Summary

Fourier analysis is used to present conclusions about stability and dispersion in finite-difference modelling. The most elementary finite-difference model is presented, one dimension in space with second-order differencing in space and time. For this one spatial dimension case, formulae are derived to correct for the dispersion and instability caused by finite grid sampling. The conclusions drawn are compatible with other discussions of stability in one dimension, but an increased understanding of the mechanisms for instability is obtained.

Introduction

There has been much work done on the stability of finite difference algorithms. In the literature on seismic modelling, an example can be found in Aki and Richards (1980). These studies usually follow the von Neumann approach with Fourier analysis of the finite-difference error function.

The method outlined here extends the von Neumann approach in Press et al. (1992) (p836) by making a direct Fourier analysis of the finite-difference method as used in the wave-equation. A single frequency wave is operated upon by the sequence of steps required to obtain a single finite-difference time step, and this is compared to the continuous case. The continuous case is then used as a standard, and the adjustment required to make the finite difference step equal to the continuous case is regarded as a correction.

Theory of finite difference velocities and amplitudes

The scalar wave equation in one dimension can be written as

$$\frac{\partial^2 \phi}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 \phi}{\partial t^2}.$$  \hspace{1cm} (1)

A single wavenumber/frequency solution for this continuous equation can take the form

$$\phi = \sin(kx \pm \omega t) \hspace{1cm} (2)$$

where $k = \omega/v$. A solution of this form can be substituted into the finite-difference second-order equation to show that the continuous and finite difference second derivatives in $x$ are related by

$$D_{\Delta x}^2 \sin(kx \pm \omega t) = \text{sinc}^2 \left( \frac{k\Delta x}{2} \right) \frac{\partial^2}{\partial x^2} \sin(kx \pm \omega t) \hspace{1cm} (3)$$

where $D_{\Delta x}^2$ is the second-order finite-difference in $x$ and $\Delta x$ is the sample rate. Similarly, the second derivatives in time are related by

$$D_{\Delta t}^2 \sin(kx \pm \omega t) = \text{sinc}^2 \left( \frac{\omega \Delta t}{2} \right) \frac{\partial^2}{\partial t^2} \sin(kx \pm \omega t). \hspace{1cm} (4)$$

If the continuous solution is considered to be perfect, equations (3) and (4) may be divided by the sinc$^2$ functions and substituted into equation (1), and the finite differences expanded into
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\[
\frac{\sin(kx \pm \omega(t + \Delta t)) - 2\sin(kx \pm \omega t) + \sin(kx \pm \omega(t - \Delta t))}{\text{sinc}^2\left(\frac{\omega\Delta t}{2}\right)} = \\
\frac{1}{\text{sinc}^2\left(\frac{k\Delta x}{2}\right)} \frac{(\Delta t)^2}{(\Delta x)^2} \left[\sin(k(x + \Delta x) \pm \omega t) - 2\sin(kx \pm \omega t) + \sin(k(x - \Delta x) \pm \omega t)\right]
\] (5)

This equation can be solved for the advanced time step

\[
\sin(kx \pm \omega(t + \Delta t)) = 2\sin(kx \pm \omega t) - \sin(kx \pm \omega(t - \Delta t))
\]

\[
\frac{\text{sinc}^2\left(\frac{k\nu\Delta t}{2}\right)}{\text{sinc}^2\left(\frac{k\Delta x}{2}\right)} \frac{(\Delta t)^2}{(\Delta x)^2} \left[\sin(k(x + \Delta x) \pm \omega t) - 2\sin(kx \pm \omega t) + \sin(k(x - \Delta x) \pm \omega t)\right]
\] (6)

where kv has been substituted for \(\omega\) in the upper sinc function. This is an improvement over the usual second-order finite-difference time-stepping equation because the ratio of squared sinc functions makes it exact for a particular wavenumber and frequency. The usual expression corresponds to setting this ratio to unity. Therefore this ratio can be regarded as a correction factor to be applied after spatial differencing to make the finite time stepping exact.

The errors that result from setting the correction factor to unity may be examined to compare with other studies. There are three possible cases for the values of the correction factor:

1. \(\Delta x = \nu\Delta t\) - the perfect sampling case where the correction factor equals one. Inspection of Figure 1 shows that with the same number of samples per wavelength in x and t, the second differential approximations will be equal.

2. \(\Delta x > \nu\Delta t\) - the dispersive but stable case (see Figure 2), where the correction factor is greater than one. The way that equation (6) is satisfied is with a ‘pseudo’ velocity equal to the square root of the correction factor multiplied by the material velocity as shown in the box. High wavenumbers then propagate more slowly than low wavenumbers as given by the lower curve in Figure 5 and as shown by the narrow wavenumber wavelets in Figure 3.

3. \(\Delta x < \nu\Delta t\) – the dispersive and unstable case, where the correction factor is less than one and where the pseudo velocity propagates high wavenumbers more quickly than low (see Figures 4 and 5). However, with the correction factors omitted, the left-hand side of equation (5) can not be made equal to the right-hand side under all conditions because the \((\nu\Delta t/\Delta x)^2\) factor is greater than 1. In the cases where \(k > 1/(2\nu\Delta t)\) the sinusoidal functions must be combined with growing exponential functions for solutions. The exponential growth (enhanced at the highest wavenumbers) builds formerly insignificant high wavenumbers into dominant amplitudes. This analysis is compatible with those from previous studies, for example in Lines et al. (1999). At the same point (where \(k > 1/(2\nu\Delta t)\)) the sinusoidal components alias and wrap in phase. The upper curve in Figure 5 gives a pseudo velocity for the unstable case and shows the wrapping.

Tests on ‘unstable’ models

A set of model parameters was chosen to illustrate normally unstable conditions (as in Figure 4). In this case the time sample rate was set at .005 seconds, twice the value for stability. Figure 6 shows the correction factor as a function of spatial wavenumber, and it is less than one everywhere as discussed above. This correction factor could be applied to the Fourier transform of the finite difference spatial calculations, or the Fourier transform of this factor could be convolved with the finite difference spatial calculations.
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A broad band wavelet was put into the model with the above parameters and propagated only four steps in the uncorrected mode. The result is shown in Figure 7. It shows the original and propagated wavelets at low amplitude at about 500 offset, and two typical unstable artifacts of high wavenumbers.

The same wavelet and model were used for the corrected modelling procedure. The correction was made by convolving the transform of the operator of Figure 6 with the result of the finite difference spatial calculations. The input wavelet and its propagated version (propagated through 100 steps) are shown in Figure 8. It is clear that the propagation is visually flawless. The attenuation caused by the correction factors in the high wavenumbers is sufficient to eliminate the undesirable side effects.

**Conclusions**

By extending von Neumann analysis of the wave equation, qualitative analysis can be enhanced to derive realistic correction factors to simulate the continuous solution from the finite difference solution.

Correction factors can be combined with the velocity of the material to obtain pseudo velocities which give the rate at which energy is propagated by the uncorrected equations.

**Acknowledgements**

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**References**

Lines, L. R., Slawinski, R. & Bording, P. R., 1999, Geophysics, Vol. 64, No. 3, P. 967-969.

**Figures**

![Figure 1: Sampling of \( \sin(kx + \omega t) \) at perfect ratio \( (\Delta t = \Delta x / v) \)](image1)

![Figure 2: Fine time sampling of \( \sin(kx + \omega t) \) (\( \Delta t = \Delta x / 2v \))](image2)
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Figure 3: Travel of 30 Hz and 60 Hz waves with fine time sampling ($\Delta t = \Delta x / 2v$)

Figure 4: Unstable sampling of $\sin(kx + \omega t)$, low wavenumber ($\Delta t = 2\Delta x / v$)

Figure 5: Stable and unstable (wrapped) pseudo velocity curves

Figure 6: Correction factor for $\Delta t = 2\Delta x / v$

Figure 7: Unstable propagation after 4 steps ($\Delta t = 2\Delta x / v$)

Figure 6: Corrected propagation after 100 steps ($\Delta t = 2\Delta x / v$)