Dual extrapolation algorithms for Fourier shot record migration
Yanpeng Mi* and Gary F. Margrave, CREWES Project, Geology and Geophysics Department, The University of Calgary, 2500 University Drive, Calgary, Alberta T2N 1N4, Canada

Summary
In heterogeneous media, Fourier domain wavefield extrapolators are typically applied recursively over depth to perform migration or modeling. Nonstationary Fourier extrapolators are both highly accurate and able to take large extrapolation steps. To reduce computations costs, a dual algorithm that uses a highly accurate integral wavefield extrapolation to compute reference wavefields at a grid coarser than the imaging grid and employs algorithms of lower accuracy to compute the intermediate wavefields, is presented.

Nonstationary Fourier wavefield extrapolators
The wavefield at depth \( z \) can be obtained by extrapolation the wavefield at depth 0, given knowledge of the velocity field between 0 and \( z \). Consider an \((\alpha k)_z\) domain wavefield at depth 0, \( \phi(k_z, z = 0, \omega) \), assuming constant velocity, the wavefield at depth \( z \) can be written as (Gazdag, 1978),

\[
\phi(k_z, z, \omega) = \phi(k_z, 0, \omega) \alpha(k_z, z, \omega), \tag{1}
\]

where \( \alpha(k_z, z, \omega) = e^{-\omega \frac{2\pi}{c^2} z} \) is the \((\alpha k)_z\) domain wavefield extrapolator operating from 0 to \( z \). The plus and minus signs denote upward and downward extrapolation when the \( z \)-axis points downward. For heterogeneous media, the extrapolation operator is known only approximately and the step size is limited. The extrapolation is done recursively with small steps, so that local wave propagation can be approximated by homogeneous propagation theory.

There are several ways to develop an approximate wavefield extrapolator for heterogeneous media. Typical algorithms are phase-shift-plus-interpolation (PSPI) originally developed by Gazdag and Sguazzero (1984) and the split-step Fourier algorithm originally developed by Stoffa et al. (1990), which is also called the phase-screen algorithm by Wu and Huang (1992). There are many further developments of the above algorithms in order to deal with extreme lateral velocity variation and achieve high efficiency (Jin and Wu, 1998; Popovici, 1996.)

For a complete set of reference velocities, the PSPI algorithm becomes an integral over horizontal wave number \( k_x \), which performs wavefield extrapolation simultaneously with an inverse Fourier transform (Margrave and Ferguson 1999a)

\[
\psi(x, z, \omega) = \frac{1}{2\pi} \int \alpha(x, k_z, z, \omega) \phi(k_z, 0, \omega) \exp(-ik_x x) dk_x, \tag{2}
\]

where \( \alpha(x, k_z, z, \omega) = e^{-\omega \frac{2\pi}{c^2} z} \) is the nonstationary wavefield extrapolator operating from 0 to \( z \). \( \psi(x, z, \omega) \) is the \((\alpha x)\) domain expression of the wavefield at depth \( z \) and \( \phi(k_z, 0, \omega) \) is the \((\alpha k_z)\) domain expression of the wavefield at depth 0. Though there is no interpolation in equation (2), the expression PSPI is used for equation (2) because it is a limiting form of Gazdag’s PSPI.

Equation (2) has a complementary form, called nonstationary phase-shift (NSPS), which performs wavefield extrapolation simultaneously with a forward Fourier transform (Margrave and Ferguson, 1999a)

\[
\psi(x, z, \omega) = \int \alpha(x, k_z, z, \omega) \psi(x, 0, \omega) \exp(ik_x x) dk_x, \tag{3}
\]

where \( \alpha(x, k_z, z, \omega) \) is expressed as before.

Equation (2) can be approximately computed by matrix-vector multiplication, which can be written as

\[
\psi_z = A \psi_0, \tag{4}
\]

where \( \psi_0 \) and \( \psi_z \) are column vectors representing a monofrequency wavefield in \((\alpha k)_z\) domain at depth 0 and the extrapolated wavefield in \((\alpha x)\) domain at depth \( z \), respectively. Matrix \( A \) is the combination of the wavefield extrapolator and the simultaneous inverse Fourier-transform kernel

\[
A = \begin{bmatrix}
\overbrace{e^{-\omega \frac{2\pi}{c^2} z k_{x1} - \omega k_{x1}} & \ldots & e^{-\omega \frac{2\pi}{c^2} z k_{xn} - \omega k_{xn}}} \\
\overbrace{\ldots & \ldots & \ldots} \\
\overbrace{e^{\omega \frac{2\pi}{c^2} z k_{x1} - \omega k_{x1}} & \ldots & e^{\omega \frac{2\pi}{c^2} z k_{xn} - \omega k_{xn}}} 
\end{bmatrix} \tag{5}
\]

Similarly, integral (3) can also be approximated by matrix-vector multiplication, however with \( k_x \) varying in the column direction while \( x \) varying in the row direction.

Margrave and Ferguson (1999b) showed that the NSPS and PSPI can be naturally combined into a symmetric wavefield extrapolator (SNPS) by first performing NSPS for the upper half \( z \) and PSPI for the lower half \( z \) within a single step. A Taylor series derivation of PSPI and NSPS and related error analysis showed that the first-order errors of PSPI and NSPS oppose one another, so that SNPS has a smaller error and is more stable than either PSPI or NSPS alone (Margrave and Ferguson, 2000).
Large-step formulae for PSPI and NSPS

In media of constant velocity, considering downward extrapolation only, the overall phase shift that carries wavefield from 0 to \( z \) can be split into the sum of a static phase shift in \((\omega, x)\) domain and a focusing phaseshift in \((\omega, k_x)\) domain

\[
\Phi = z \sqrt{\frac{\omega^2}{v^2} - k_x^2} = \Phi_{\text{static}} + \Phi_{\text{focus}}
\]

\[
\Phi_{\text{static}} = \frac{\alpha x}{v} \left( 1 - \frac{k_x^2 v(x)^2}{\omega^2} \right)
\]

\[
\Phi_{\text{focus}} = \frac{\alpha x}{v} \left( 1 - \frac{k_x^2 v(x)^2}{\omega^2} \right)
\]

(6)

where \( \Phi_{\text{static}} \) is applied in \((\omega, x)\) domain, and the focusing term \( \Phi_{\text{focus}} \) is applied in \((\omega, k_x)\) domain.

Equation (6) is exact for homogeneous media. It is also approximately applicable for \(v(z)\) media by replacing the static phaseshift velocity with the average velocity and the focusing phaseshift velocity with a depth average velocity (mean velocity). Equation (6) can be easily generalized to media with weak lateral velocity variation. Both the static and the focusing phase-shift become a function of spatial location \( x \)

\[
\Phi(x) = z \sqrt{\frac{\omega^2}{v^2(x)} - k_x^2} = \Phi_{\text{static}}(x) + \Phi_{\text{focus}}(x).
\]

(7)

Similar to the case of constant velocity, the static phaseshift \( \Phi(x)_{\text{static}} \) is applied in \((\omega, x)\) domain, which corresponds to a spatially varying vertical time shift. The focusing phaseshift \( \Phi(x)_{\text{focus}} \) is applied in \((\omega, k_x)\) domain, and causes a time-shift that depends on propagation direction.

Bringing equation (7) to the nonstationary wavefield extrapolator \( \alpha(x,k_x,z,\omega) = e^{-i(\Phi(x)_{\text{static}} + \Phi(x)_{\text{focus}})} \) gives a split nonstationary operator

\[
\alpha(x,k_x,z,\omega) = e^{-i(\Phi(x)_{\text{static}} + \Phi(x)_{\text{focus}})}.
\]

(8)

The PSPI and NSPS integral extrapolation can then be written as

\[
\psi(x,z,\omega) = e^{-i\Phi(x,z,\omega)} \left[ \frac{1}{2\pi} \int \alpha(x,k_x,z,\omega) \exp(-ik_x x) dk_x \right].
\]

(9)

and

\[
\varphi(k_x,z,\omega) = \int \alpha'(x,k_x,z,\omega) \psi(x,0,\omega) e^{-i\omega_0 x} \exp(-ik_x x) dx.
\]

(10)

respectively. \( \alpha'(x,k_x,z,\omega) \) is the nonstationary focusing operator

\[
\alpha'(x,k_x,z,\omega) = e^{-i\Phi'(x,k_x,z,\omega)}.
\]

(11)

Equations (9) and (10) can be written approximately in the form of a matrix-vector multiplication with the \((\omega, x)\) domain static phase-shift term applied before (for NSPS) and after (for PSPI) application of the nonstationary focusing term.

For \(v(z)\) media, both the static and focusing phase-shift become independent of \(x\) and they can be accumulated over depth. The accumulated static phase-shift can be written as

\[
\Phi_{\text{static}} = \int_0^\infty \frac{\omega}{v(z')} dz' = \frac{\alpha x}{v_{\text{ave}}(z)}.
\]

(12)

The accumulated focusing term is

\[
\Phi_{\text{focus}} = \int_0^\infty \left( \frac{k_x^2 v(z')^2}{\omega^2} - 1 \right) dz'.
\]

(13)

For weak lateral velocity variation, these results approximately generalized to

\[
\Phi_{\text{focus}} = \int_0^\infty \left( \frac{k_x^2 v(x,z')^2}{\omega^2} - 1 \right) dz',
\]

(14)

and

\[
\Phi_{\text{static}} = \int_0^\infty \frac{\omega}{v(x,z')} dz' = \frac{\alpha x}{v_{\text{ave}}(x,z)}.
\]

(15)

Expanding the square-root term in equation (14) with power series leads to,

\[
\Phi_{\text{focus}} = -\frac{k_x^2}{2\omega^2} \int_0^\infty \frac{v(x,z')}{z'} dz' - \frac{k_x^2}{8\omega^4} \int_0^\infty v(x,z')^3 dz' + \frac{k_x^6}{16\omega^6} \int_0^\infty v(x,z')^5 dz' - ...
\]

(16)

Note the \(\int_0^\infty v(x,z') dz'\) is exactly the depth average velocity (mean velocity) defined as

\[
v_{\text{ave}} = \frac{1}{z} \int_0^z v(z') dz'.
\]

(17)

This suggests that the equation (14) can be approximated by
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\[ \phi'_{\text{focus}} = \frac{\omega z}{v_{\text{mean}}(x,z)} \left( \sqrt{1 - \frac{k^2}{\omega^2} v_{\text{mean}}(x,z)^2} - 1 \right). \]  \hspace{1cm} (18)

Expanding the equation (18) with power series and then subtracting equation (16) leads to the error term of the focusing phase-shift

\[ \Phi_{\text{error}} = \frac{k^2 z}{8\omega^4} \left[ \int_0^{\infty} v(x,z') dz' - v_{\text{mean}}(x,z) \right] + \frac{k^2 z}{16\omega^4} \left[ \int_0^{\infty} v(x,z') dz' - v_{\text{mean}}(x,z) \right] + \ldots . \]  \hspace{1cm} (19)

This error can generally be ignored when the depth z or the lateral velocity gradient is not too large.

For large-step extrapolation, equations (9) and (10) becomes

\[ \psi(x,z,\omega) = e^{-ik\omega z} \left[ 1 + \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \alpha'_{\text{mean}}(x,k\omega) \exp(-ik\omega z) dk^2 \right] \]  \hspace{1cm} (20)

and

\[ \phi(k, z, \omega) = \int_{-\infty}^{\infty} \alpha'_{\text{mean}}(k,x\omega) \left[ \psi(x,0,\omega) e^{-ik\omega z} \right] \exp(i k\omega x) dx . \]  \hspace{1cm} (21)

where \( \alpha'_{\text{mean}}(x,k\omega,\omega) \) is now the large-step nonstationary focusing operator

\[ \alpha'_{\text{mean}}(x,k\omega,\omega) = e^{-ik\omega z} \left[ 1 - \frac{k^2 v_{\text{mean}}(x,z)^2}{\omega^2} \right] . \]  \hspace{1cm} (22)

Similar to the natural combination of the recursive NSPS and PSPI (Ferguson and Margrave, 1999b), equations (20) and (21) can also be combined naturally to form a one-step symmetric large-step extrapolator. For example, a one-step symmetric large-step extrapolation from depth 0 to z can be easily formulated as

\[ \psi'(x,z,\omega) = e^{-i(k\omega z)_{\text{mean}(z/2-\omega z)}} \left[ \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \alpha'_{\text{mean}}(z/2-\omega z,\omega) \phi(k, z/2, \omega) \exp(-ik\omega x) dx \right] \]  \hspace{1cm} (23)

where

\[ \phi(k, z/2, \omega) = \int_{-\infty}^{\infty} \alpha'_{\text{mean}(0-\omega z/2)} \left[ \psi(x,0,\omega) e^{-ik\omega z} \right] \exp(i k\omega x) dx . \]  \hspace{1cm} (24)

Symbols \((0 \rightarrow z/2)\) and \((z/2 \rightarrow z)\) denote depth interval from 0 to z/2 and z/2 to z.

Equations (23) and (24) require the same amount of computation as the recursive PSPI and NSPS integral as described before. However they allow a much larger extrapolation step. The maximum allowable depth step under a condition of limited phase-error is dependent on the complexity of the velocity field and the maximum angle to be imaged.

Dual algorithms for depth imaging

Consider two \((\omega, x)\) domain mono-frequency reference wavefields at \(z_1\) and \(z_2\): \(\psi(x, z_1, \omega)\). \(\psi(x, z_2, \omega)\), which are computed by the large-step integral extrapolator. An intermediate wavefield between \(z_1\) and \(z_2\): \(\psi(x, z, \omega)\), can be computed by either forward extrapolation from the wavefield at \(z_1\) or by inverse extrapolation from the wavefield at \(z_2\). However, computation of the focusing term with laterally varying velocity is time consuming. It requires the same computing effort as that of a normal PSPI or NSPS integral extrapolator. Since the distance between the upper and lower reference wavefields is small, after a vertical travel time shift, the upper reference wavefield produces an under-focused version of the intermediate wavefield and the lower reference wavefield gives an over-focused version of the intermediate.

Consider forward extrapolation only, the intermediate wavefield at depth \(z\) can be written either as

\[ \psi'(x, z, \omega) = \psi(x, z_1, \omega) e^{-i(z-z_1)_{\text{mean}(z/2-\omega z)}} , \]  \hspace{1cm} (25)

or as

\[ \psi_2(x, z, \omega) = \psi(x, z_2, \omega) e^{-i(z-z_2)_{\text{mean}(z/2-\omega z)}}, \]  \hspace{1cm} (26)

where \(v_{\text{ave}1}\) and \(v_{\text{ave}2}\) are average velocities from \(z_1\) to \(z\) and \(z\) to \(z_2\).

A linear interpolation of equations (25) and (26) estimates the intermediate wavefield if the difference between \(v_{\text{ave}1}\) and \(v_{\text{ave}2}\) is small

\[ \psi(x, z, \omega) = \frac{z-z_2}{z_2-z_1} \psi'(x, z, \omega) + \frac{z-z_1}{z_2-z_1} \psi_2(x, z, \omega) . \]  \hspace{1cm} (27)
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For a larger difference between $v_{ave1}$ and $v_{ave2}$, a single reference velocity split-step Fourier algorithm can be used to roughly correct the focusing phase error.

**Numerical example**

Figure 1. shows a prestack depth imaging of the Marmousi data set (Bourgeois et al, 1991). A 40 m extrapolation step was used to compute the reference wavefields and the depth sampling rate is 4 m. The original 96-trace shot records were padded to 256-traces to accommodate migrated energy. A single-reference velocity split-step Fourier algorithm, using the mean interval velocity between the reference and intermediate levels, and a linear interpolation were used to compute the 9 intermediate wavefields between each pair of reference wavefields. Note the three main fault planes and the top and bottom of the reservoir are well imaged. It took about 2-3 minutes to migrate a shot gather on a single-CPU 128 Mb memory Alpha XP1000 workstation, which is about 8-9 times faster than the recursive PSPI or NSPS integral.

**References**


Margrave, G. F. and Ferguson, R. J., 1999a, Wavefield extrapolation by nonstationary phase shift: Geophysics, 64, no. 04, 1067-1078.


**Figure 1.** Prestack depth imaging of the Marmousi data set with the dual algorithm. The distance between the reference wavefields is 40 m and a single-reference-velocity split-step-Fourier algorithm and a linear interpolation were used to compute the intermediate wavefields.