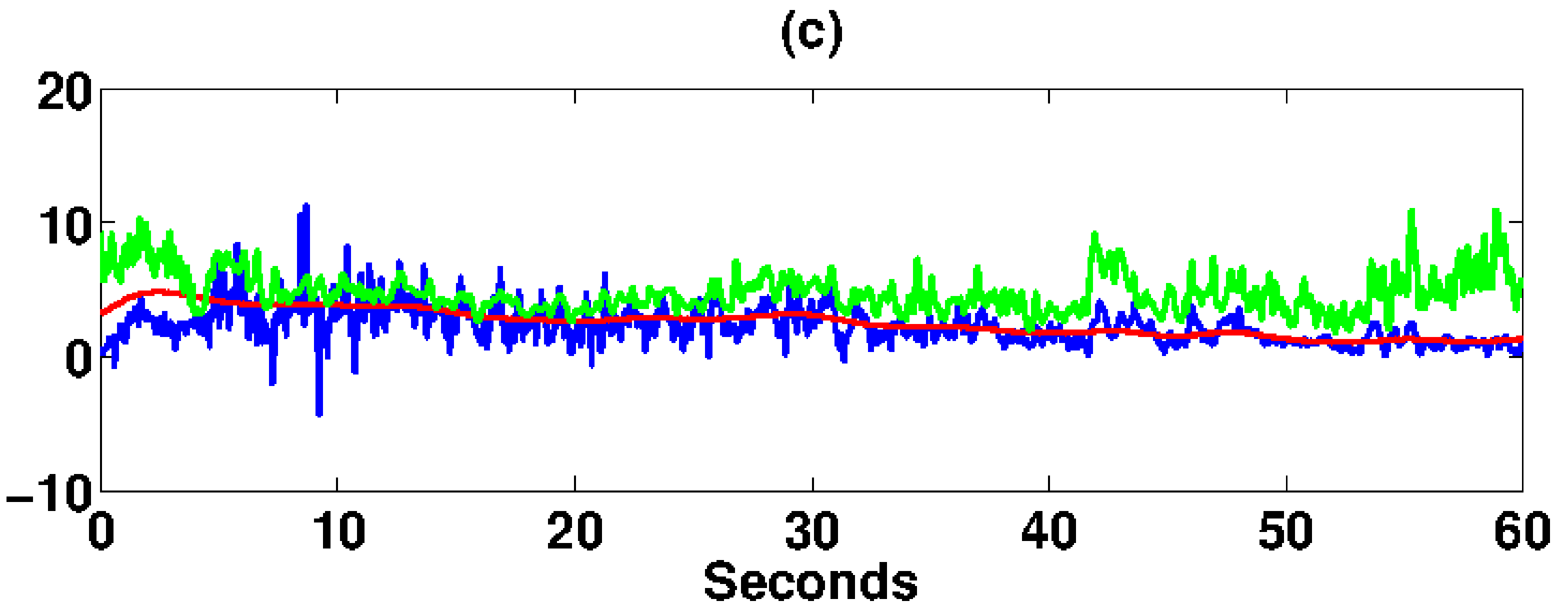
Instantaneous Frequency



f_{loc} Fomel
f_{loc} Gabor
f_{loc} Stockwell

Zoom of Blast Analysis (c)

f_{loc} Fomel
f_{loc} Gabor
f_{loc} Stockwell



5. Conclusions

1. There is instability in the instantaneous frequency – it can go negative or have a value greater than the Nyquist frequency;
2. Cohen's theorem provides an intuitive connection between instantaneous frequency and the Gabor spectrum;
3. We need to find an objective means of choosing the Gaussian window;
4. We need to find an objective means for choosing the optimum amount of regularization in the Fomel method.