Predicting the statistical properties of the fluid stack
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Summary
The signal to noise ratio effects the apparent mudrock trend when cross-plotting AVO reflectivity sections such as the P and S impedance reflectivity sections. The effect of noise can be predicted using the covariance matrix and traditional error analysis methodologies. The slope of the data in cross-plot space is influenced by both the geologic trend defined by the mudrock relationship and the error in estimating the reflectivity series. For signal to noise ratios greater than one, this second influence is relatively minor. The fluid stack provides statistically independent information about the fluid and rock properties in a robust fashion.

Introduction
AVO is starting to be used regularly as part of the conventional interpretation methodology. Yet there still remains controversy about how useful it is and how much information can actually be extracted from it. Papers by Cambois (1998), Verm and Hilterman (1995) exemplify two opposite viewpoints. Cambois argues that since the data is noisy, the near and far offset stacks are effectively the only independent information that can be obtained yielding only qualitative measures. Verm and Hilterman suggest it is possible to predict both fluid content and lithology from estimates of elastic parameters such as the delta-Poisson section. The applicability of AVO probably lies somewhere in between these positions. Data is noisy and our knowledge of the earth uncertain. This will create error and uncertainty in our estimates of AVO sections. This paper analyzes the effect of noise on the variance of AVO reflectivity sections such as the P and S impedance reflectivity (Fatti et al, 1994) and linear combinations of them such as the fluid stack (Smith & Gidlow, 1986).

Specifically, this paper examines the problem of measuring AVO parameter estimate uncertainty due to noisy data using linear inversion techniques. The AVO problem is linearized using the Zoeppritz equations (Aki and Richards, 1980) approximation of the Zoeppritz equations assuming that the P-wave and S-wave velocities are known apriori. The elastic parameters may then be inverted using least squares inversion. At the same time, the parameter uncertainty may be estimated from the covariance matrix (Menke, 1984). To simplify the analysis, this paper assumes the only uncertainty is in the data. Further, it is assumed that the noise is random, uncorrelated, with zero mean. The variance of the estimated AVO reflectivity series is a linear combination of the variance due to geology and the variance due to error in inversion process. The slope and correlation coefficient that describe the multivariate data are influenced by both these factors. Model data is used to help understand the relative importance of these influences. A synthetic gather is created with different signal to noise ratios varying from zero to infinity. AVO extractions are performed on these gathers and compared with the theoretical predications to better understand the effect of noise on the ability of the fluid stack to make meaningful predictions about the geology.

AVO Inversion
In order to linearize the AVO problem an approximate form (Aki and Richards, 1980) of the Zoeppritz equations is used
\[
R(\theta) = \frac{1}{2} \left( 1 - 4 \frac{\beta}{\alpha} \sin^2 \theta \right) \frac{\Delta \rho}{\rho} + \frac{1}{2} \frac{\Delta \alpha}{\alpha} \cos^2 \theta - \frac{\beta}{\alpha} \sin^2 \theta, \tag{1}
\]
where \( \alpha, \beta, \rho, \theta \) are the average p-wave velocity, s-wave velocity, density, and the angle of incidence respectively. Similarly, \( \Delta \alpha, \Delta \beta, \Delta \rho \) are the change in p-wave velocity, s-wave velocity and density. Alternatively \( \Delta \alpha/\alpha, \Delta \beta/\beta, \) and \( \Delta \rho/\rho \) are defined to be the P-wave, S-wave and density reflectivity coefficients. Equation (1) may be solved using linear inverse techniques. Generally, the linear inverse problem may be written as
\[
Gm = d, \tag{2}
\]
where \( G \) is the Jacobian of the problem, \( m \) the unknown parameter vector and \( d \) the input data vector. Equation (1) may be written in the form of equation (2) by treating only the density, compressional and shear velocity reflectivity coefficients as unknowns. The \( \beta/\alpha \) ratio and the average angle of incidence \( \theta \) are considered to be known as part of the initial model of the problem. Since the data is recorded as a function of offset, raytracing must be done to transform the functional dependence of the data from offset to average angle of incidence. The Jacobian, \( G \), is defined by the initial model (\( \beta/\alpha \) ratio and the average angle of incidence \( \theta \)) and the acquisition geometry. The unknown parameter vector \( m \) represents the reflectivity coefficients to be solved for and the data vector \( d \) contains the data recorded as a function of offset.
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Since it is assumed that the noise is random and uncorrelated, the linear inverse problem is optimally solved in a least squares fashion \( \mathbf{m} = [\mathbf{G}^T \mathbf{G}]^{-1} \mathbf{G}^T \mathbf{d} \), which may be written more succinctly as \( \mathbf{m} = \mathbf{F} \mathbf{d} \), where \( \mathbf{F} \) is the least squares operator \( \mathbf{F} = [\mathbf{G}^T \mathbf{G}]^{-1} \mathbf{G}^T \). The exact solution for \( \mathbf{m} \) is obtained in the case of noise free data \( \mathbf{d} \). However, only an estimate of the model parameters \( \hat{\mathbf{m}} \) may be obtained when data has noise. The noisy data \( \hat{\mathbf{d}} \) can be written as the sum of the actual data \( \mathbf{d} \) and a noise vector \( \mathbf{\epsilon} \) so \( \hat{\mathbf{d}} = \mathbf{d} + \mathbf{\epsilon} \). The parameter error \( \delta \mathbf{m} = \hat{\mathbf{m}} - \mathbf{m} \) and data misfit \( \mathbf{\epsilon} = \hat{\mathbf{m}} - \mathbf{d} \) may be calculated if the parameter vector \( \mathbf{m} \) is known. In real data, this is never the case. However the data misfit may be estimated using \( \hat{\mathbf{\epsilon}} = \hat{\mathbf{m}} - \hat{\mathbf{d}} \). Since the problem is linear the parameter misfit or uncertainty \( \delta \hat{\mathbf{m}} \) may be estimated from \( \hat{\mathbf{\epsilon}} \).

Based on this analysis, the parameter uncertainty can be estimated by calculating the parameter covariance matrix (Menke, 1984). If the data has uncertainty characterized by the covariance matrix \([\text{cov} \, \mathbf{d}]\), the parameter covariance matrix will be \([\text{cov} \, \mathbf{m}] = \mathbf{F}[\text{cov} \, \mathbf{d}] \mathbf{F}^T\) if the data errors are uncorrelated, with zero mean and the same standard deviation \(\sigma_d\). Then \([\text{cov} \, \mathbf{d}] = \sigma_d^2 \mathbf{I}\) resulting in

\[
[\text{cov} \, \mathbf{m}] = \sigma_d^2 [\mathbf{G}^T \mathbf{G}]^{-1}. \tag{3}
\]

The diagonal of the covariance matrix \([\text{cov} \, \mathbf{m}]\) contains the variance of the parameter estimates. The off diagonal elements show the amount of correlation between the variance of the uncertainty of the variables. The parameter uncertainty is the standard deviation, which is the square root of the parameter variance. Note that this matrix cannot say anything about the statistical independence of the input variables. It is assumed that they are independent going into the analysis. It does contain information about how correlated the parameter uncertainty is.

**Shuey Approximation**

Using the proceeding parameter uncertainty analysis, it becomes clear that if there is even a small amount of noise and one attempts to invert for all 3 unknowns in equation (1), the parameter uncertainty will be unacceptably large. This is shown by Lines (1998) using the Edgehog approach to estimate uncertainty. To address this issue, there are a variety of papers in the literature suggesting various reformulations of equation (1) to make the problem more stable. To illustrate the use of the covariance matrix, this paper will use the two term Shuey (1985) approximation, since it is relatively intuitive to understand. The two term Shuey equation is

\[
R(\theta) = A + B \sin^2 \theta, \tag{4}
\]

where \(A\) is the Intercept and \(B\) is the Gradient. Generally, it is possible, making certain approximations, to transform the output from one formulation to another using the transform matrix

\[
q = C y_1 + D y_2, \tag{5}
\]

where \(q\) is the new output parameter, \(y_1\) and \(y_2\) are the input parameters, and \(C\) and \(D\) are scalars. Assuming \(V_p/V_s = 0.5\), the output parameters of Shuey equation can be transformed to \(P\) and \(S\) impedance reflectivity using

\[
Rp = A, \tag{6}
\]

and

\[
Rs = \frac{A - B}{2}. \tag{7}
\]

The uncertainty associated with \(A\) and \(B\) can be transformed in a similar fashion.

**Sensitivity Analysis of the Covariance Matrix**

The covariance matrix calculated using equation (3) for the two-term Shuey approximation (equation 4) demonstrates the sensitivity of the parameter variance to the acquisition geometry

\[
\begin{bmatrix}
\sigma_A^2 & \sigma_{AB} \\
\sigma_{AB} & \sigma_B^2
\end{bmatrix} = \frac{1}{N^2 \sigma_d^2} \begin{bmatrix}
\sum_i x_i^2 & -\sum_i x_i \\
-\sum_i x_i & N
\end{bmatrix}, \tag{8}
\]

The diagonal elements of the left-hand matrix; \(\sigma_A^2\) and \(\sigma_B^2\) are the variance of the intercept, \(A\), and gradient, \(B\), parameters. The off-diagonal element \(\sigma_{AB}\) shows the degree of correlation between the variance of \(A\) and \(B\). The covariance matrix is solely a function of the data variance, \(\sigma_d^2\), the fold \(N\), the square of the sine of the angle of incidence of the \(i\)th offset, \(x_i = \sin^2 \theta_i\), and

\[
\sigma_i^2 = \langle x^2 \rangle - \langle x \rangle^2, \tag{9}
\]

where \(\langle x \rangle\) is the mean of \(x_i = \sin^2 \theta_i\) and \(\langle x^2 \rangle\) is the mean of \(x_i^2 = \sin^4 \theta_i\). The range of angles and their distribution determine the value of \(\sigma_i^2\). If the distribution is uniform then \(\sigma_i^2\) is only a function of the angle range. The form of equation (8) is the same of equation (1) of Cambois (1998). By noting that the sum of \(x_i\) is equivalent to \(N\) multiplied by the average of \(x_i\), it is easy to see that the variance of the Gradient section \(B\) is solely a function of the fold and the aperture \(\sigma_i^2\). Increasing fold and aperture both decrease the variance of \(B\). Further, the variance of the intercept section \(A\) has the same denominator as that of \(B\), so it too, is largely controlled by fold and aperture. The variance of \(R_P\) and \(R_s\) can be calculated...
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from the variance of A and B using equations (5), (6), (7) and (8). Thus the variance of Rs is

\[
\sigma_{Rs}^2 = \frac{1 + 2N\langle x \rangle + \langle x^2 \rangle}{4(N-1)\sigma_x^2}.
\]

The variance for Rp is simply

\[
\sigma_{Rp}^2 = \frac{\langle x^2 \rangle}{(N-1)\sigma_x^2}.
\]

The covariance matrix for Rp and Rs can be calculated more directly using the two term Fatti approximation. This is more accurate for arbitrary Vp/Vs ratios than the above analysis but leads to a more complex covariance matrix that is not as intuitive.

Cross-plots and Fluid stacks

In order to analyze the two reflectivity sections simultaneously, interpreters often interpret in cross-plot space. It is expected that most of the data will follow some geologic trend and cluster together. The cross-plotted data can be described statistically, by each of the reflectivities variances, the correlation coefficient and the slope. The data, which is anomalous statistically, is considered geologically interesting. The background trends can be predicted using empirical linear models linking the rock and fluid parameters to the elastic parameters. Using this model the geologic variables are the independent variables and both the elastic parameters and their corresponding reflectivity sections are dependant variables. The Fluid stack (Smith & Gidlow 1986) is an example of this. The Rp and Rs section when cross-plotted should cluster along a line whose slope is defined by the mudrock relationship (Castagna et al., 1985) and the background Vp/Vs ratio. Significant departure from the line (cluster) may indicate the presence of gas.

In interpreting the multi-variate data, it is important to understand that the variance of the AVO reflectivity series come from two distinct sources, the geology and error in the reflectivity estimate. The estimate of variance that the covariance matrix provides is an estimate of parameter uncertainty, the error in the reflectivity series. If an anomaly is identified, the interpreter needs to know that the anomaly is a result of geology, not parameter uncertainty. To understand the relative importance of both these types of variances the two extreme cases are examined, the first in which the variance is due totally to the geology and the second that the variance is due totally to noise. In first case, the p-reflectivity and s-reflectivity coefficients are linked together via the Geo-stack relationship (Fatti et al., 1994) equation (7)

\[
\Delta F(t) = Rp(t) - g(t)Rs(t),
\]

where \( \Delta F \) is the Geo-stack and g(t) is a slowly time-varying gain function dependent on the local mudrock relationship and the background velocity ratio. If Rp and Rs are cross-plotted, the slope of the cluster should be defined by g(t). In the second case where Rp and Rs are created from only noise, variances are determined by diagonal terms of the covariance matrix. The correlation coefficient, R, may be calculated using \( R = \sigma_{Rs/Rp} / \sigma_{Rs}\sigma_{Rp} \) and the slope, b, can be calculated using

\[
b = R \cdot \sigma_{Rs} / \sigma_{Rp}.
\]

The slope defined by equation (13) is totally defined by the Jacobian of the problem and gives the no information about the geology. For real data that has noise, the question is; which of the two situations is more representative, equation (12) or (13)?

Example

To help answer this question a synthetic gather was created based on a well log from Western Canada. Compressional, shear velocity and density well log data were available for the well. Noise was added to the synthetic record with different signal to noise ratios from zero to infinity. The total energy within the gather was normalized for comparison purposes. The P and S impedance reflectivity were estimated using equations (6) and (7) above and cross-plotted. The data with no noise (figure 1a) shows a slope that is consistent with that predicted by the fluid stack relationship equation (12). The line shown on each of the figures is the best fit as determined by least squares regression. The cross-plot generated using the all noise record (figure 1e) shows a slope consistent with that of equation (12). Figures 1b) through 1d) show the effect of varying the signal to noise ratio. The statistical properties for these figures are closer to that described by equation (12) than equation (13). Thus for the signal to noise ratios greater than the fluid stack should give valid information about the geology.

Conclusions

The covariance matrix may be used to calculate estimates of parameter uncertainty to help quality control parameter estimates. The covariance matrix may not be used to say anything about the statistical independence of variables being analyzed. The statistical independence of the variables is determined by the underlying geologic relationships defined by the rock properties. This is contrary to the conclusions reached by Cambois (1998).
Further, the fluid stack for signal to noise ratios greater than one should provide reasonable estimates of the geology.

References


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Figure 1: The effect of S/N ratio on slope of the apparent mudrock line. The lower right panel shows a composite display overlaying all 6 best fit lines.