New mathematical tools

…to simplify sparse CO_2 monitoring with FWI

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CREWES Annual Sponsors' Meeting

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Overview

Goal: enable FWI for low-cost seismic monitoring

…such that sparseness, targeting leveraged

Approach: engage "Projective Geometric Algebra", PGA

….capitalize on repetitiveness from sparse acq / targeting

Requirements:

- examining basic operations in adjoint-state method
- examining operations between PGA elements
- access to PGA computation

New computational tools for new problems

- Computation via alternative hardware / physical systems
    - Quantum computers, Ising computers, analog computers
- Computation via alternative algebras
    - Quaternions, Clifford algebras, Geometric Algebra

New computational tools – current and future

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Geometric Algebra (GA)

- Extension of standard vector algebra
  - Standard ops, complex analysis, division, ND generalizations
- Major recent impact on theoretical science
  - Mathematics, gravity, mechanics, visualization, graphics
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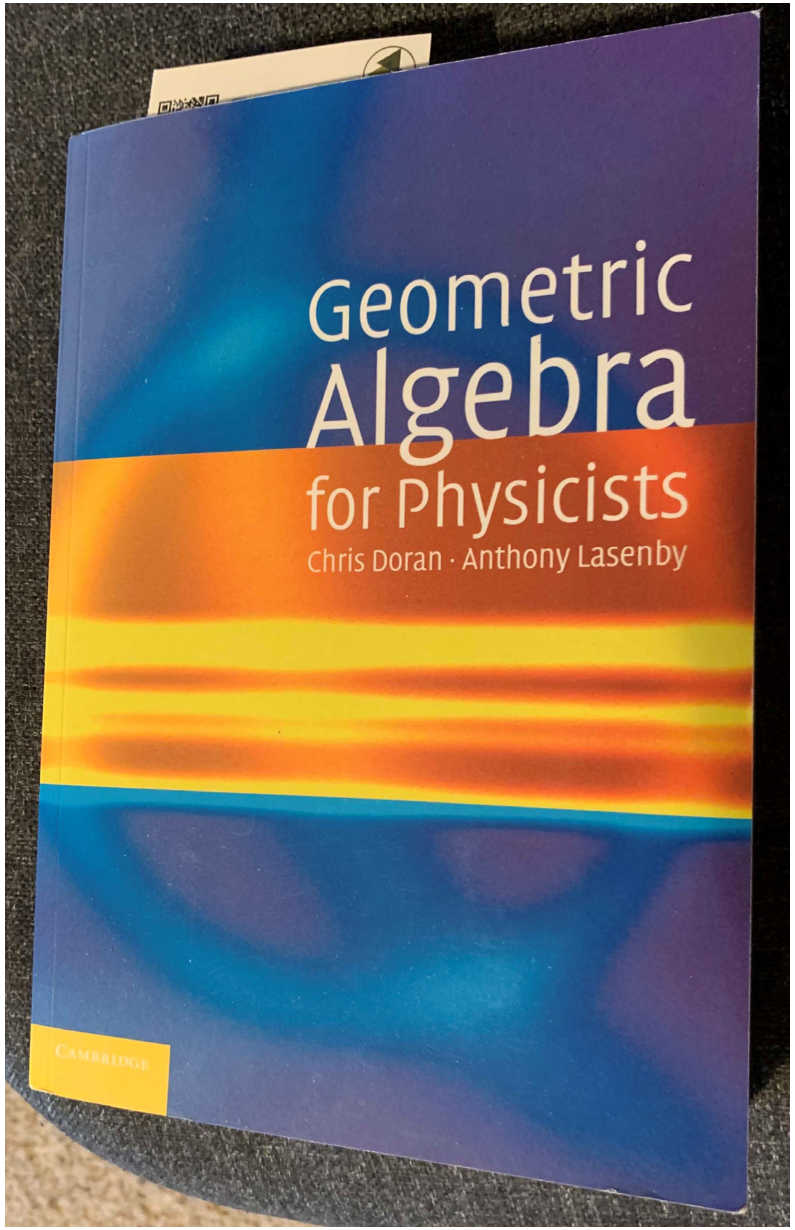
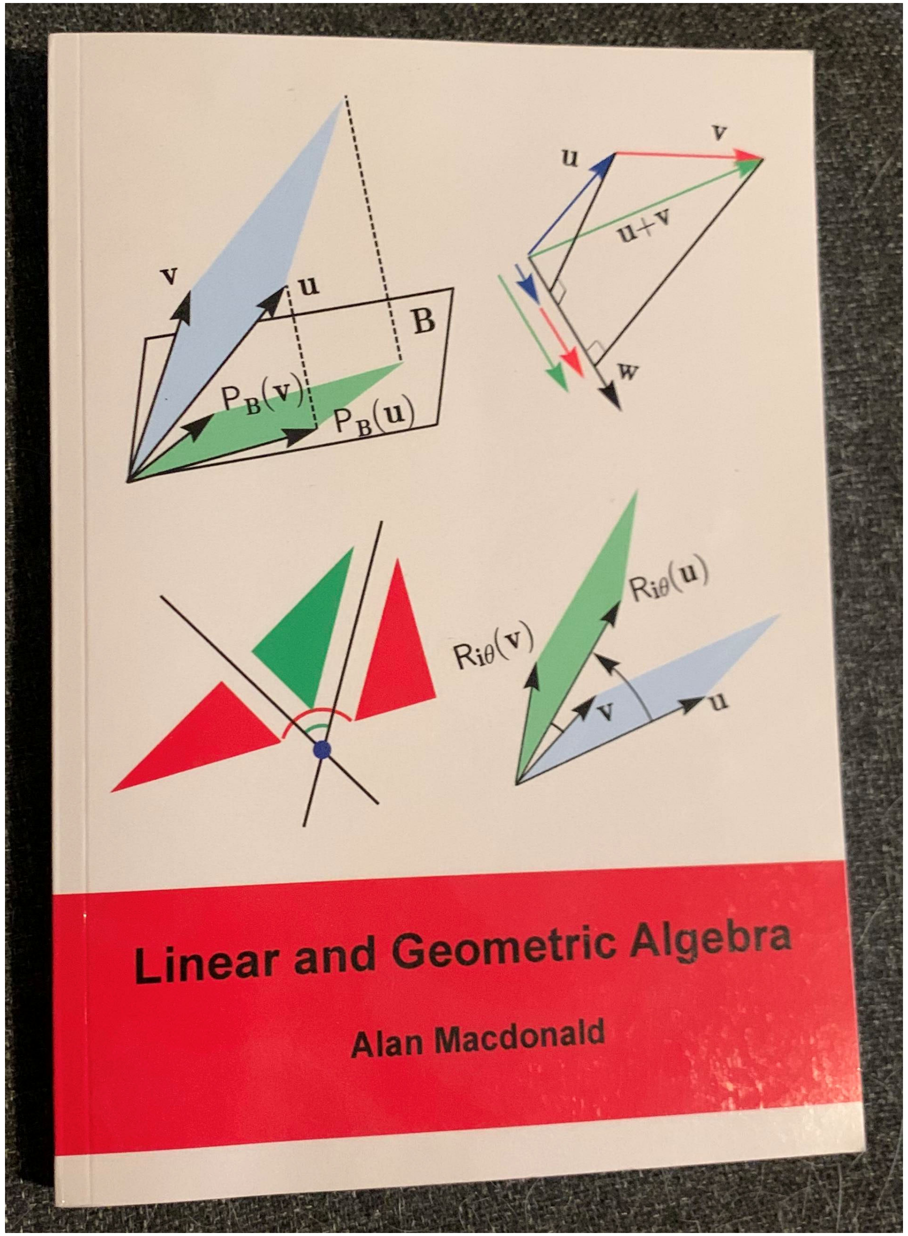
Projective Geometric Algebra (PGA)

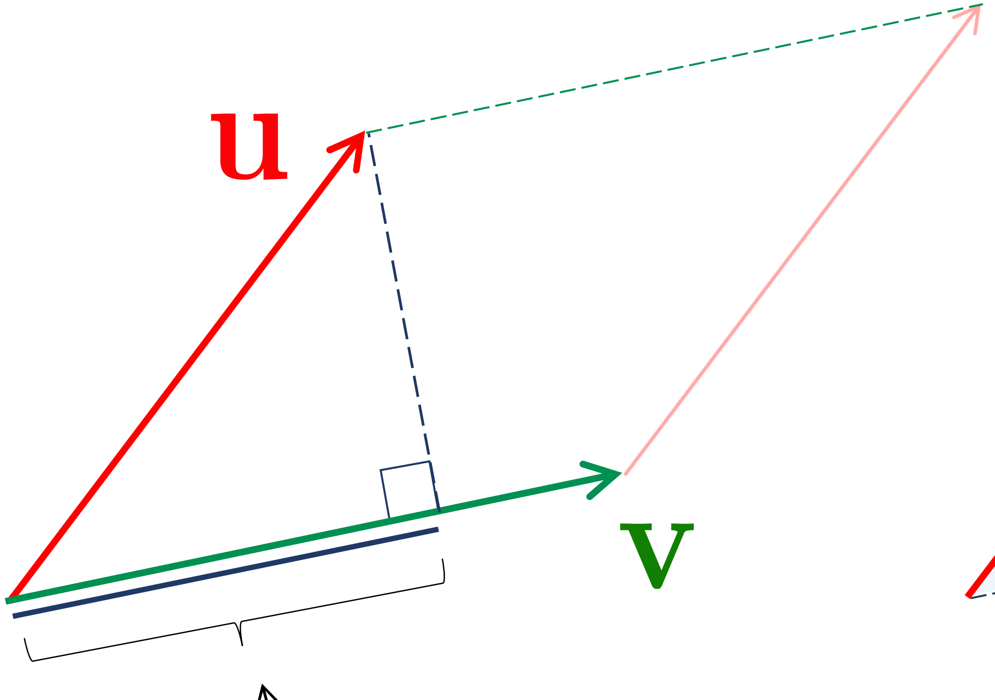
- A brand of GA with (possible) use for analyzing linear systems

Geometric algebra

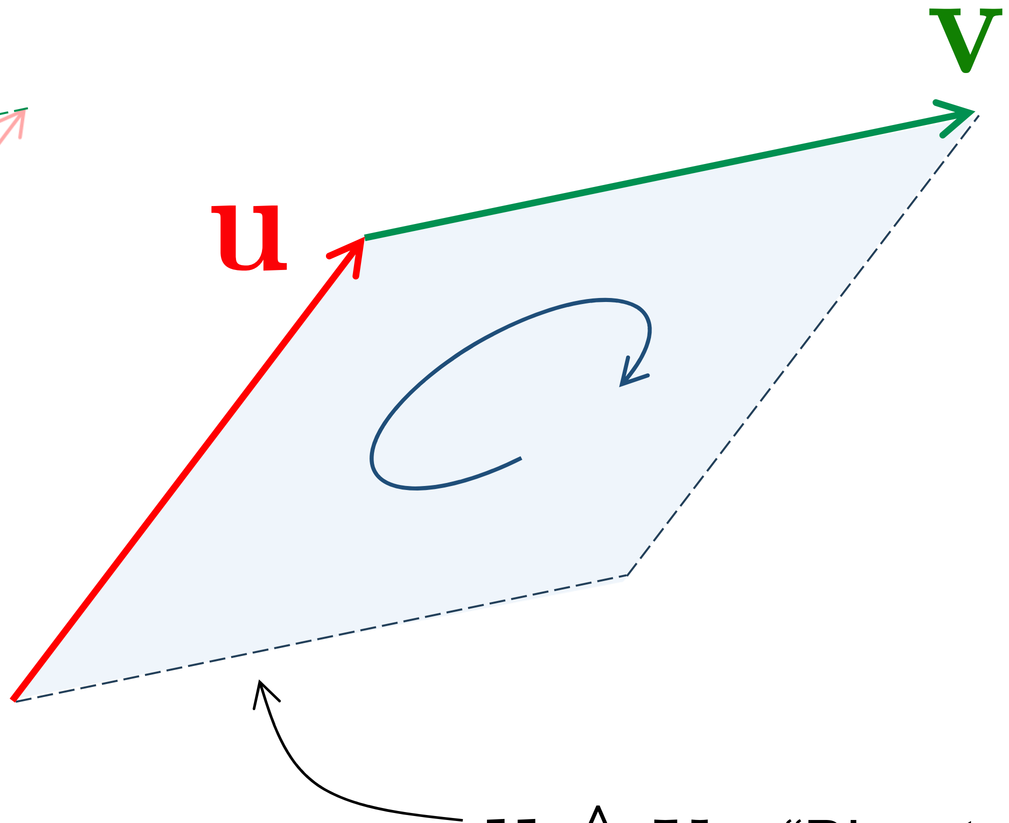




Geometric algebra – dot products, wedge products, bivectors

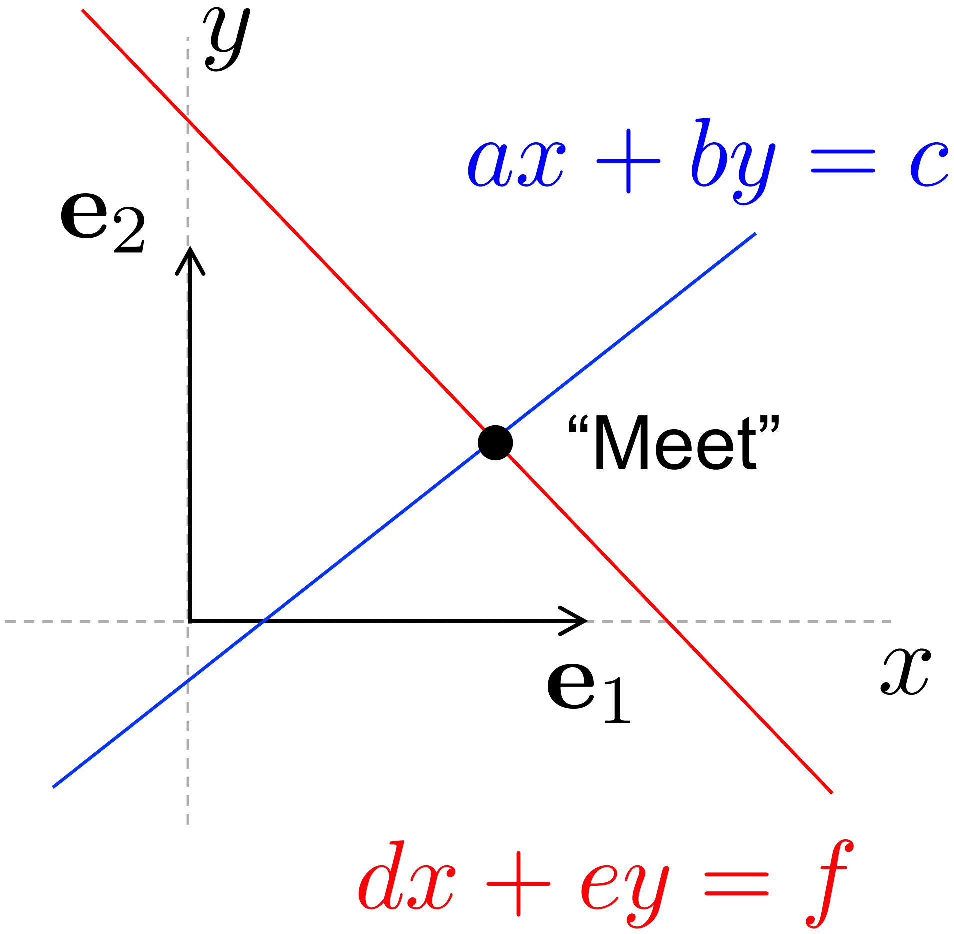


$\mathbf{u} \cdot \mathbf{v} / v$ = Scalar



$\mathbf{u} \wedge \mathbf{v}$ = “Bivector”

Projective Geometric Algebra and linear equations – the row picture



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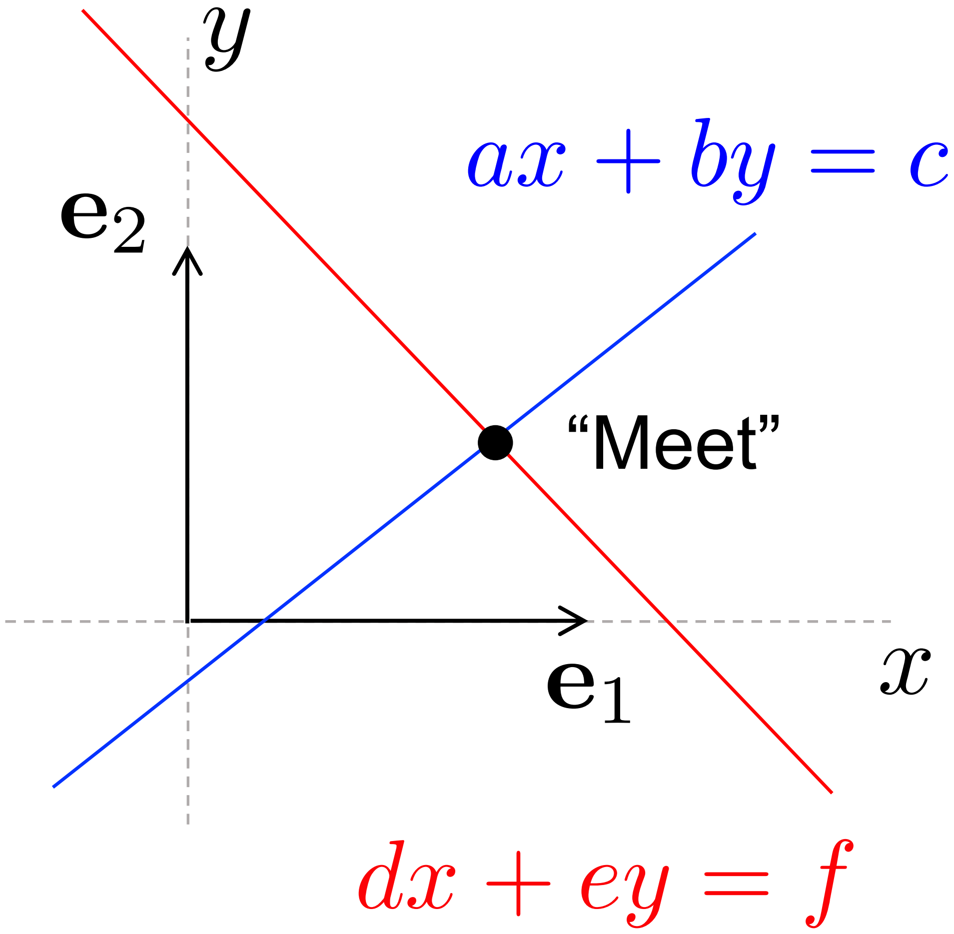
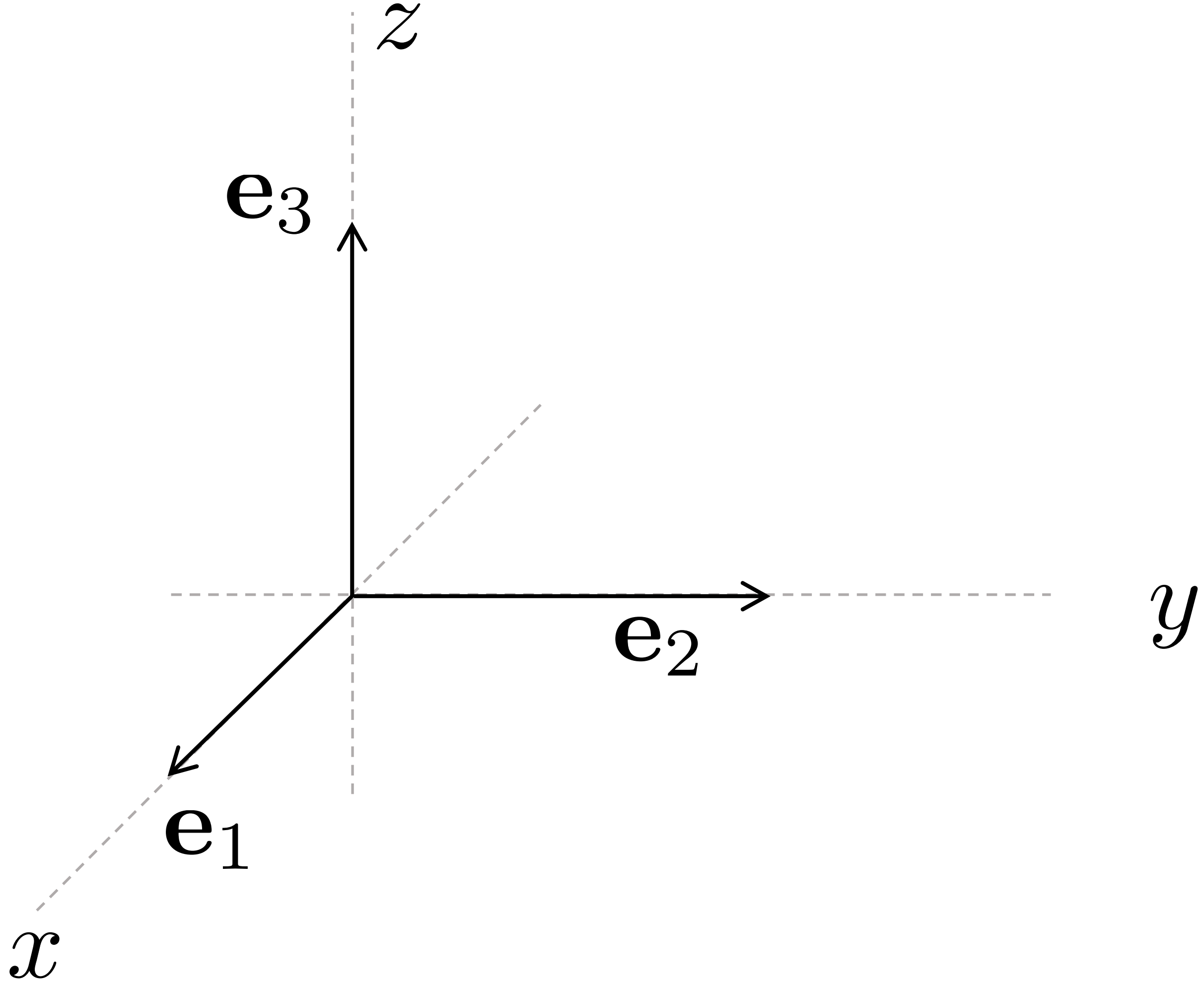
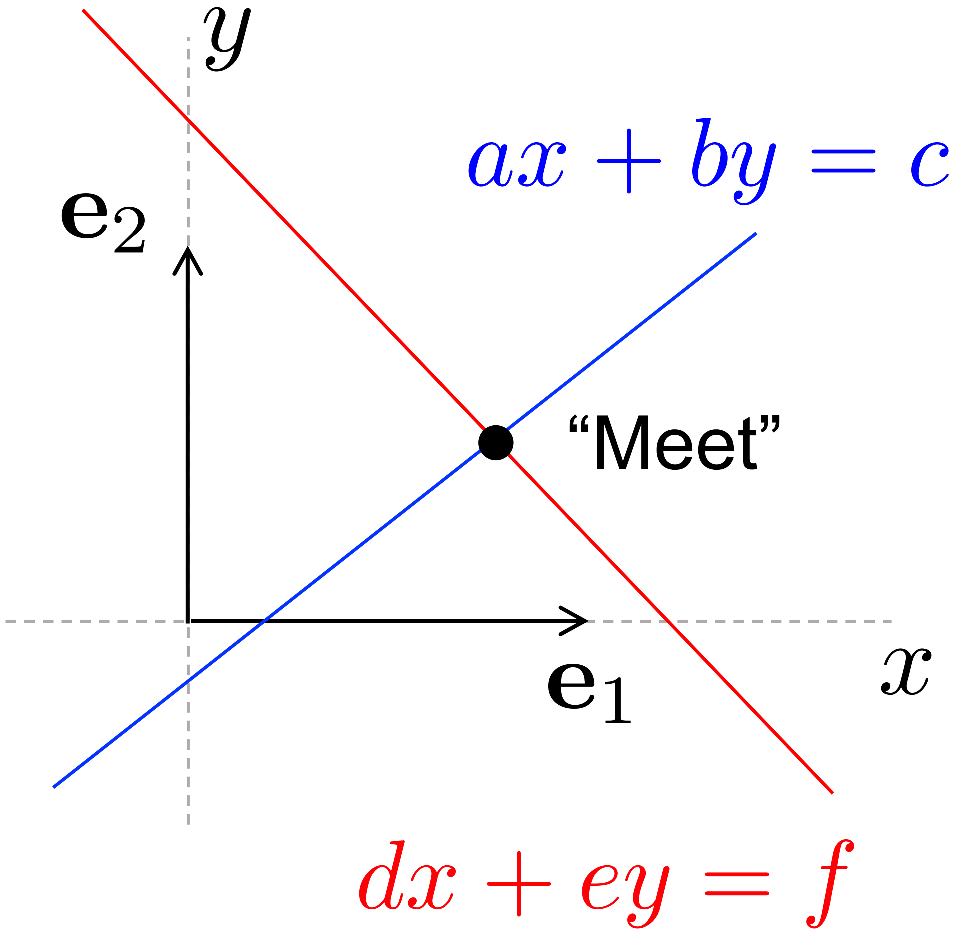
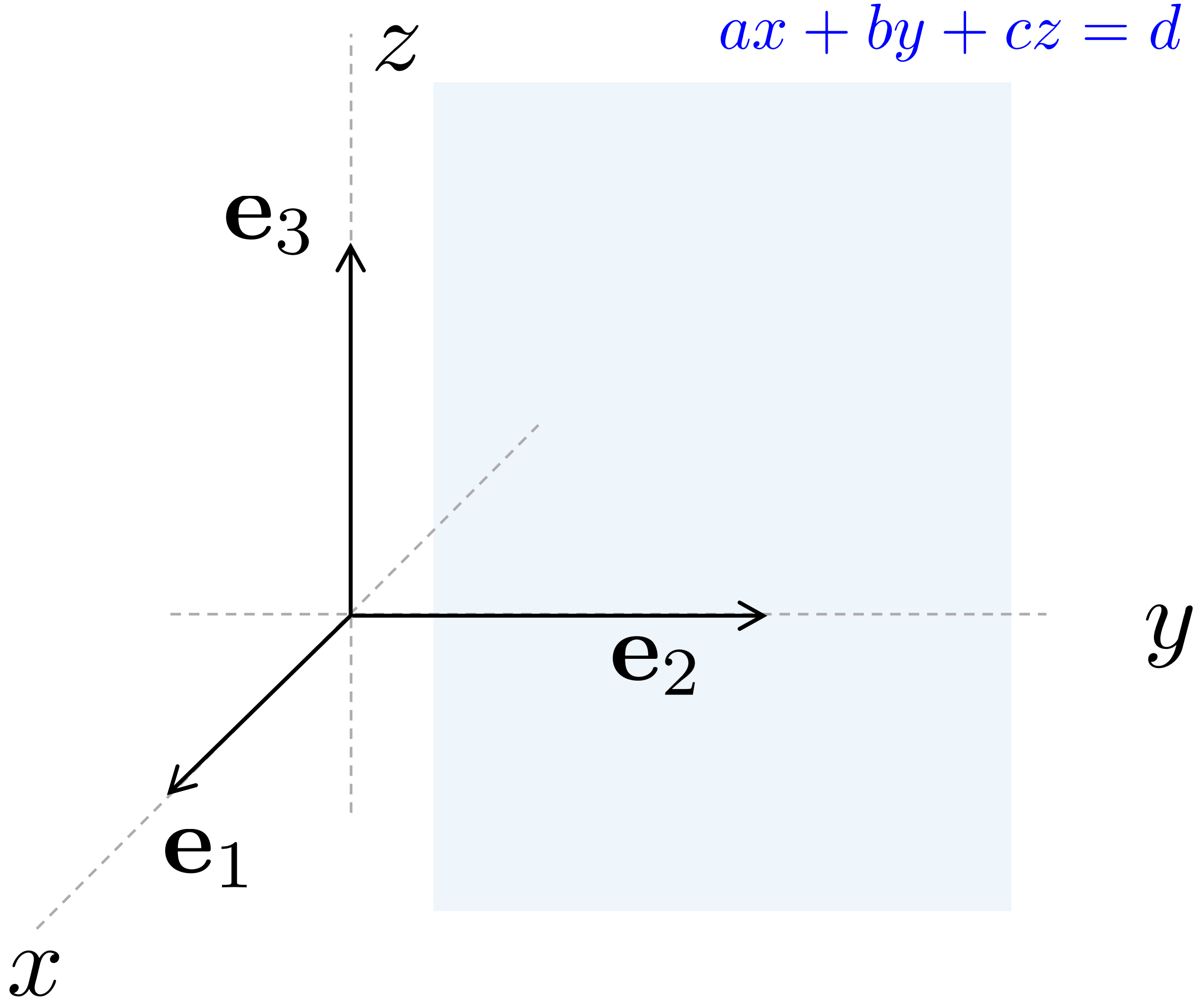





y
$ax + by = c$
$\mathbf{e}_2$
"Meet"
$\mathbf{e}_1$
x
$dx + ey = f$


z
$ax + by + cz = d$
$\mathbf{e}_3$
y
$\mathbf{e}_2$
$\mathbf{e}_1$
x

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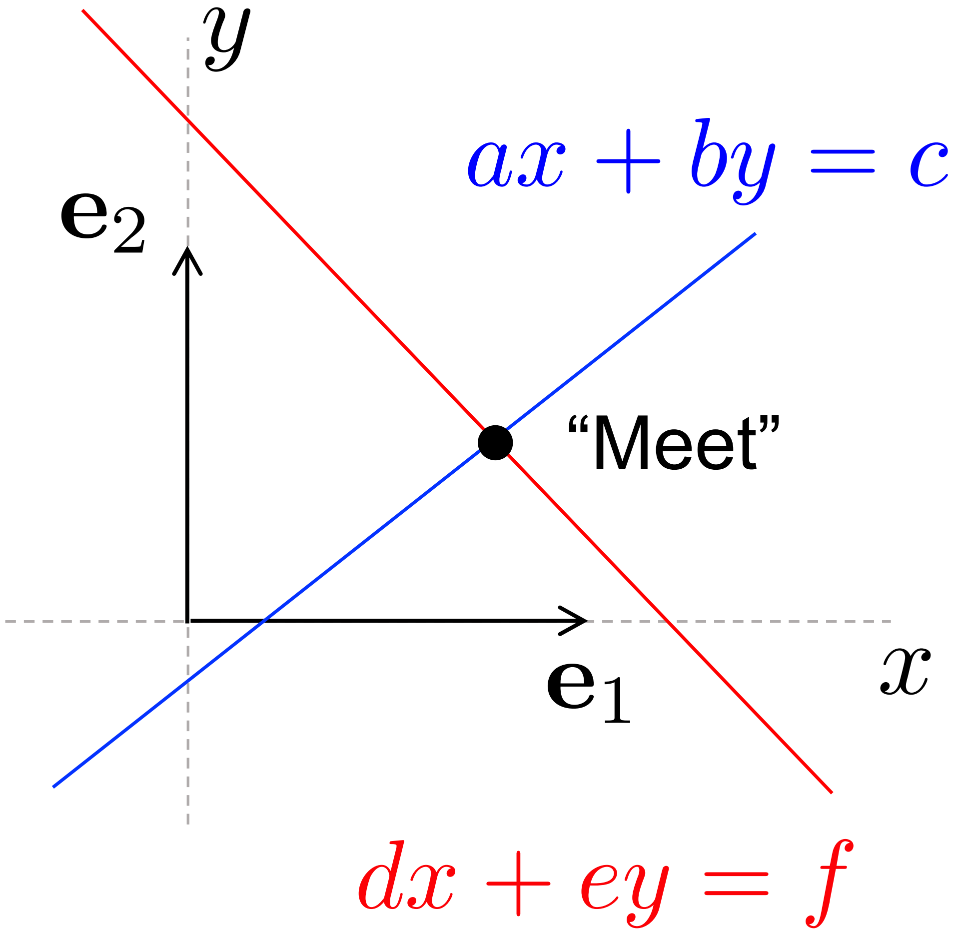

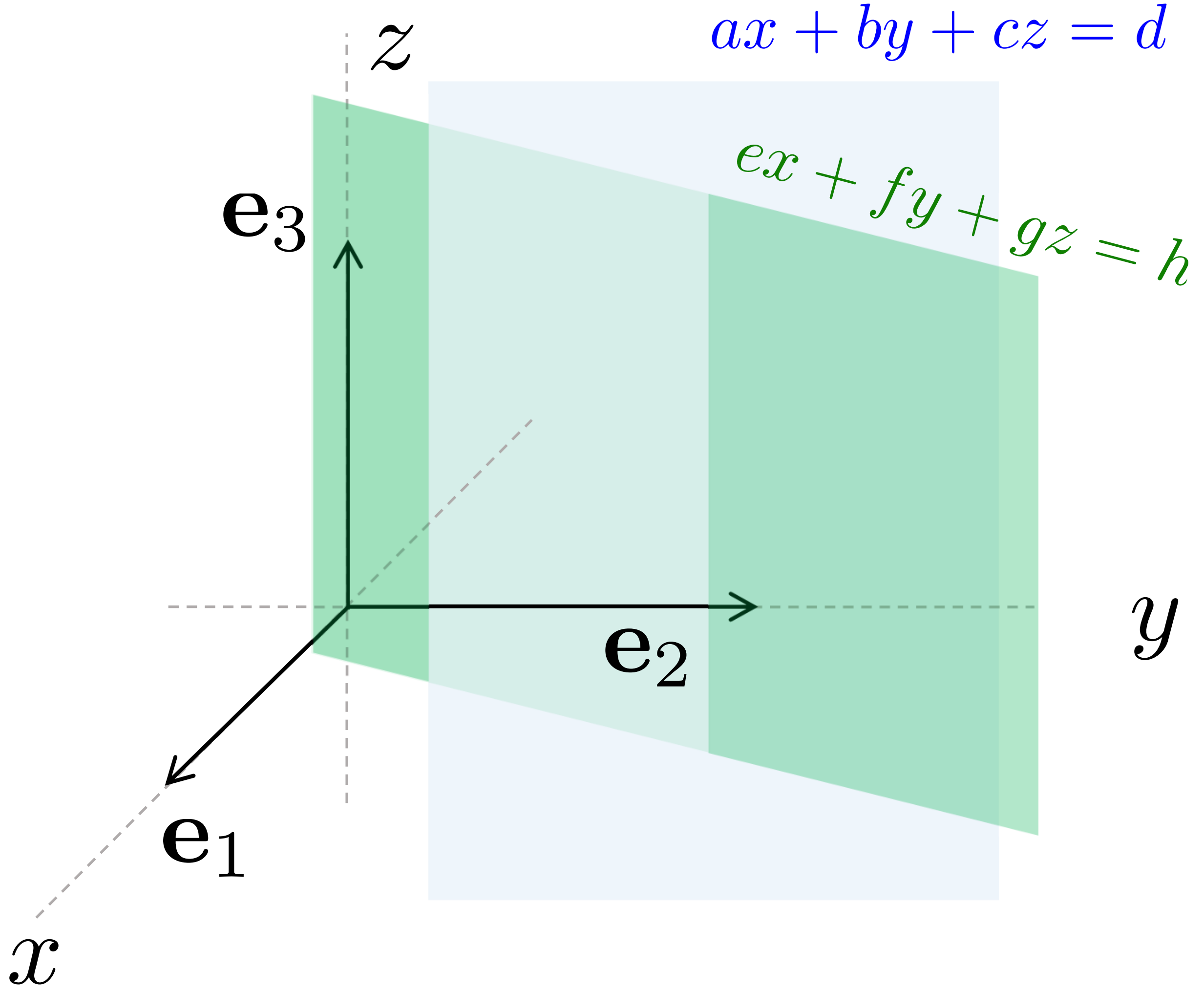

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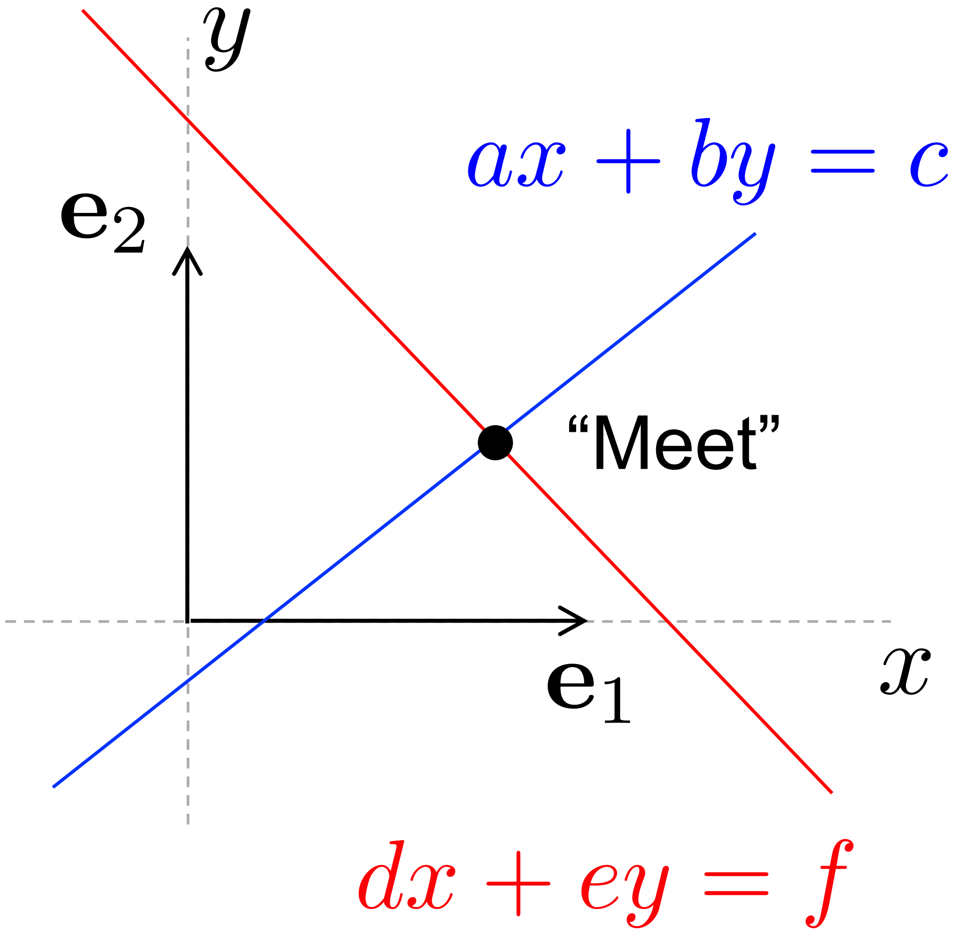

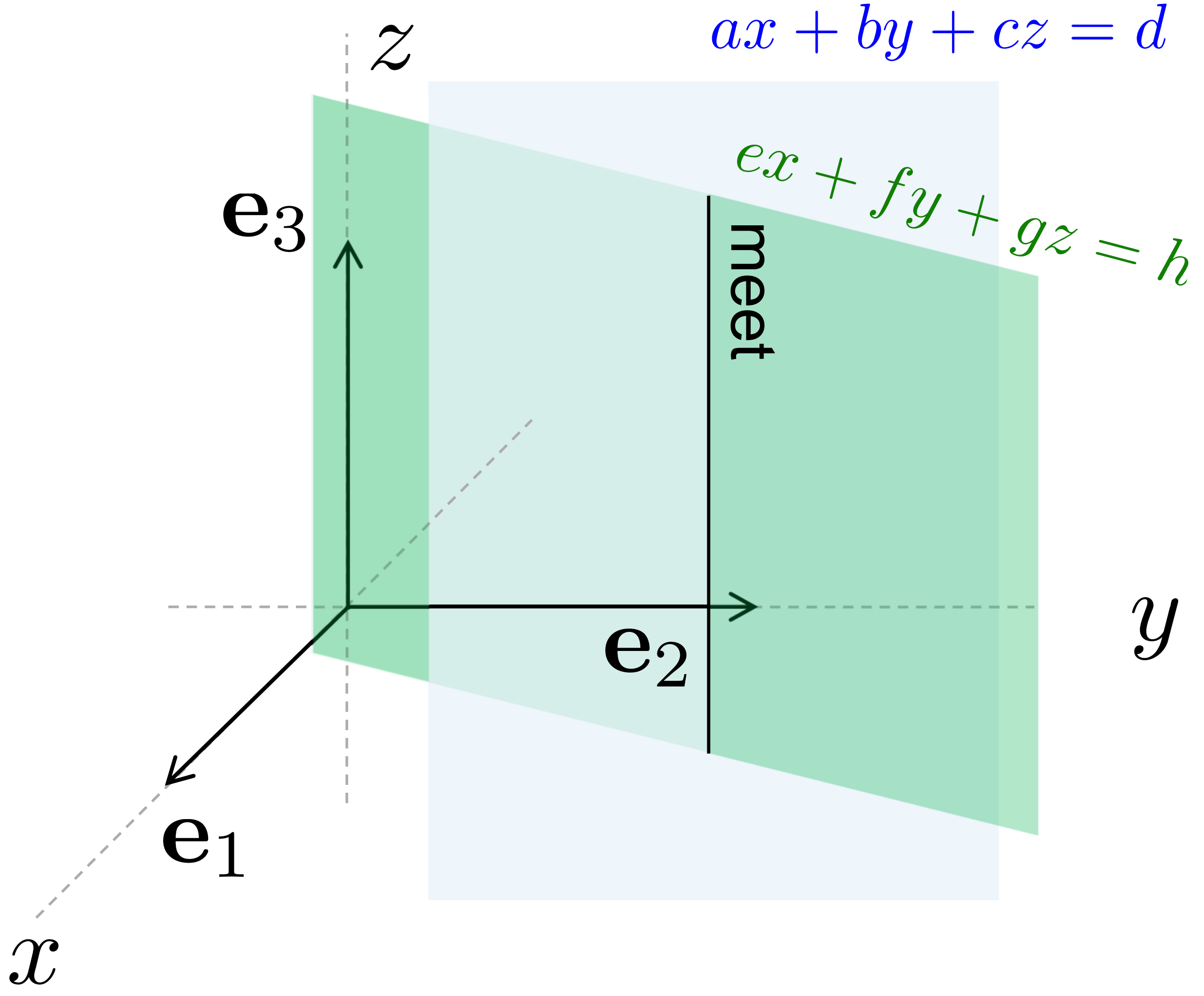

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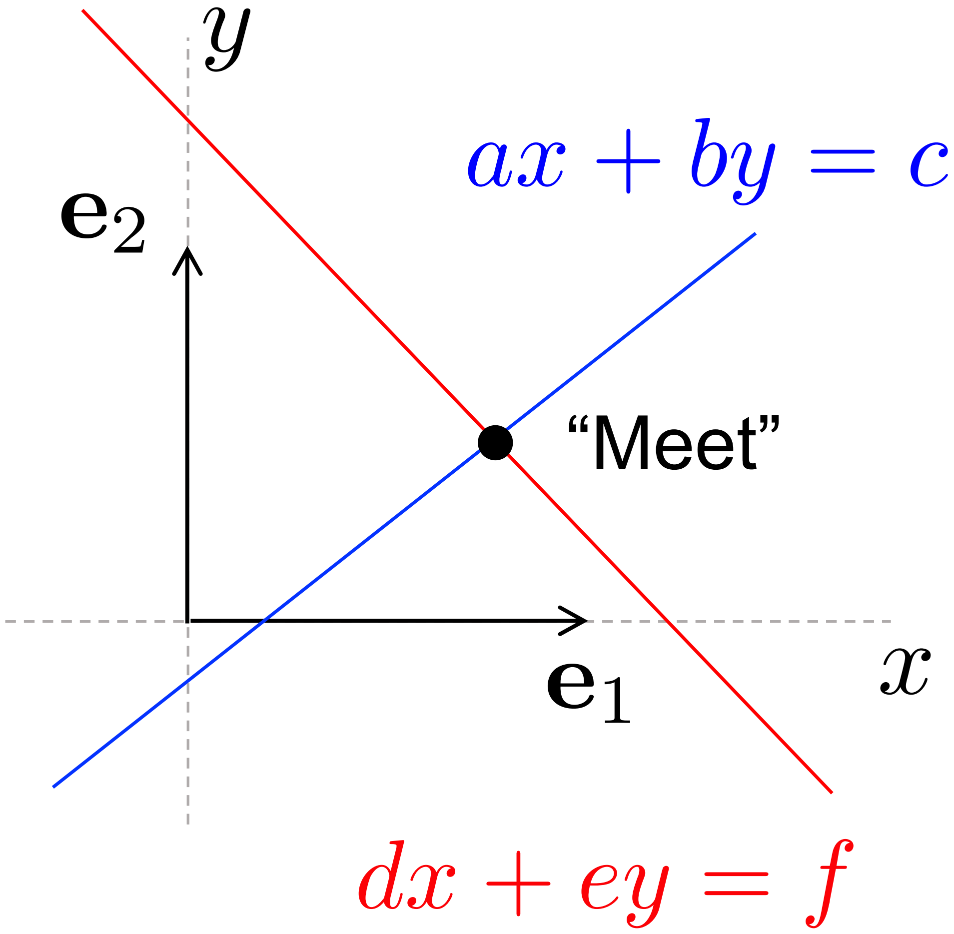

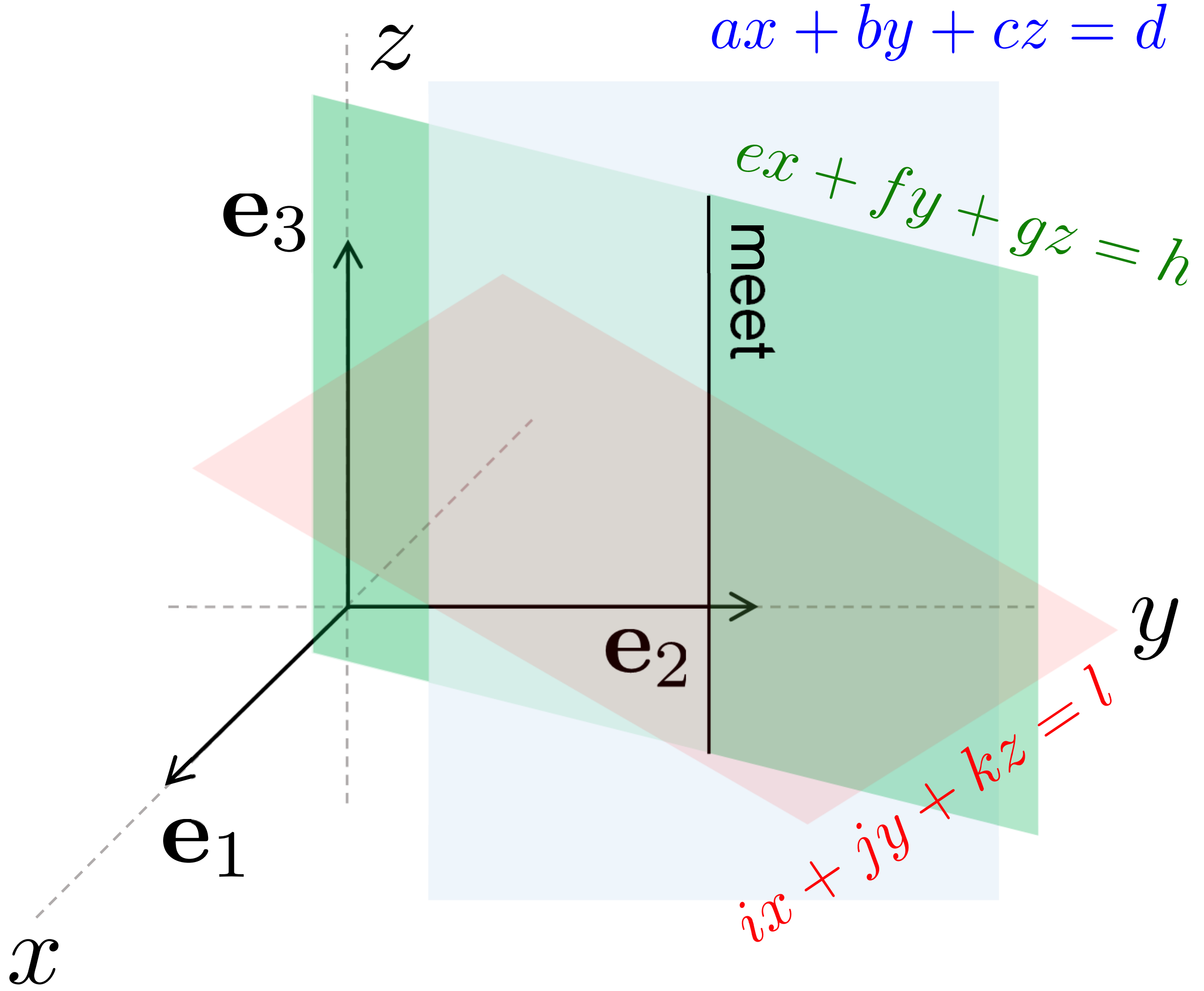

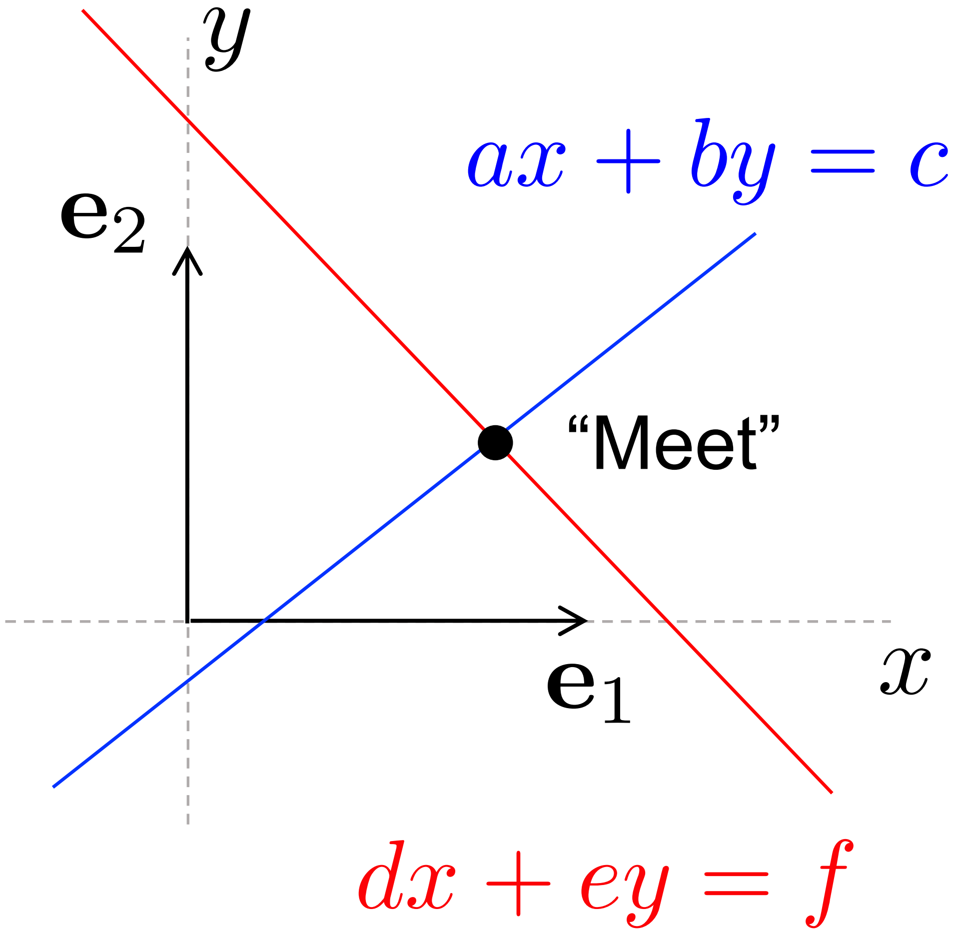

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$\mathbf{e}_2$
"Meet"
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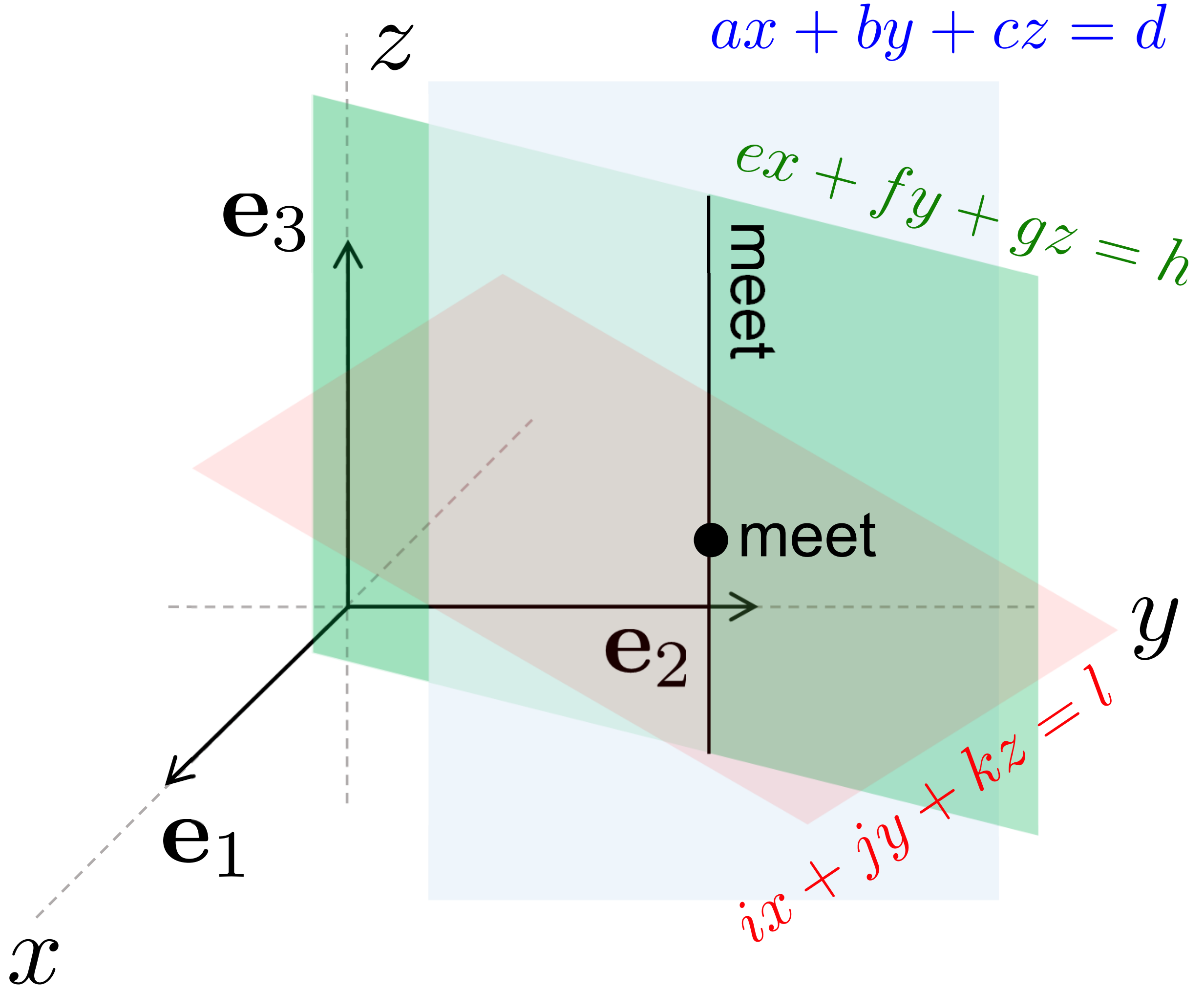
x
$dx + ey = f$



z
$ax + by + cz = d$
$ex + fy + gz = h$
$\mathbf{e}_3$
meet
meet
y
$\mathbf{e}_2$
$ix + jy + kz = l$
$\mathbf{e}_1$
x

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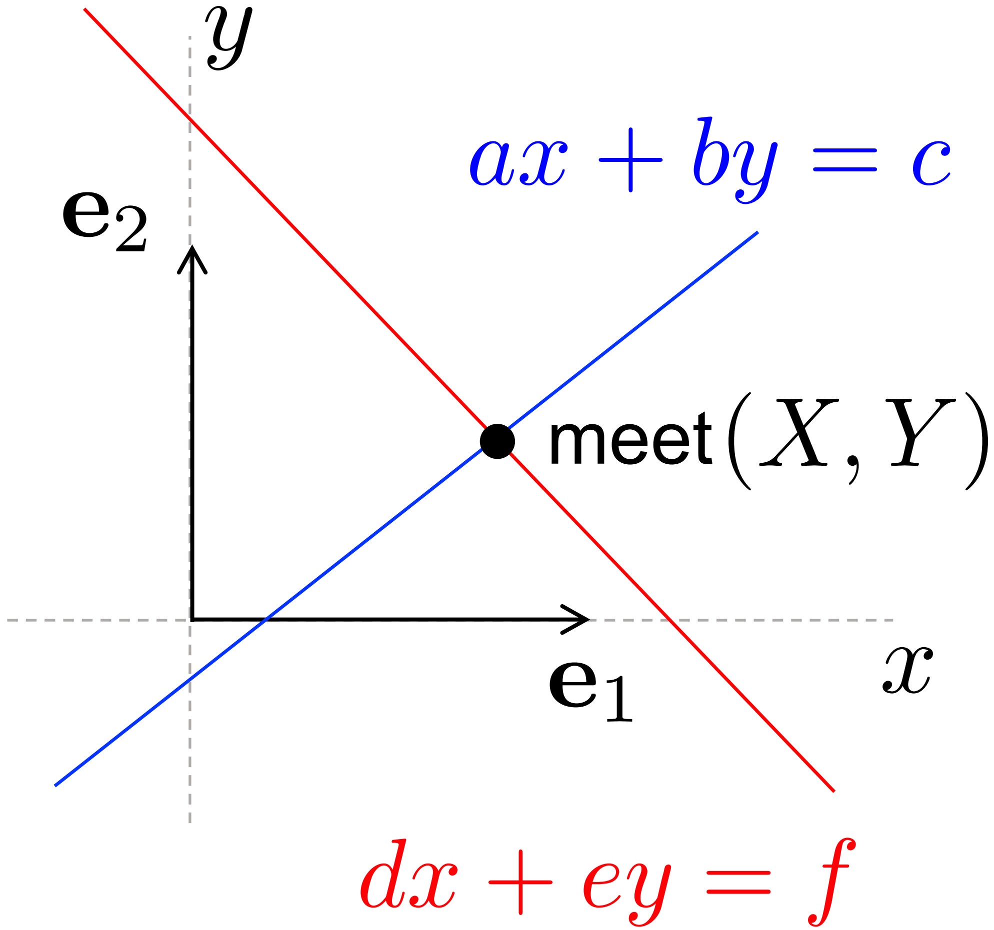

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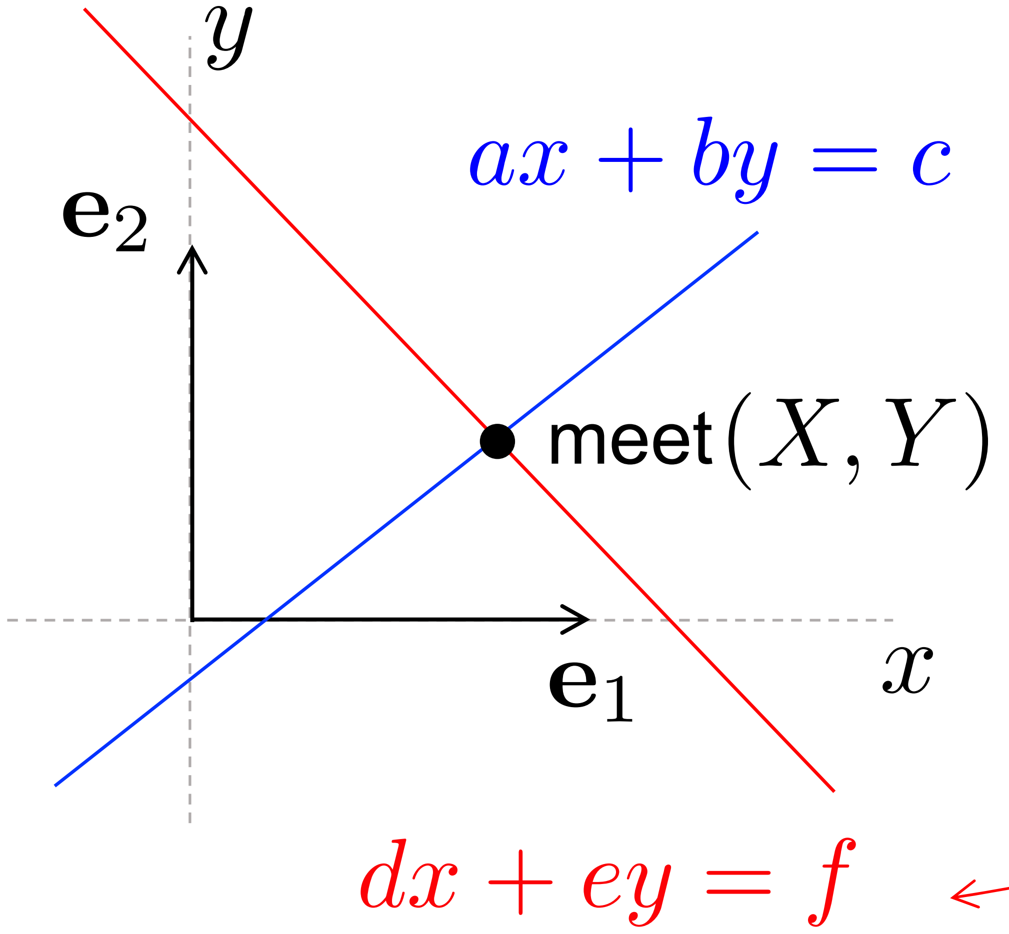


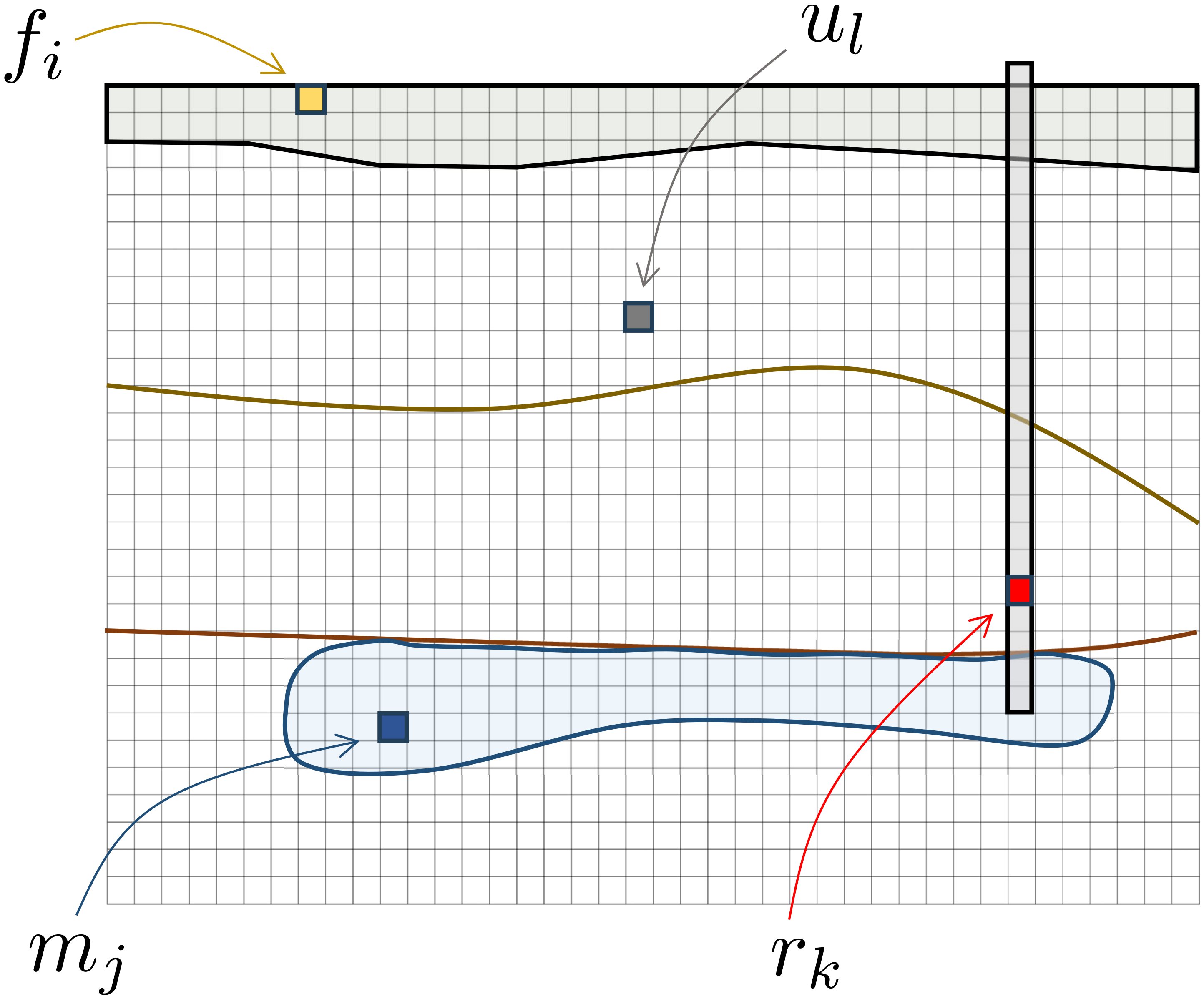
$$\mathbf{A}_1 = a\mathbf{e}_1 + b\mathbf{e}_2 - c\mathbf{e}_3$$

$$\mathbf{A}_2 = d\mathbf{e}_1 + e\mathbf{e}_2 - f\mathbf{e}_3$$

$$\mathbf{A}_1 \wedge \mathbf{A}_2 = D\mathbf{e}_{12} + XD\mathbf{e}_{23} + YD\mathbf{e}_{13}$$

PGA encourages us to adopt Strang's "row picture" of a system of equations

FWI and the gradient calculation



$$\mathbf{S}^T\boldsymbol{\lambda} = \mathbf{r}$$

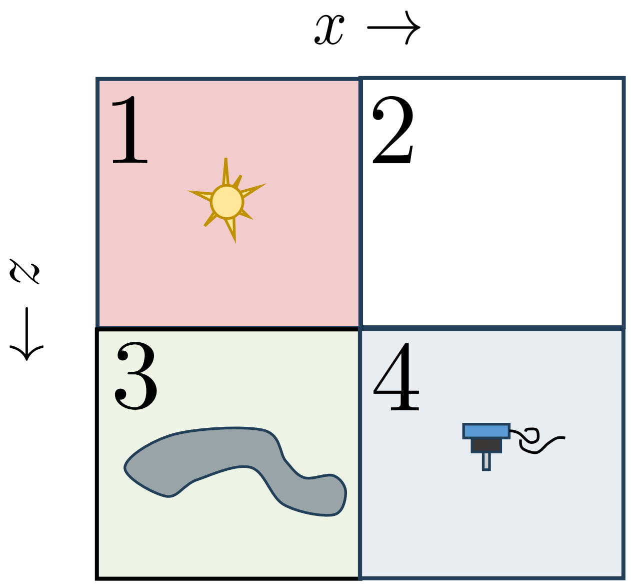
$$\mathbf{S}\mathbf{u} = \mathbf{f}$$

$$\boldsymbol{\lambda}^T \frac{\partial \mathbf{S}}{\partial \mathbf{m}} \mathbf{u}$$

FWI gradient via the adjoint-state method

$$\frac{d\phi}{dm_3} \propto \left\{ \begin{bmatrix} s_{11} & \mathrm{x} & \mathrm{x} & 0 \\ \mathrm{x} & s_{22} & \mathrm{x} & \mathrm{x} \\ \mathrm{x} & \mathrm{x} & s_{33} & \mathrm{x} \\ 0 & \mathrm{x} & \mathrm{x} & s_{44} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ 0 \\ r_4 \end{bmatrix} \right\}^T \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \left\{ \begin{bmatrix} s_{11} & \mathrm{x} & \mathrm{x} & 0 \\ \mathrm{x} & s_{22} & \mathrm{x} & \mathrm{x} \\ \mathrm{x} & \mathrm{x} & s_{33} & \mathrm{x} \\ 0 & \mathrm{x} & \mathrm{x} & s_{44} \end{bmatrix}^{-1} \begin{bmatrix} f_1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}$$

impedances — residuals — sensitivities — impedances — sources



- If certain model components are targeted…
- If source positions are sparsely occupied…
- If receiver positions are sparsely occupied…

- Then certain rows in the calculation are focused on
- Info from most rows incorporated again & again

Opportunity? Yes – but how to take advantage of it?

FWI gradient formulated within PGA

$$\mathbf{B}\boldsymbol{\lambda} = \mathbf{r}$$

$$\mathbf{A}\mathbf{u} = \mathbf{f}$$

$$\boldsymbol{\lambda}^T \frac{\partial \mathbf{S}}{\partial \mathbf{m}} \mathbf{u}$$

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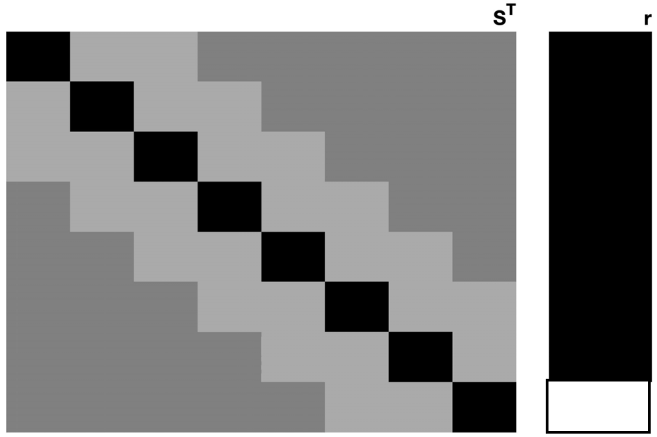

$$\frac{\partial \mathbf{S}}{\partial \mathbf{m}}$$

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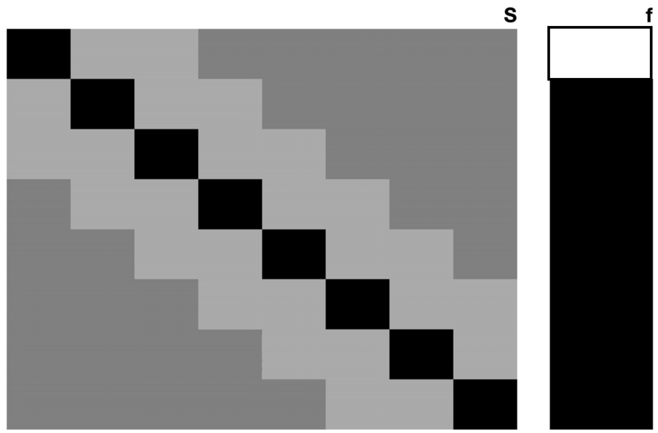

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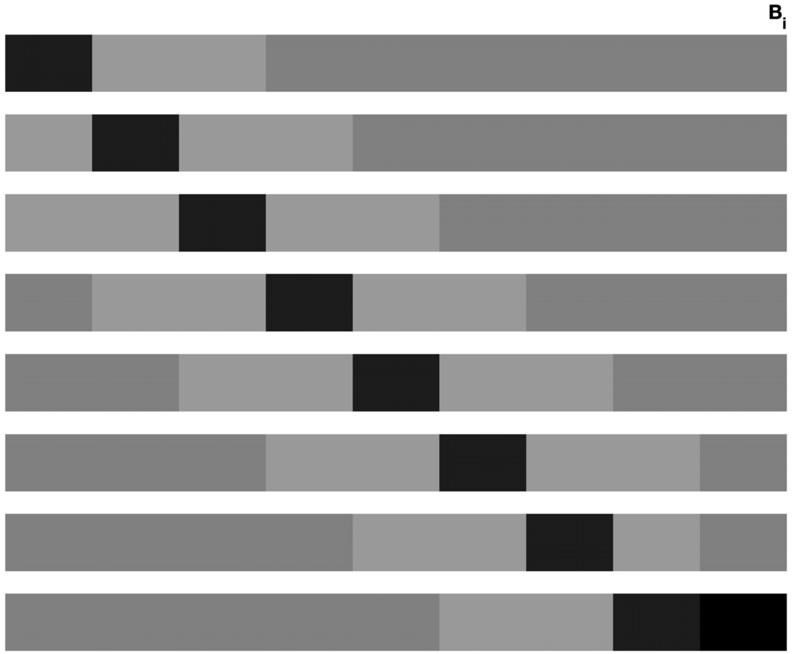
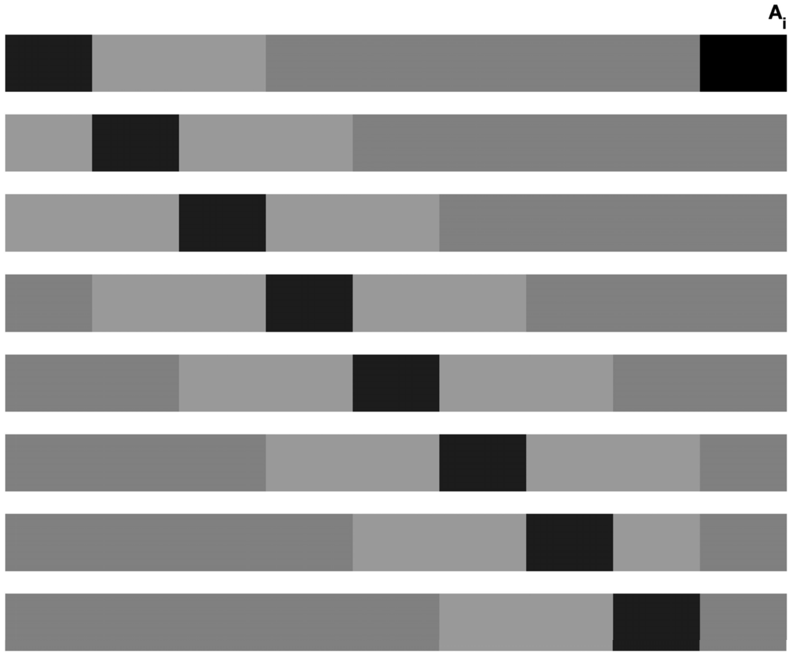


$$\frac{\partial \mathbf{s}}{\partial \mathbf{m}}$$



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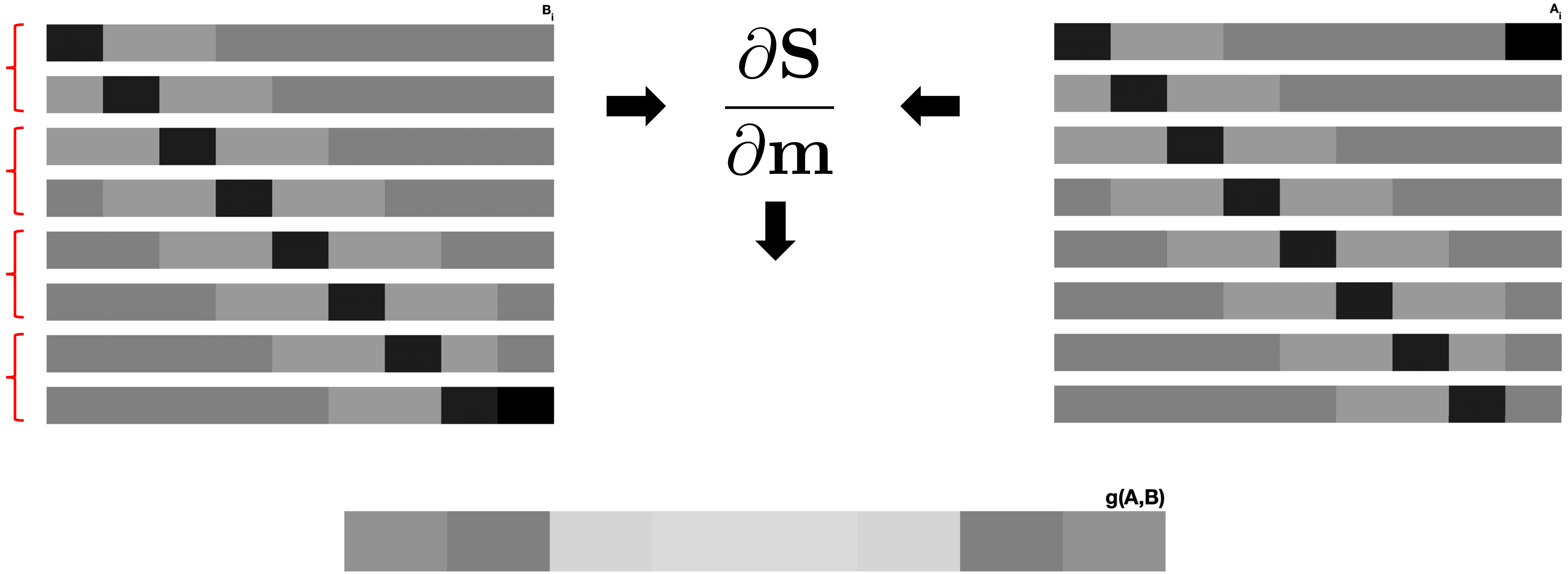

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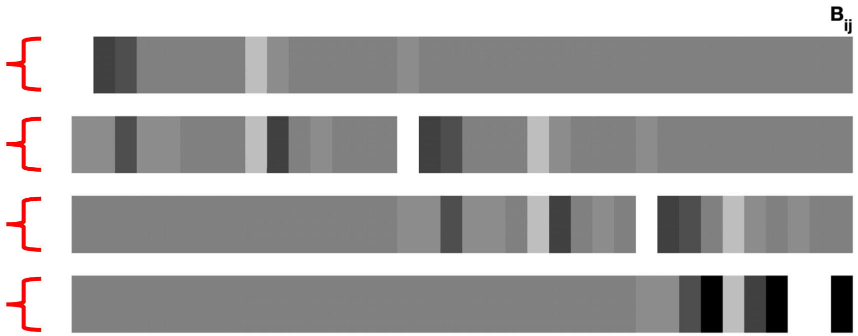

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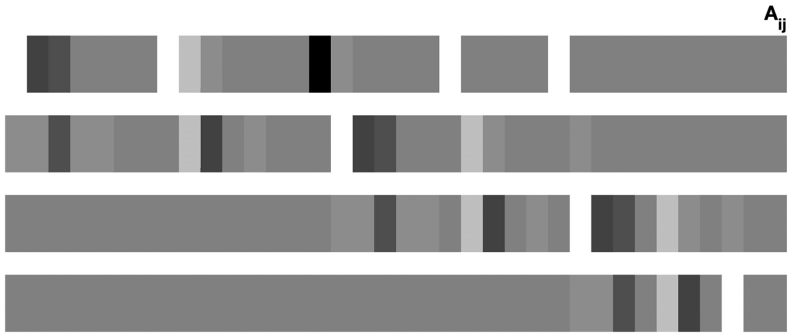

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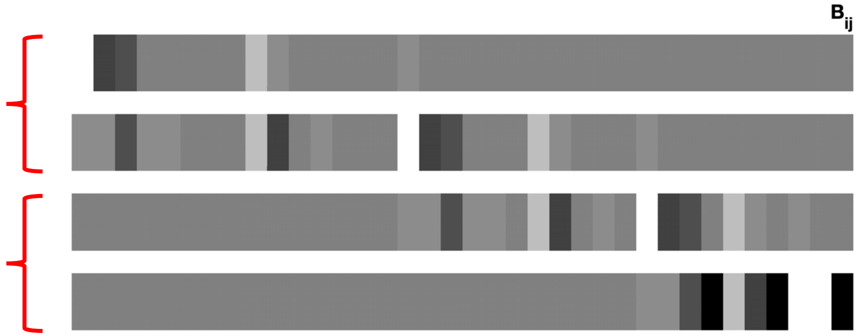

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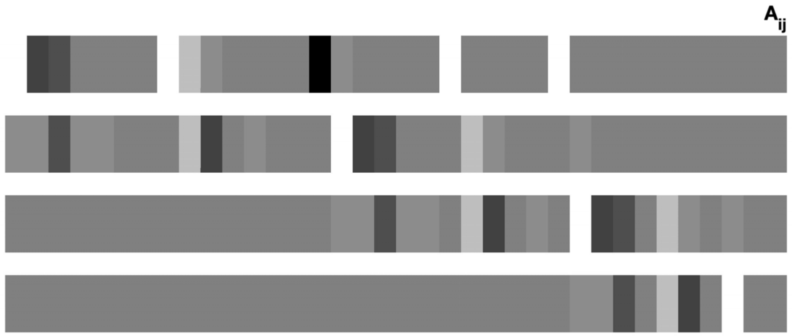

FWI gradient formulated within PGA

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$$\mathbf{Au = f}$$

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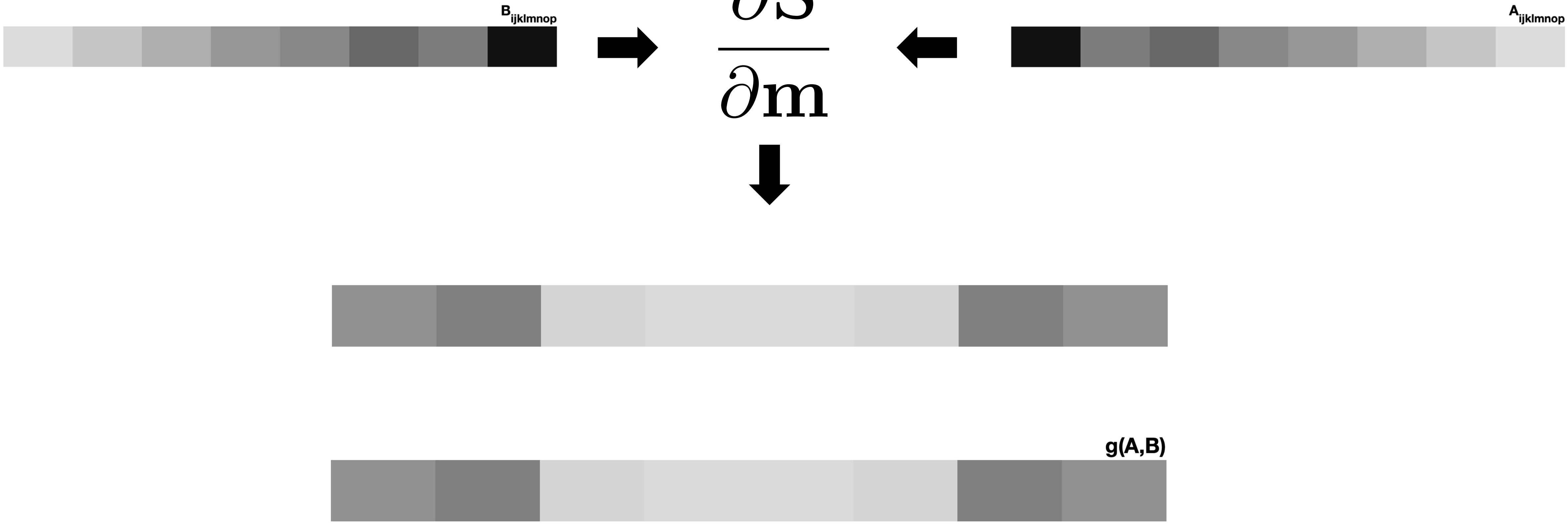
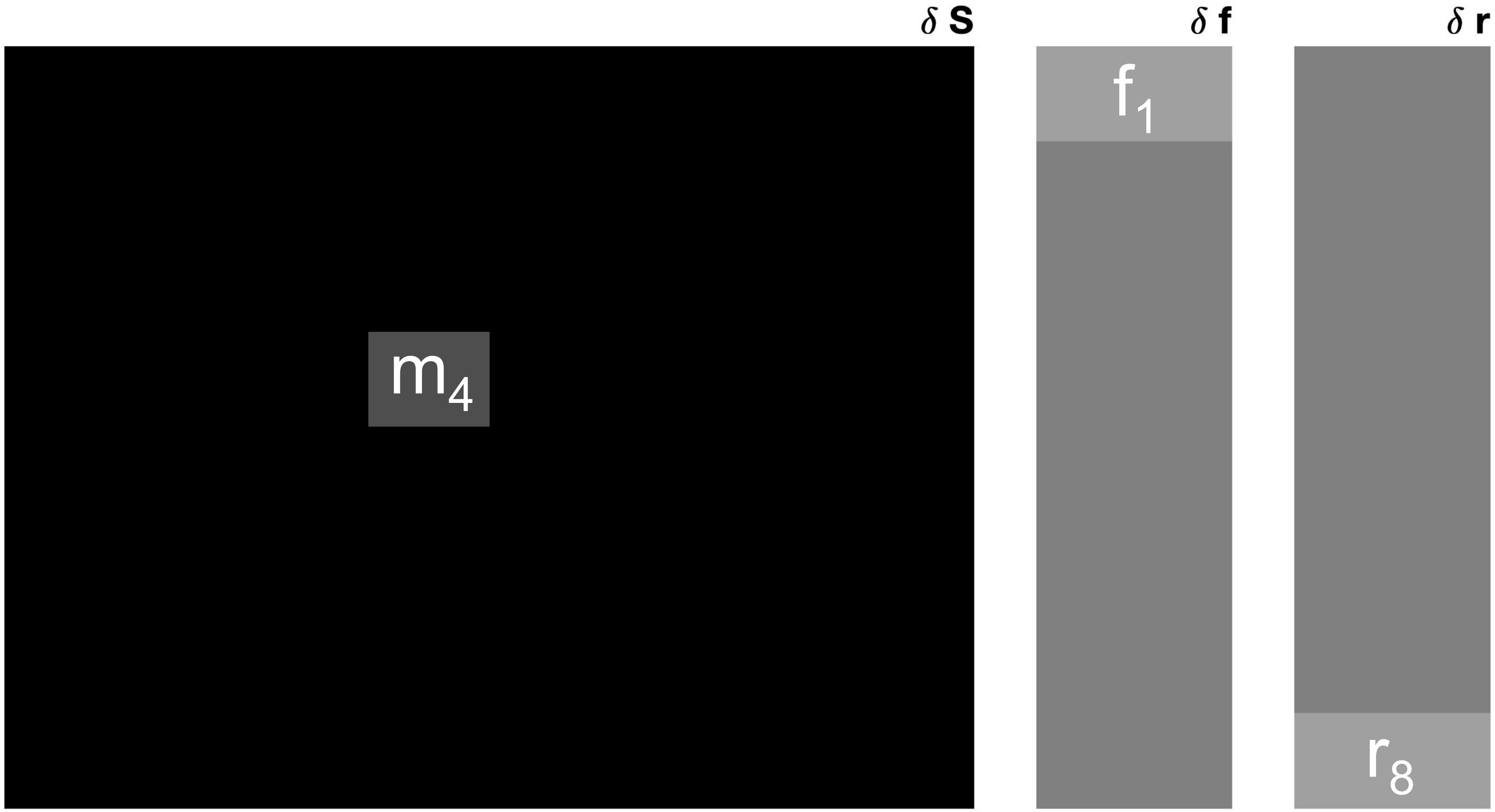


Targeting and sparseness = low computation in PGA?



By perturbing m_i, f_j and r_k, map where computational savings reside

Targeting and sparseness = low computation in PGA?

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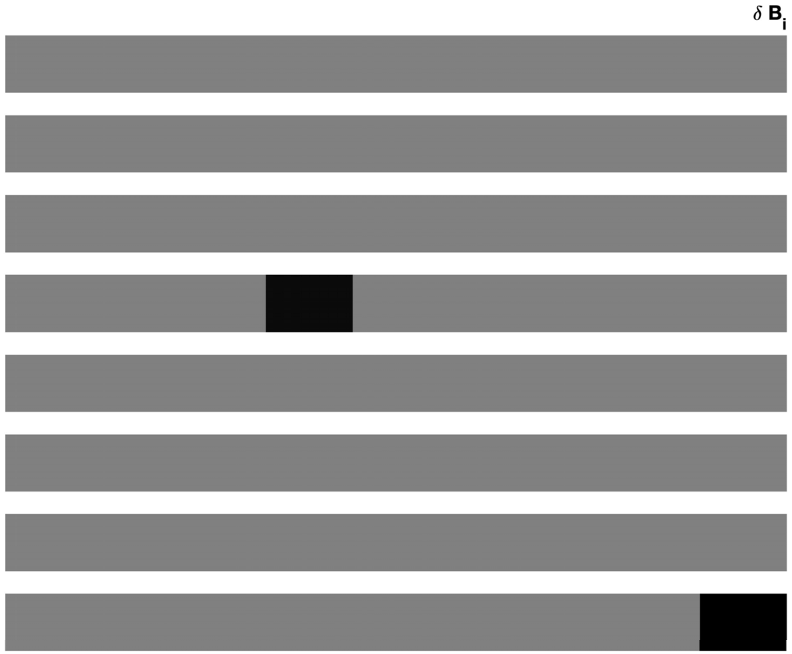

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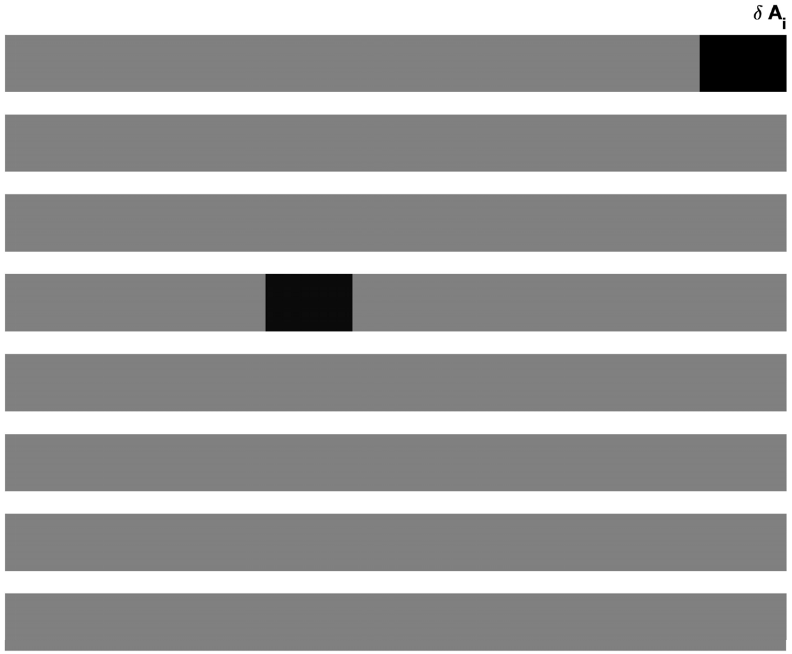

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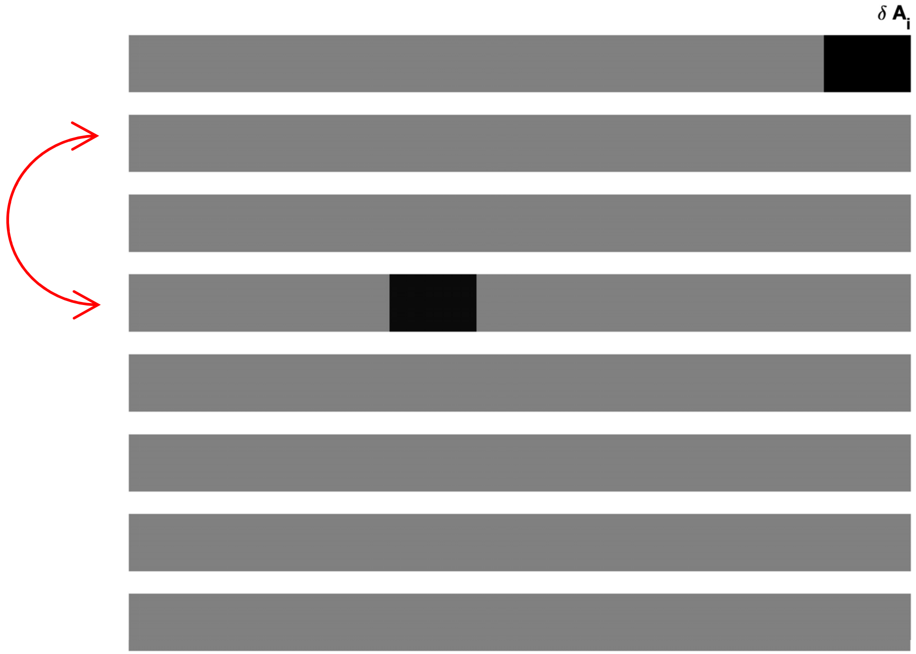

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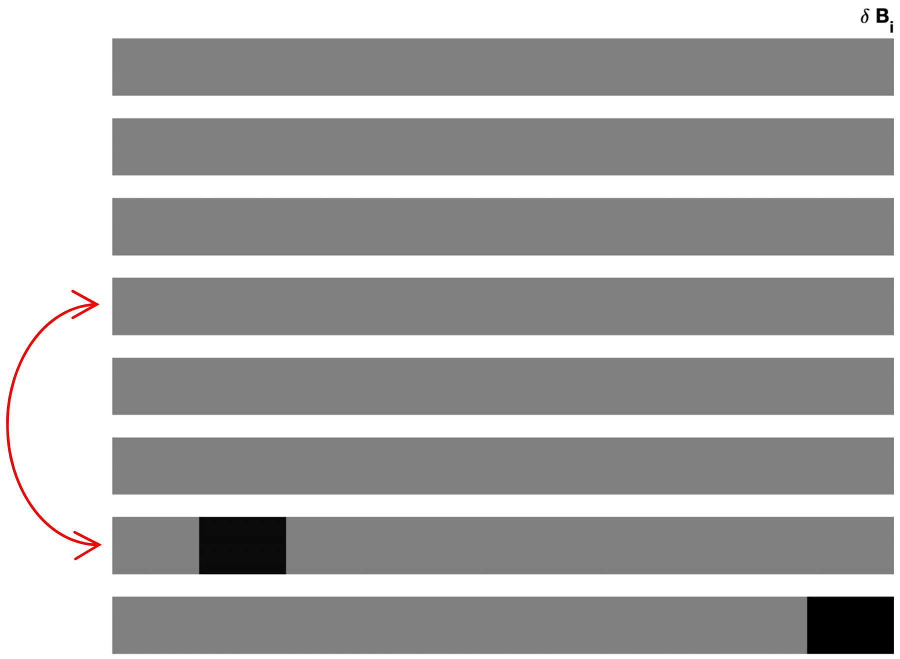

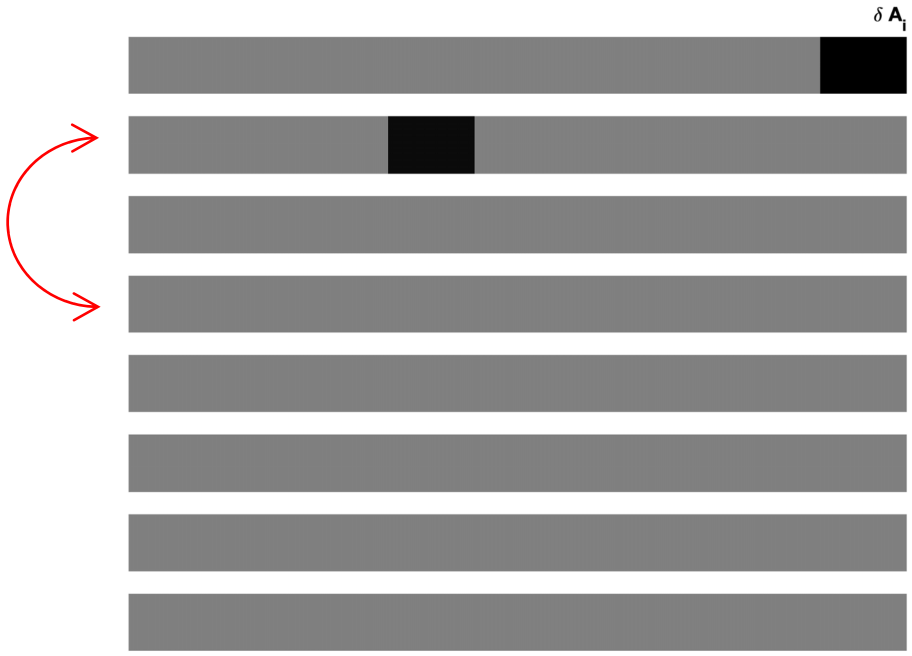

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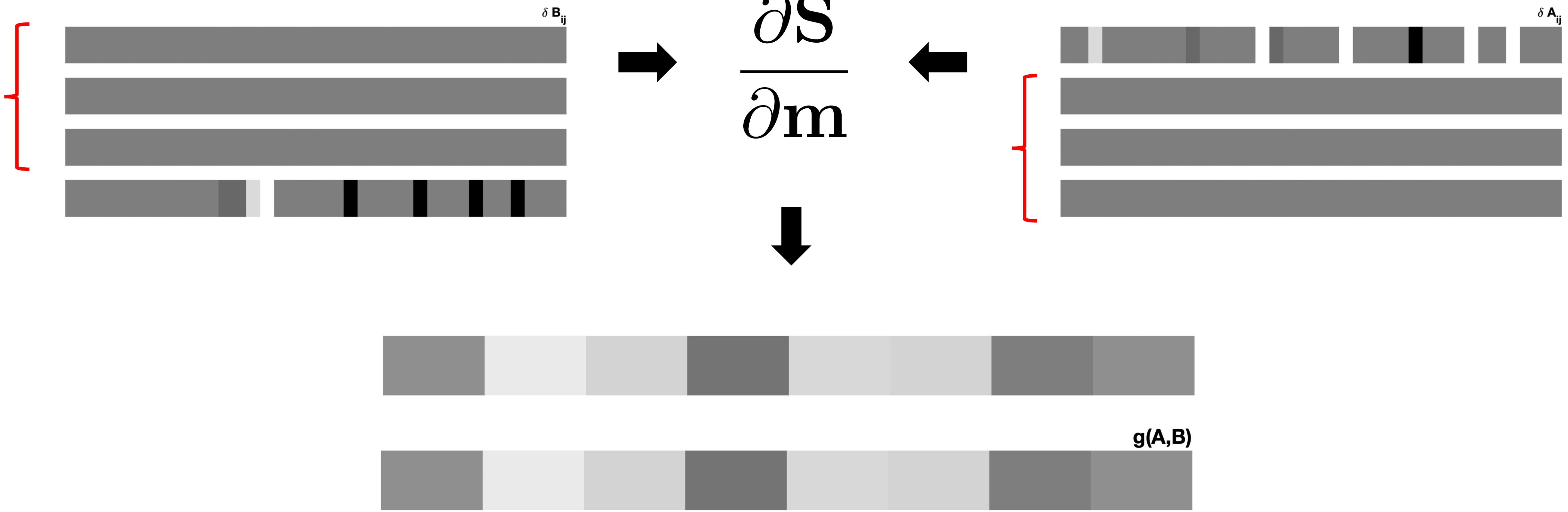

Conclusions

PGA: geometric objects (lines, planes) ➡ algebra

…e.g., solution of **Ax=b** computed in terms of "meets"

Sparse/targeted FWI invokes the same info repeatedly

….but absent "row picture" computation, hard to capitalize

Opportunities:

- Avoidance of massive duplication (~ # targeted grid cells)
- Enable acquisition design minimizing FWI cost?

Requirements:

- Libraries for Geometric Products on significant scale