Particle Swarms for Numerical Wave Equation

Michael Lamoureux & Heather Hardeman

CREWES Sponsors Meeting 2017

Acknowledgements

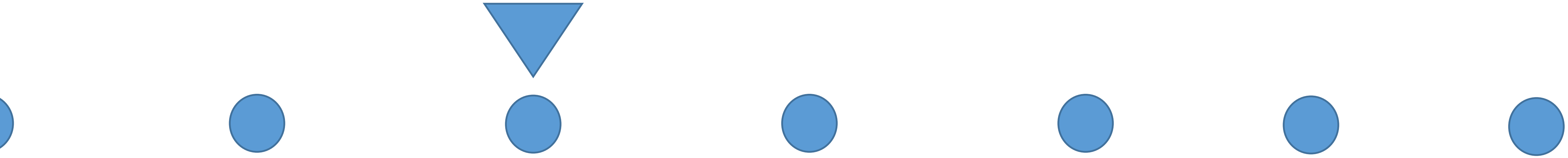
- Thanks to:

- CREWES sponsors
- CREWES staff and students
- NSERC CRD and Discovery grants
- Nvidia for GPU funding
- Probabilists at UBC

Outline

- Motivation – stochastics, random walks, computation
- Diffusion – random walks and heat equation
- Wave equation – directed random walks
- Parallelization
- Conclusions

Motivation – a random walk, one particle



A particle randomly moving on a grid

Motivation - Stochastics

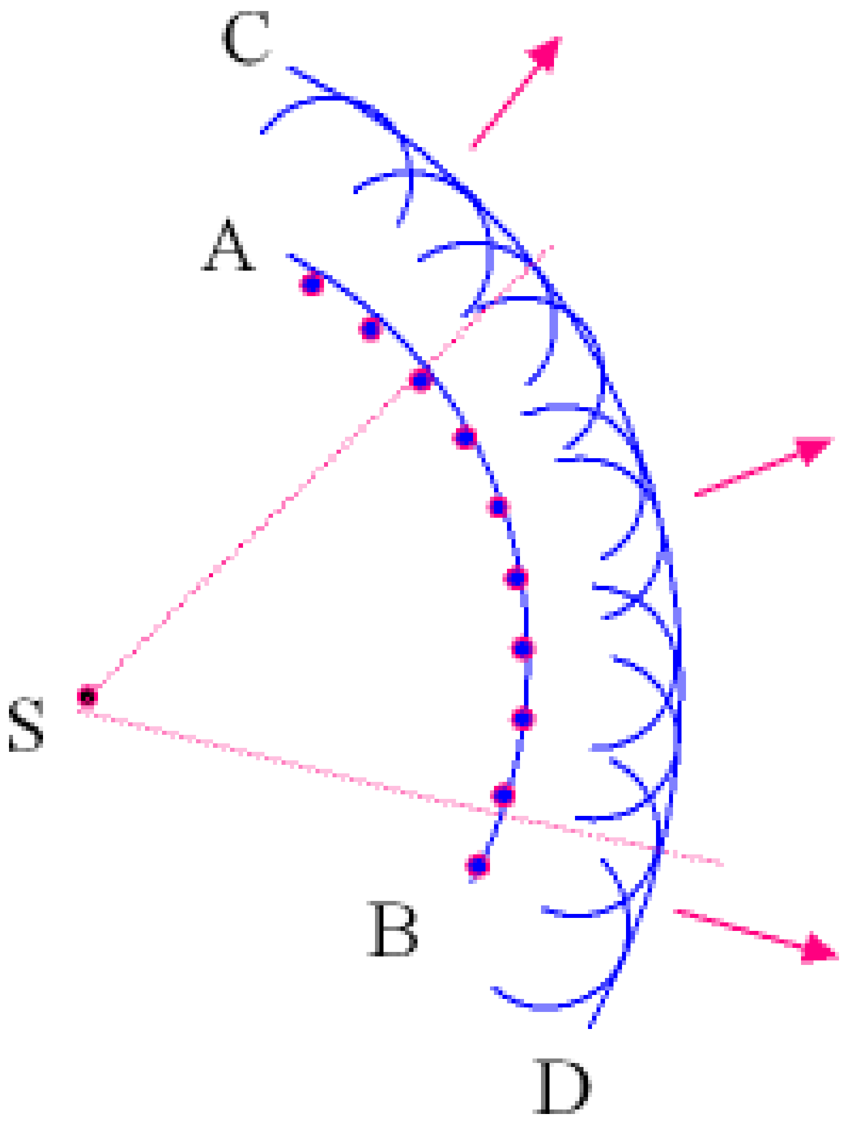
- Stochastic equations generalize PDEs

$$d\pi_t(\varphi) = \pi_t(A\varphi)dt + (\pi_t(h^*\varphi) - \pi_t(h^*)\pi_t(\varphi))(dY_t - \pi_t(h)dt)$$

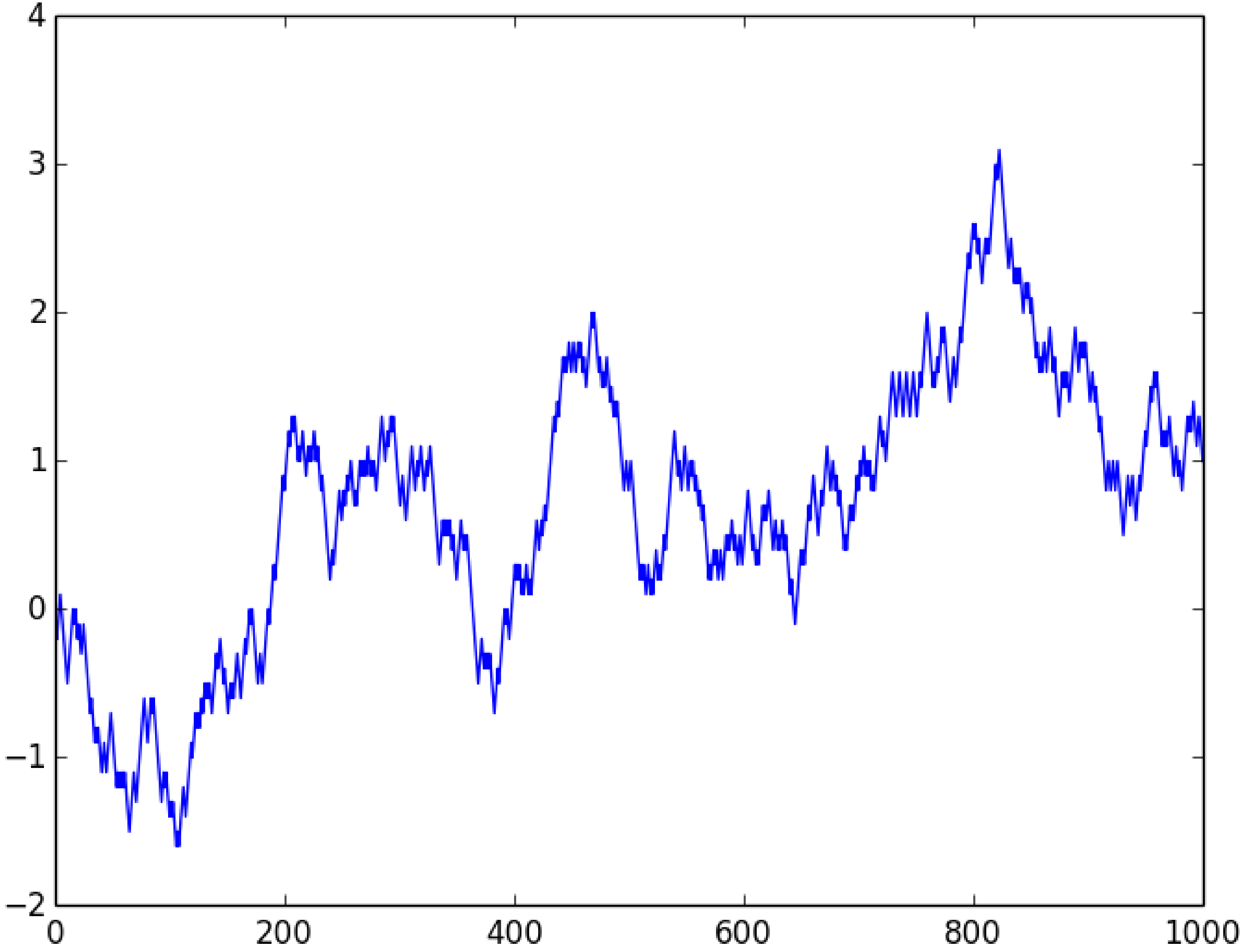
- E.g. Financial math, branching processes, probability
- Numerical solutions using random motion of “particles”
- Borrow these ideas for our work

Motivation – Computation of waves via particles

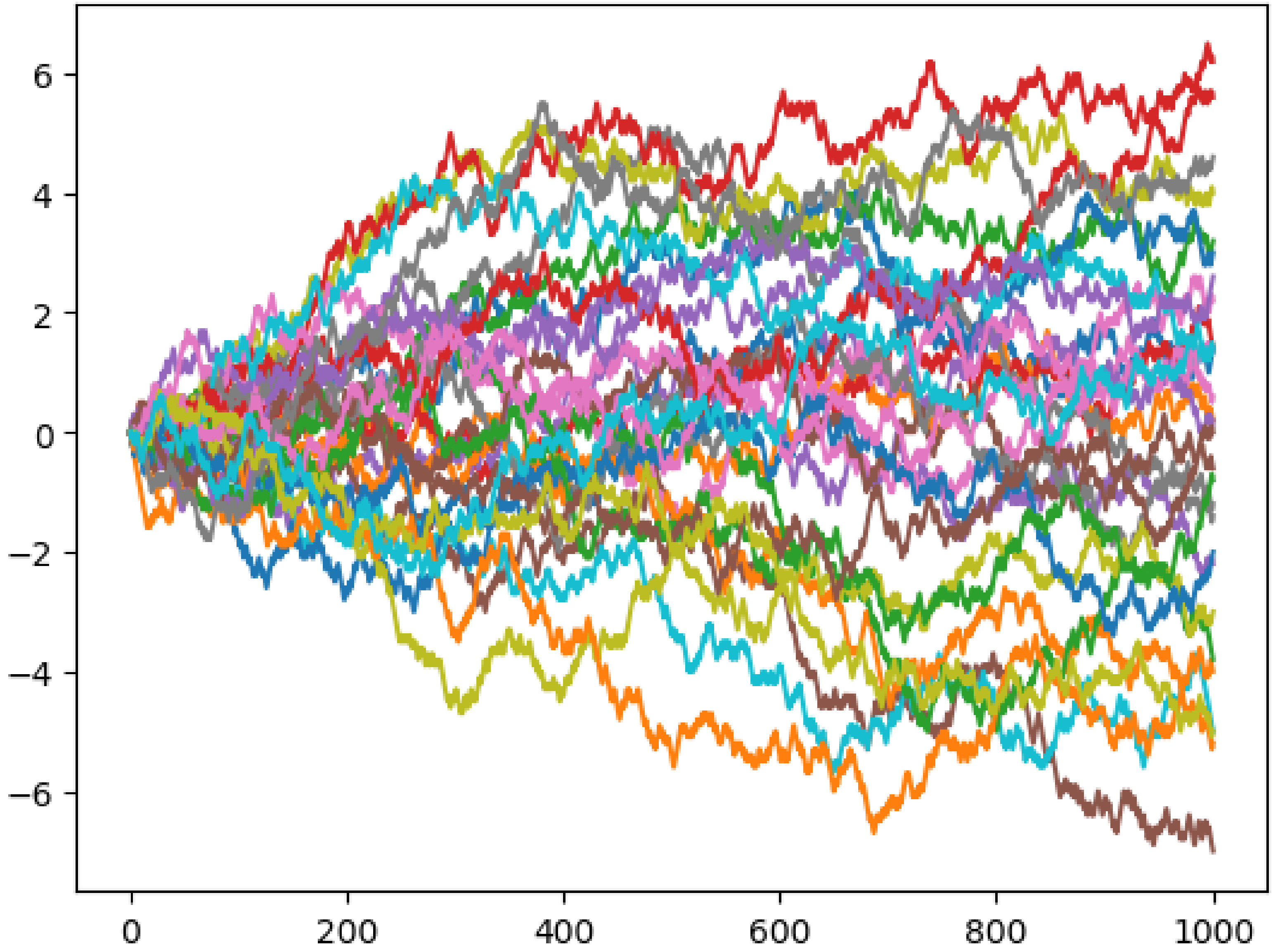
- No Grid
- No computational boundary
- Particles track the wavefront
  - Avoids "curse of dimension"
- Particles moves independently
  - Highly parallelizable (GPU)



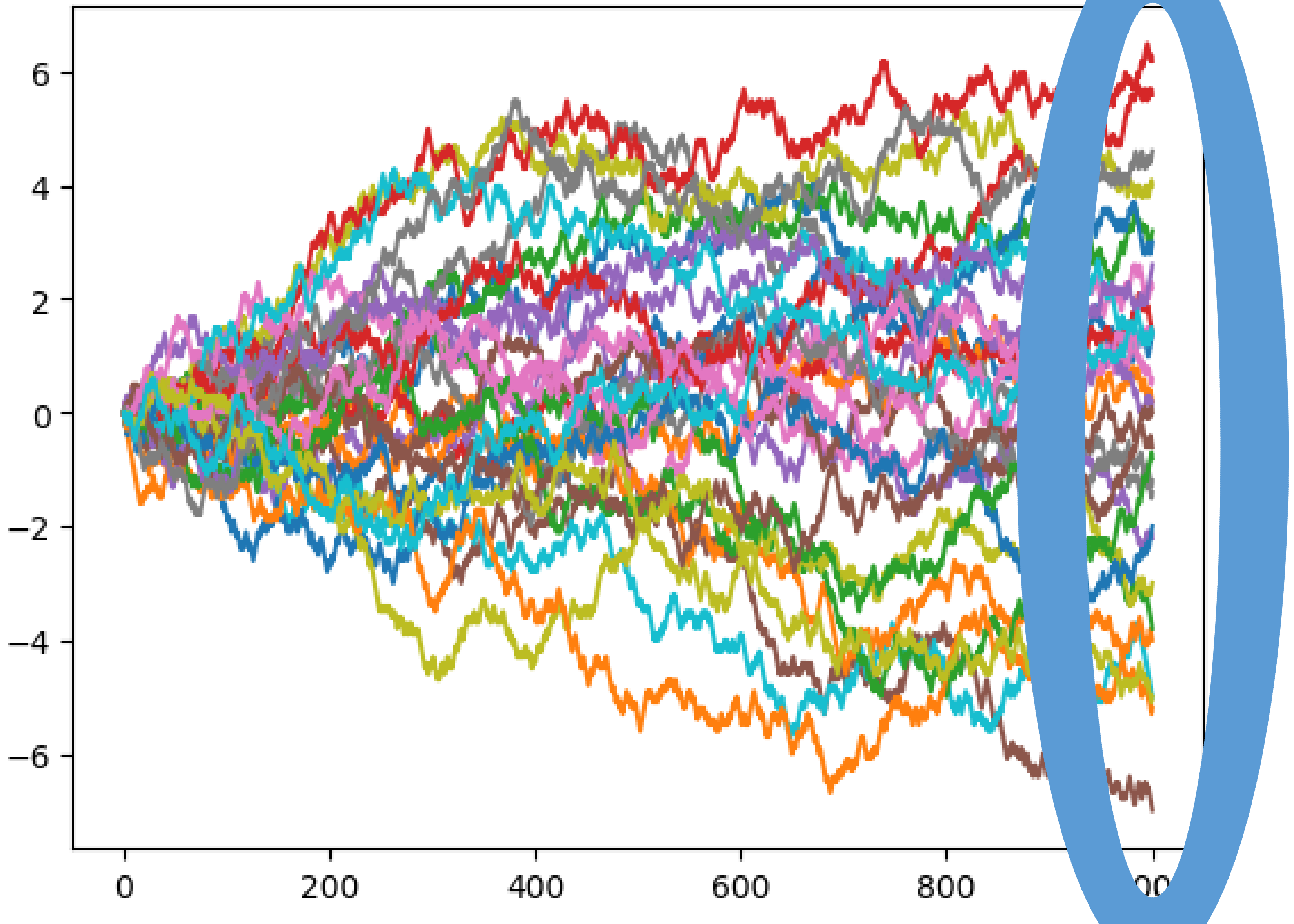
Example– a random walk in 1D



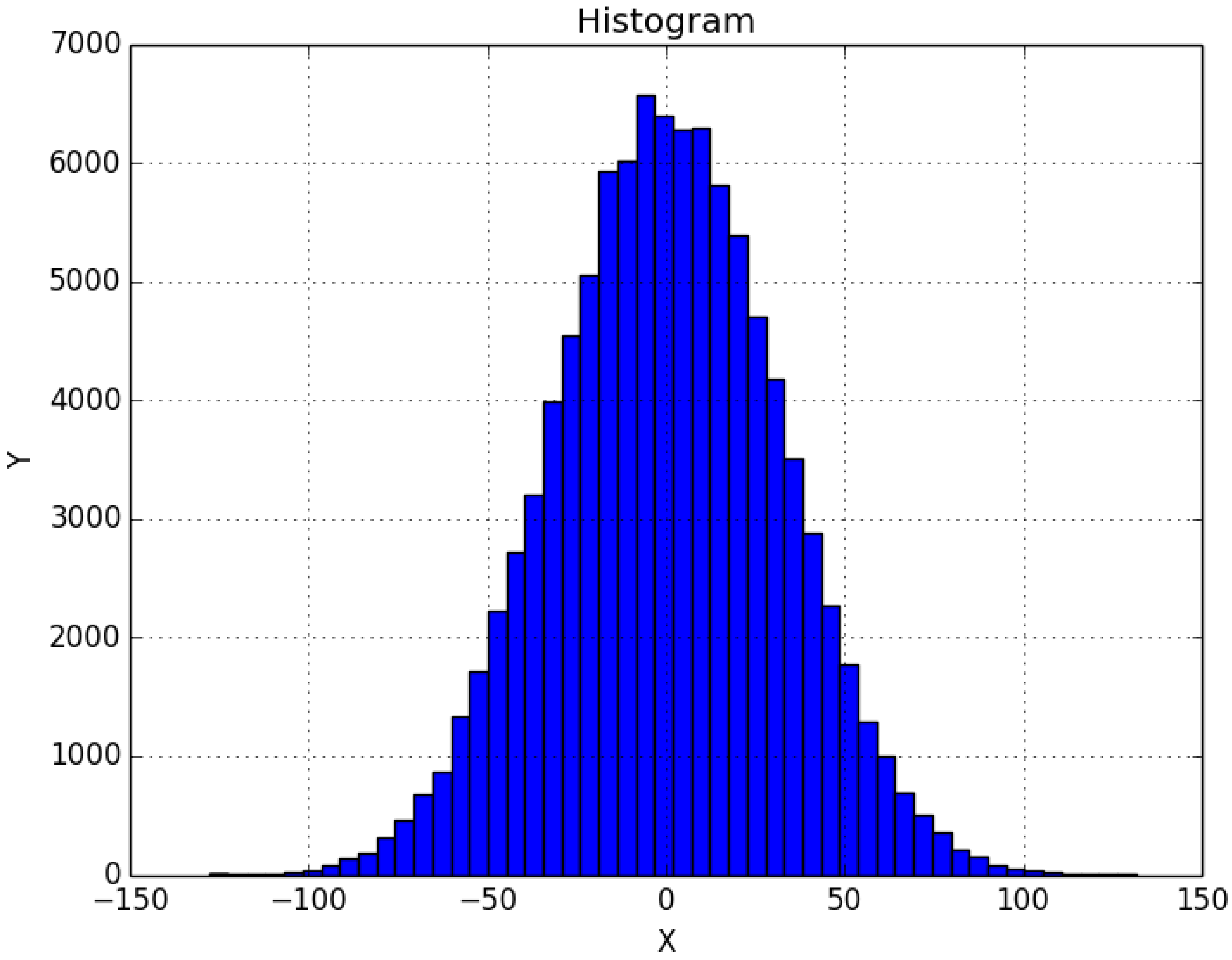
Example – many random walks in 1D



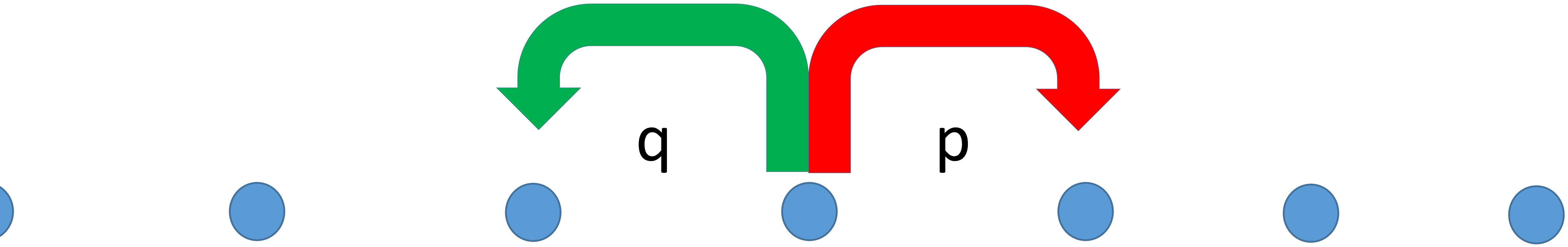
Example– examine endpoints of the paths



Example results – endpoints line up as a Gaussian



Derivation – from particles to diffusion



$v(x,t)$ = Number of particles at point x, at time t

p,q = probability of jumping right or left

$$v(x,t+\tau) = pv(x-\delta,t) + qv(x+\delta,t)$$

Derivation– diffusion equation

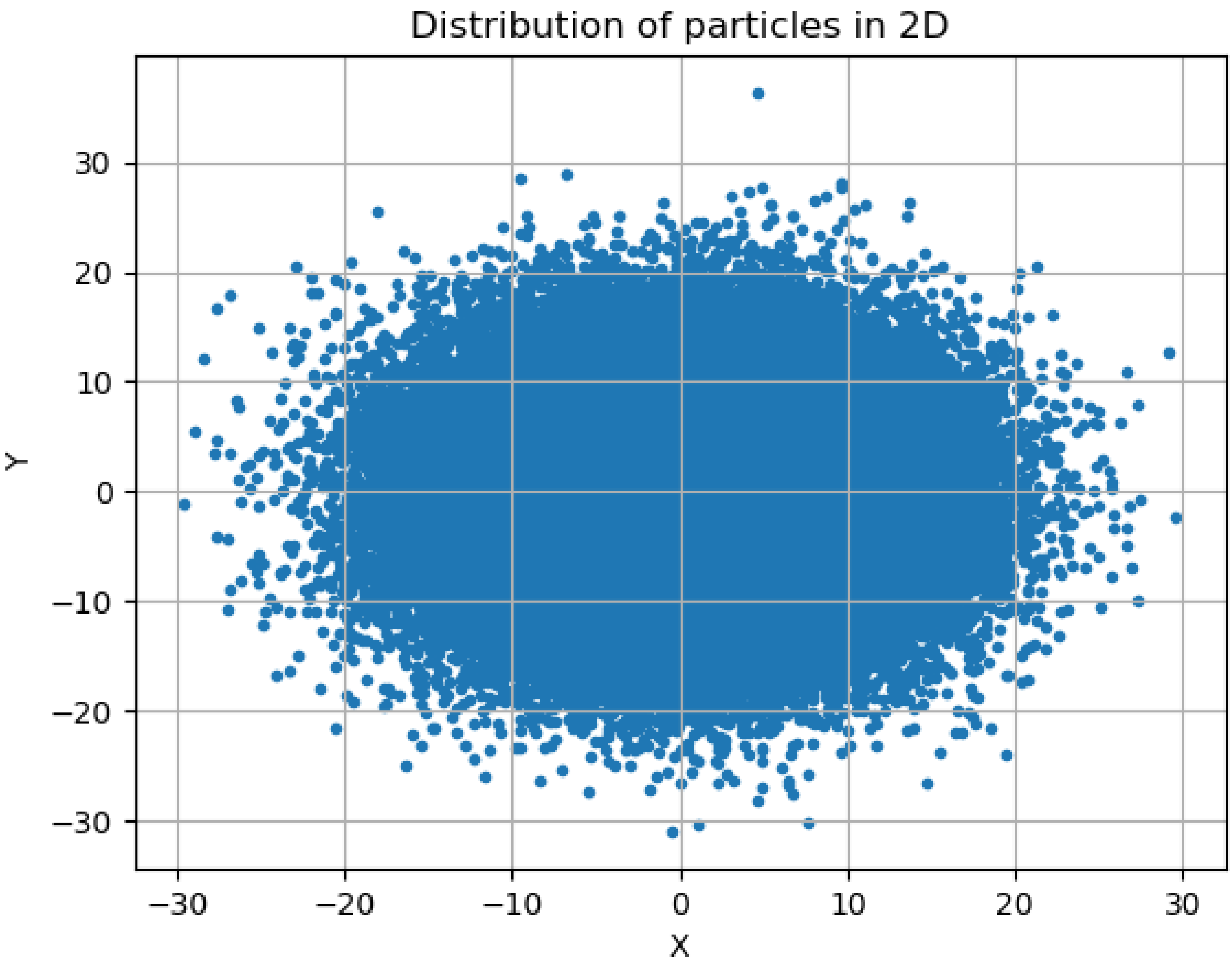
$$v(x, t+\tau) = pv(x-\delta, t) + qv(x+\delta, t)$$

$$v_t(x,t) = [(q-p)\frac{\delta}{\tau}]v_x(x,t) + \frac{1}{2}[\frac{\delta^2}{\tau}]v_{xx}(x,t)$$

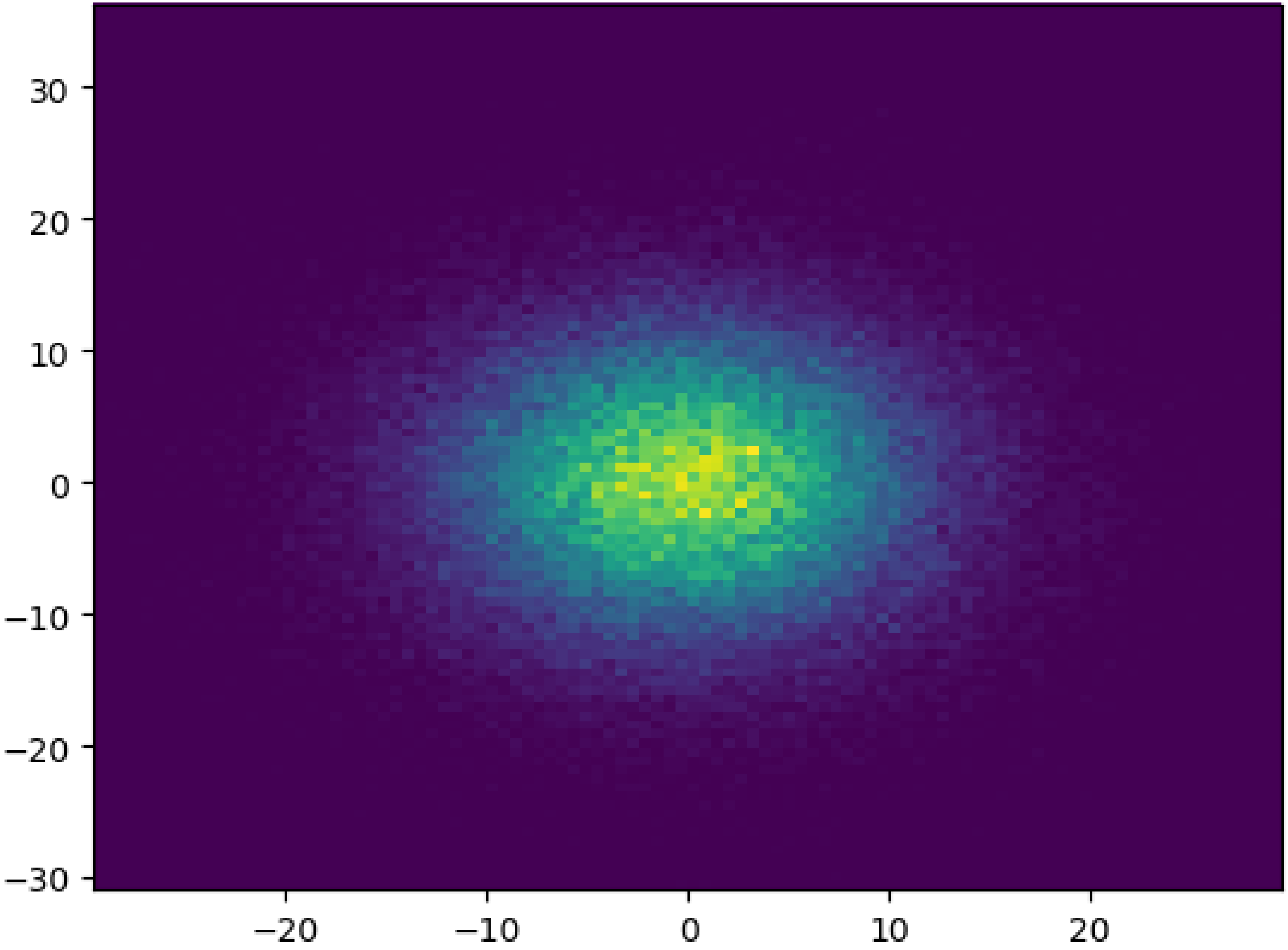
$$v_t(x,t) = -b[\frac{\delta^2}{\tau}]v_x(x,t) + \frac{1}{2}[\frac{\delta^2}{\tau}]v_{xx}(x,t)$$

$$v_t(x,t) = -cv_x(x,t) + \frac{1}{2}Dv_{xx}(x,t)$$

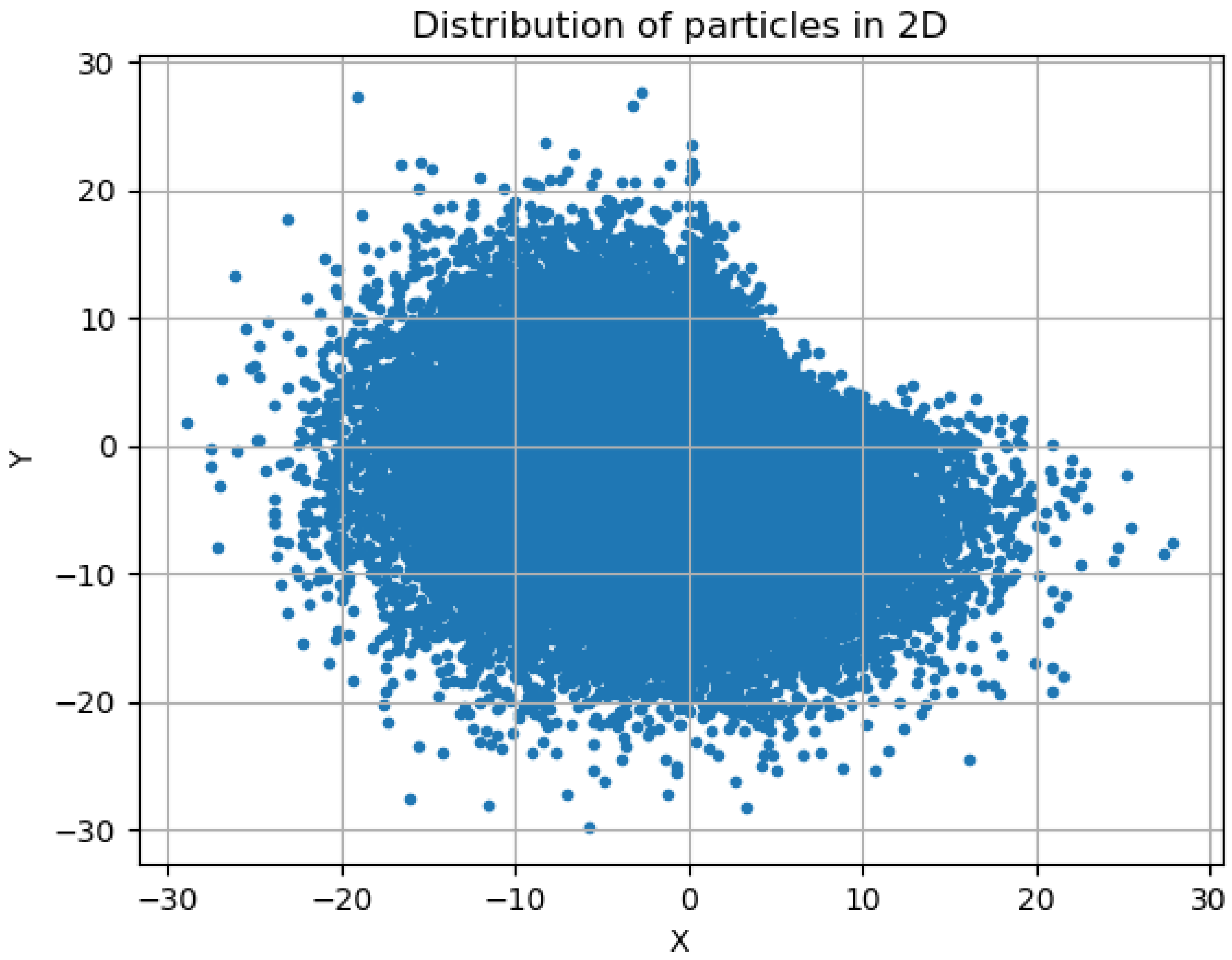
Example – diffusion simulation in 2D



Example – diffusion in 2D, heat map



Example – variable diffusion coefficient



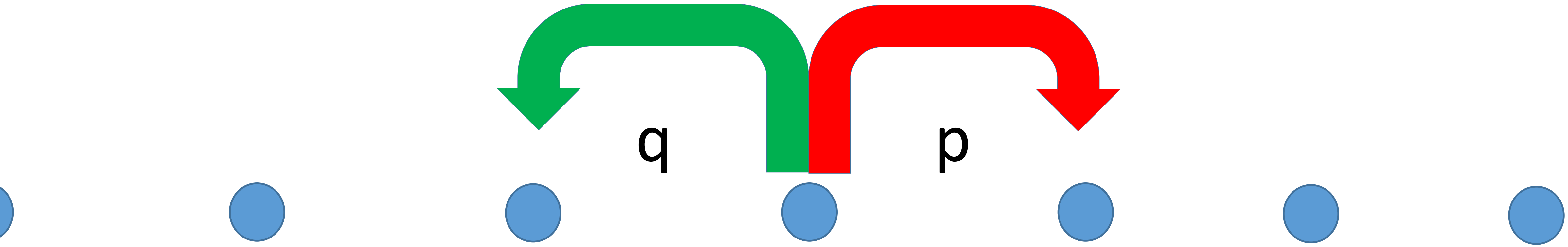
Wave equation

- We expect the particles to model the Green's function
- e.g. in 2D:

$$G^{(1)}(\vec{r}, t) = \begin{cases} \frac{1}{2\pi c} \frac{1}{\sqrt{c^2t^2 - r^2}}, & \text{for } ct > r \\ 0, & \text{otherwise} \end{cases}$$

- There is no wave, there is no frequency

Wave equation– via random particles



$v(x,t)$ = Number of particles at point x, at time t

p,q = probability of jumping right or left

= correlated with previous jump (inertia)

Wave equation – derivation

Left-going $$\alpha(x, t+\tau) = p\alpha(x-\delta, t) + q\beta(x-\delta, t)$$

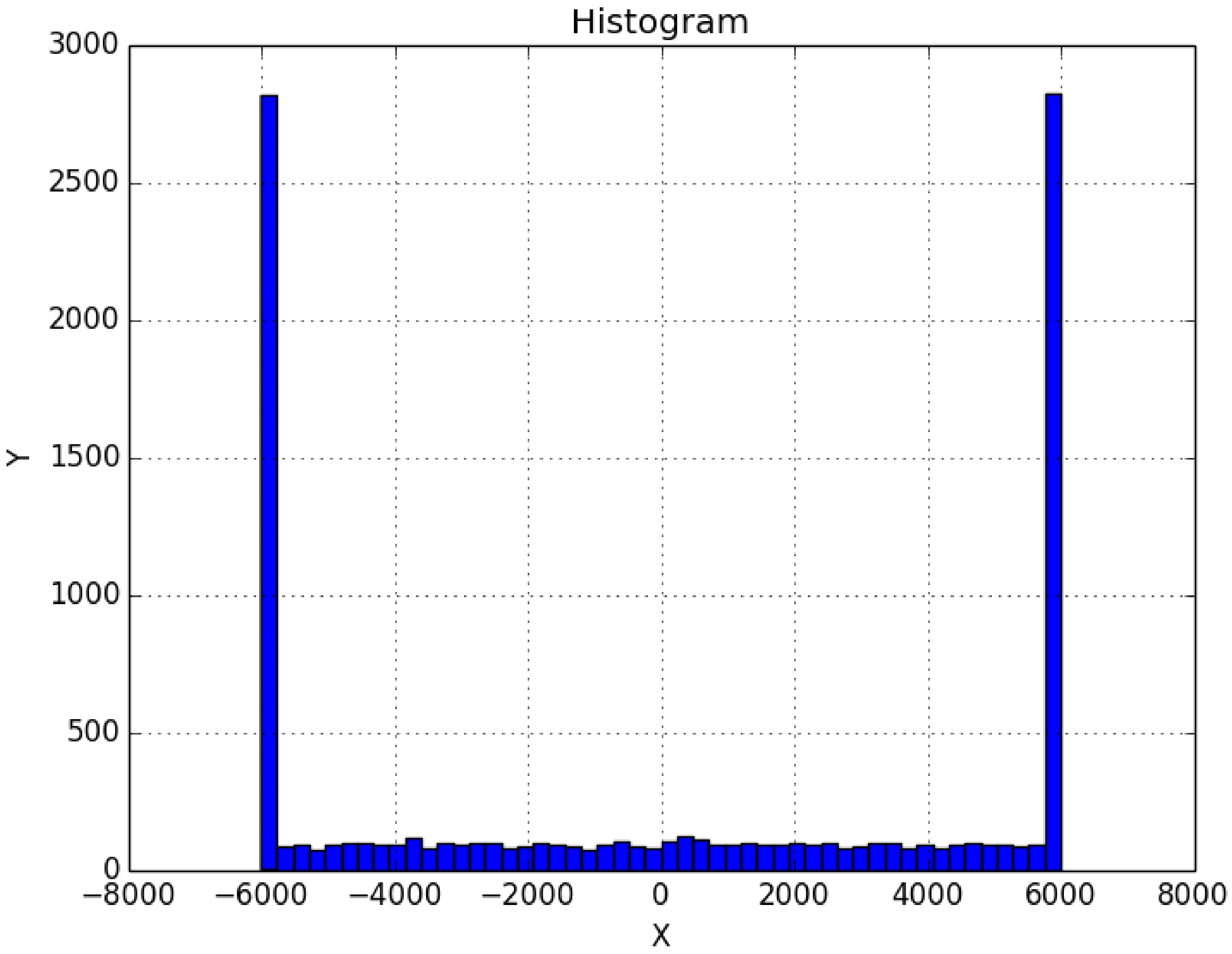
Right-going $$\beta(x, t+\tau) = p\beta(x+\delta, t) + q\alpha(x+\delta, t)$$

Total $$v(x,t) = \alpha(x,t) + \beta(x,t)$$

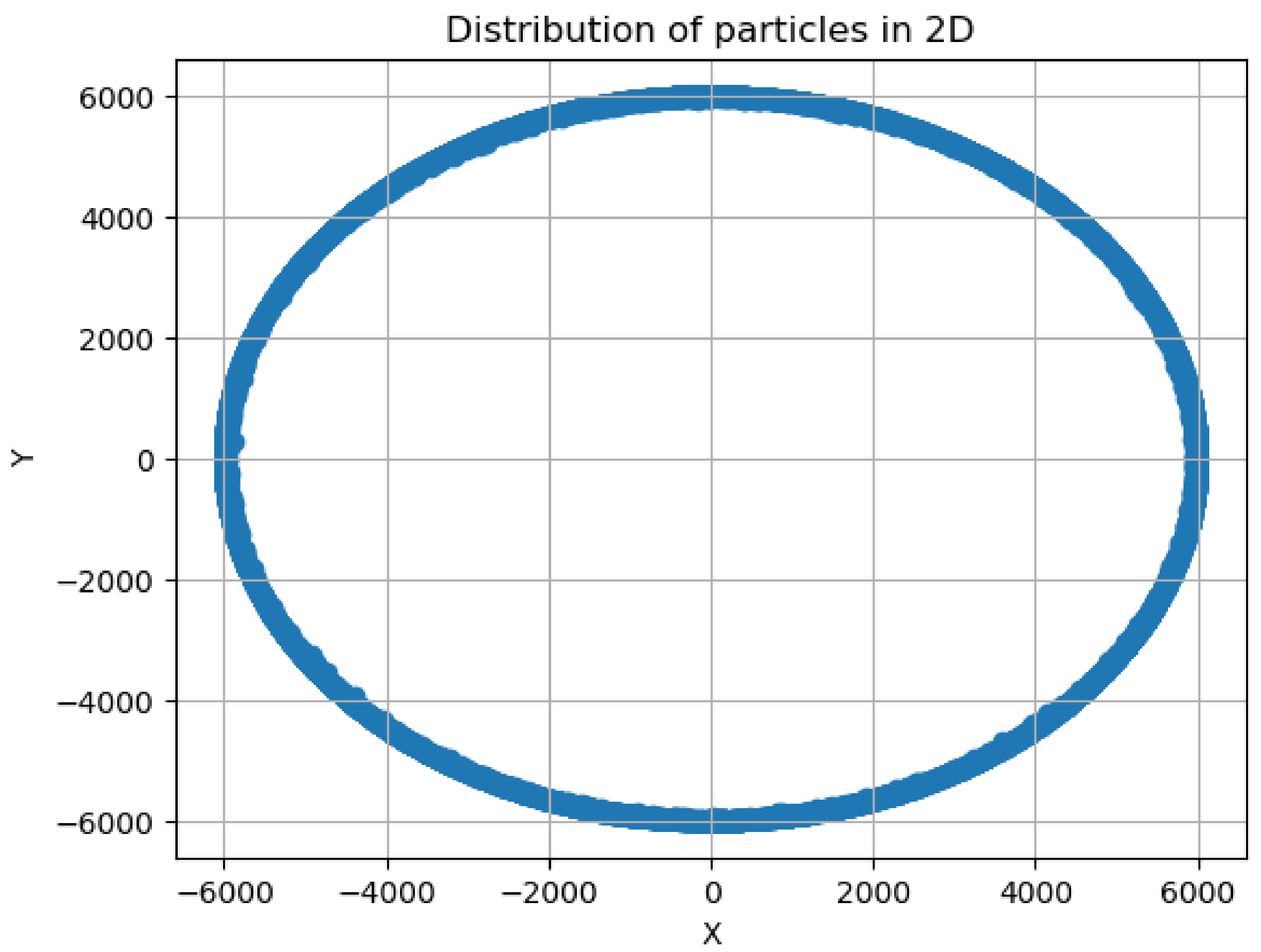
Taylor series, simplify, take limit -- Telegrapher's Equation

$$v_{tt} - c^2 v_{xx} + 2\lambda c v_t = 0$$

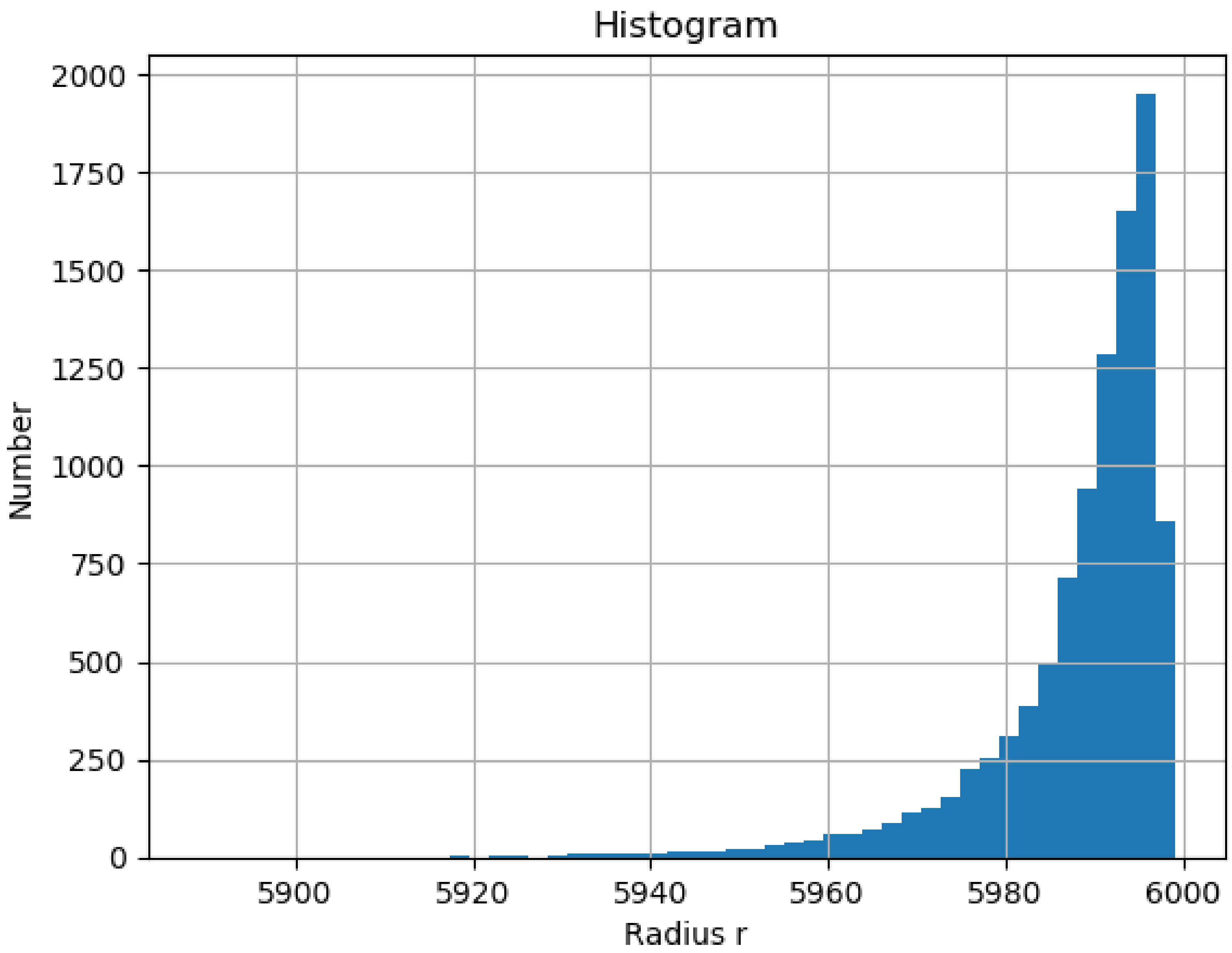
Wave equation in 1D – simulation with particles



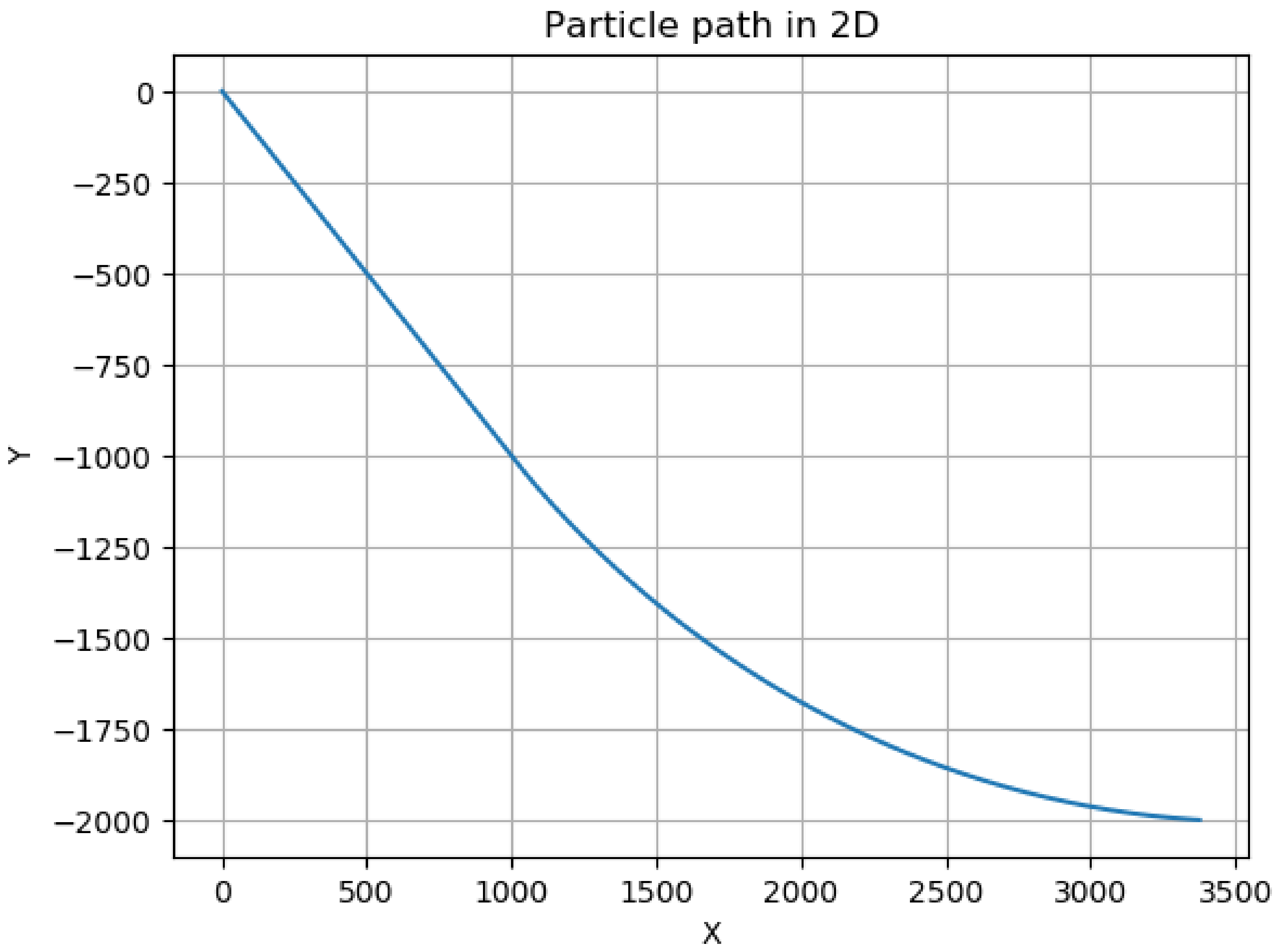
Wave equation in 2D – simulation with particles



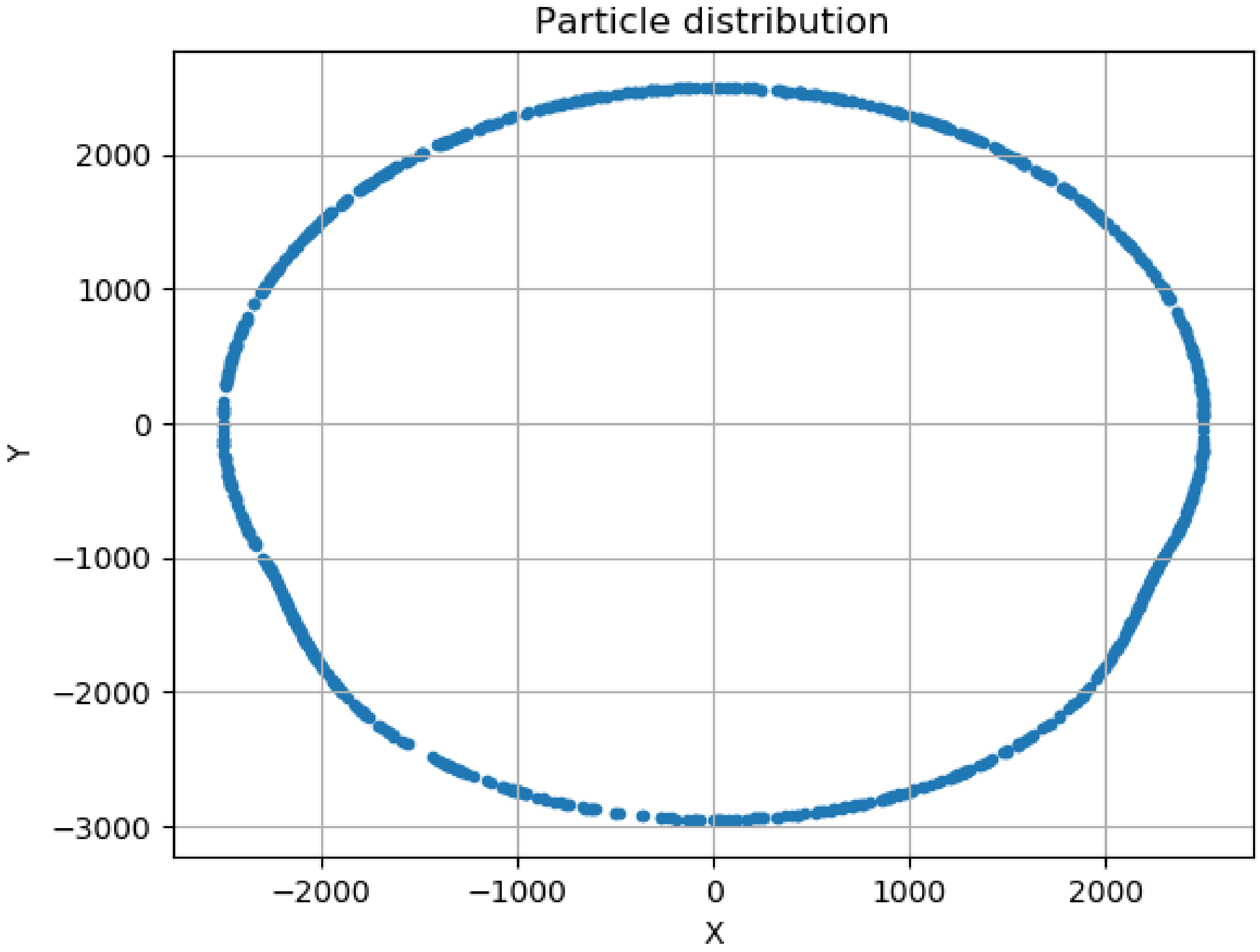
Sanity check – 2D Green's function has a tail



Example – Velocity ramp in 2D



Example – Velocity ramp in 2D



Computations issues

- GPU card has 100 – 1000 processors
- Particles move independently – no communication needed
- Simulating 1,000,000 to 1,000,000,000 particles not a problem.
- Limit to particular source/receiver pairs.

Discussion and conclusions

- Random particles simulate diffusions equation.
- Modifications shows potential to simulate wave equation.
- Potential for computational speed up.
- An early work-in-progress.

End

- Thank you – questions?