Generalized Gardner Relations
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Summary
Gardner’s relation is extended to include dependence on both shear and compressional velocity simultaneously. Data for fitting are generated from work on Gardner-type relations by Wang (2000a). It is found that a single relation of this form can reasonably represent several lithologies for either water-saturated or gas-saturated rocks.

Introduction
Gardner’s relation \( \rho = C \alpha^4 \), \( \rho \) = density, \( \alpha \) = compressional velocity has been widely used as a means of providing approximate densities or compressional velocities, when one was available and the other not (Gardner et al., 1974). Other empirical forms for a density-velocity relation are possible, such as Birch’s Law and Lindseth’s relation. One of the advantages of Gardner’s form however is in its application to AVO, as demonstrated by Smith and Gidlow (1987). In traditional AVO one uses information on P-P reflection coefficients as input to an output (\( \beta \) = shear velocity, and \( \Delta \alpha / \alpha \) is the difference over the average of the quantity \( x \) across an interface). This procedure is strongly stabilized by reducing the number of output variables from three to two. One way to accomplish this for linear inversion is by substituting for the density contrast with a relation of the form

\[
\frac{\Delta \rho}{\rho} = c_1 \frac{\Delta \alpha}{\alpha} \quad (1)
\]

Integrating this yields

\[
\ln \rho = c_1 \ln \alpha + c_2 \quad (2)
\]

or

\[
\rho = c_2 \alpha^{c_1} \quad (3)
\]

This is exactly the form of Gardner’s relation, with \( c_1 = \frac{1}{4} \), and the substitution of Equation (1) is precisely what Smith and Gidlow proposed. Such a simple manipulation is not possible in general with other empirical density-velocity relations.

There are a variety of changes one could propose and still retain the essential form of Gardner’s relation. Castagna et al. (1993) for instance proposed a set of lithology specific relations. Work by Dey & Stewart (1997), Potter & Stewart (1998), and Potter (1999) has used Blackfoot log data to develop \( \rho - \beta \) relations. More recently, Wang (2000a) has used laboratory data to develop \( \rho - \alpha \) and \( \rho - \beta \) relations for six different lithologies, for both gas- and water-saturated rocks.

Dey (2001) noted the possibility of developing an empirical relation involving all three variables. One might reason that because of the greater flexibility in fitting permitted by use of an additional variable, this method may have the ability to represent multiple lithologies with accuracy, using only a single expression. Individual lithologies have well-defined relations between \( \alpha \) and \( \beta \) (see, for instance, Mavko et al., Section 7.8), so it is hoped that fitting \( \rho \) to both \( \alpha \) and \( \beta \) will allow the \( \alpha - \beta \) relations to be incorporated implicitly into a generalized Gardner relation.

One use of such a relation would be to predict shear velocity logs from the more commonly obtained density and sonic logs. When lithology is known, \( \alpha - \beta \) relations are generally used to predict \( \beta \) from \( \alpha \). Wang (2000b) has also given a different \( \beta \) prediction technique for sandstones. However if lithology is not known, a single lithology-independent \( \beta(\rho, \alpha) \) relation would be very useful. Another potential application, given the appropriate functional form, is an analogy to the Smith-Gidlow AVO method. In this paper we propose a function of the form

\[
\rho = C \alpha^A \beta^B \quad (4)
\]

which yields the relation

\[
\frac{d \rho}{\rho} = A \frac{d \alpha}{\alpha} + B \frac{d \beta}{\beta}. \quad (5)
\]

Equation (5) clearly has potential use in AVO.

Method & Results for Liquid Saturation
We develop an lithology-independent empirical expression of the form of Equation (4) using Wang’s empirical lithology-specific expressions as input. He gives expressions both for water-saturated rocks (the same case as the original Gardner relation) and for gas-saturated rocks. We will develop an expression for each of these cases. We follow a similar procedure for both. First, for each lithology, we generate a set of densities in a range normal for that lithology. The ranges actually employed (for both gas and water cases) are shown in Table I:

<table>
<thead>
<tr>
<th>Lithology</th>
<th>Range of ( \rho ) (liquid-saturated)</th>
<th>Range of ( \rho ) (gas-saturated)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dolostone</td>
<td>2.45-2.85</td>
<td>2.45-2.85</td>
</tr>
<tr>
<td>Limestone</td>
<td>1.85-2.75</td>
<td>2.30-2.70</td>
</tr>
<tr>
<td>Sandstone (sst)</td>
<td>2.10-2.60</td>
<td>2.10-2.60</td>
</tr>
<tr>
<td>Shale</td>
<td>2.05-2.30</td>
<td>2.20-2.55</td>
</tr>
<tr>
<td>Unconsolidated sst</td>
<td>2.05-2.30</td>
<td>2.05-2.30</td>
</tr>
<tr>
<td>Shale</td>
<td>2.30-2.70</td>
<td>Not used</td>
</tr>
</tbody>
</table>

Table I: Ranges of densities values employed for generating input data for each lithology.
From these density values we employ Wang’s equations [Equations (6) and (7) from Wang, 2000a] to generate values of \( \alpha \) and \( \beta \). These are of the form \( \alpha = a \rho^b \), which is simply a rearrangement of Gardner’s relation, and similarly \( \beta = c \rho^d \). We then use a least-squares procedure to fit the resulting \((\rho, \alpha, \beta)\) triples from all lithologies to the expression

\[
\ln \rho = \ln C + A \ln \alpha + B \ln \beta \tag{6}
\]
to obtain \( A \), \( B \), and \( \ln C \). Associated with the least-squares minimization is a correlation coefficient describing the degree to which the data is well described by the linear function Equation (6). The expression for the correlation coefficient is well known for systems with one dependent variable. By analogy we generalize to two dependent variables, and for completeness include the expression here. The correlation coefficient is given formally by

\[
R = \frac{\left[ \Sigma_i (\ln \rho_i - < \ln \rho >) \right]^2 \times \Sigma_i (A[\ln \alpha_i - < \ln \alpha >] + B[\ln \beta_i - < \ln \beta >])^2 + \Sigma_i (\ln \rho_i - < \ln \rho >)^2 \times \Sigma_i (A[\ln \alpha_i - < \ln \alpha >] + B[\ln \beta_i - < \ln \beta >])^2}{\sqrt{\Sigma_i (\ln \rho_i - < \ln \rho >)^2 \times \Sigma_i (\ln \rho_i - < \ln \rho >)^2}}
\]

where \(<...>\) denotes an average. For units of g/cm\(^3\) and km/s, this yields \( A = -0.091, B = 0.319, \) and \( C = 2.10 \) for the water-saturated state.

These are not our final suggested results however. A test was carried out to see how well \( \alpha \) and \( \beta \) predicted by Wang’s equations satisfy extant \( \alpha-\beta \) relations (Mavko et al., 1998, Sec. 7.8). In fact they deviate somewhat, as illustrated for dolomite in Figure 1 below:

There is generally much less data scatter about a best-fit \( \alpha-\beta \) relation than about a best-fit \( \rho-\alpha \) or \( \rho-\beta \) relation. Thus it seems important to constrain the fitting to satisfy the \( \alpha-\beta \) relations. To accomplish this we replaced each \((\rho, \alpha, \beta)\) in the data set with \((\rho, \alpha, Pa+Q, Q, R, S)\) and \((\rho, R, S, Pa+Q)\), where \((P,Q,R,S)\) are appropriate constants extracted from the \( \alpha-\beta \) relations (Mavko et al., 1998). After including these points in the fit we obtained \( A = 0.0799, B = 0.164, \) and \( C = 1.87 \) using 1000 generated density points for each lithology. The correlation coefficient had a value of 0.83. The sum of \( A \) and \( B \) is 0.244, very close to the \( 1/4 \) exponent of Gardner’s relation. This is reasonable that they are of similar size, but likely a coincidence that they are so nearly identical. \( C \) is also of similar magnitude to the corresponding constant in Gardner’s relation, which is 1.741 (for units of g/cm\(^3\) and km/s).

A visual representation of the fit is presented in Figures 2 and 3 below. Density is plotted against shear and compressional velocity in the following way. The horizontal and vertical axes represent \( \alpha \) and \( \beta \) respectively. For each data triple, the density is plotted at the appropriate location as a short line, whose angle with the horizontal represents the magnitude of the density. A horizontal line represents the minimum density in Table I, and a vertical line represents the maximum density. The solid lines represent input densities from Wang’s lithology-specific relations, and the dotted lines over top represent the density predicted by Equation (4) after fitting to all lithologies simultaneously. For each \((\alpha, \beta)\) pair, the input and output \( \rho \) values cross over at their midpoints. For a very good fit the dashed line lies over the black line and is not visible. For clarity, the displayed results present only 11 density points for each lithology instead of the 1000 used in the fitting procedure. One observes a few results that are noticeably less satisfactory than others. These correspond to low
densities in the limestone lithology, which are predicted to be higher than the input value. In general though, this appears to be a reasonable method for fitting data from several lithologies.

Method & Results for Gas-Saturation and Combined Case

The method for obtaining an expression for gas-saturated rocks was similar to that of the liquid, but with a differing treatment of the $\alpha/\beta$ ratio. Gas-saturated rocks typically have an $\alpha/\beta$ similar to that of their mineral constituents. Ranges of $\alpha/\beta$ predicted by Wang’s relations were compared to values for mineral $\alpha/\beta$ ratios (Mavko, 1998; Christensen, 1982) and an empirical relation for dry sandstone (Wang, 2000b). It was concluded that generation of additional points was not necessary for the gas-saturated case. The other difference is that the shale lithology has not been included (Wang, 2000a).

Using this data we obtained values of $A = 0.192$, $B = 0.106$, $C = 1.612$, and $R = 0.911$ using 100 generated data points. Once again the sum of $A$ and $B$, 0.298, is relatively close to the Gardner value of $\frac{1}{4}$. A visual representation of the fit is shown below in Figures 4:

Figure 4: Similar to Figure 2 but for gas-saturated rocks. Note the poor overlap for low-density clean sandstones.
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Figure 5: An expansion of the lower left corner of Figure 4. Note that the densities of clean and shaley sandstones (equal to the slope of the solid lines) are unequal in the overlapping region. The densities from the fitted function (the slopes of dashed lines) are continuous, fitting as closely as possible to the solid lines.

Finally, for completeness we have obtained a fitting using both water-saturated and gas-saturated values (without generation of any extra points). The result is $A = 0.205$, $B = 0.053$, $C = 1.688$, and $R = 0.847$. We note that the sum of $A$ and $B$, 0.258, is intermediate between the values from only gas- or water-saturated data. The results of this fitting are displayed in Figure 6 below:

![Graph showing comparison of input $\rho$ and output $\rho$.](image)

Figure 6: Similar to Figures 2 and 4 but for both water- and gas-saturated rocks.

Conclusions

Gardner’s relation, $\rho = 1.741 \alpha^{1/4}$ (for $\rho$ in g/cm$^3$ and $\alpha$ in km/s), is generalized herein to $\rho = C \alpha^A \beta^B$, where $C$ is similar in size to 1.741 and $A + B \approx \frac{1}{4}$. Specifically for water-saturated rocks we obtain values of $A = 0.0799$, $B = 0.164$, and $C = 1.87$, while for gas-saturated we obtain $A = 0.192$, $B = 0.106$, and $C = 1.612$. Drawing on both types of saturations simultaneously we obtain $A = 0.205$, $B = 0.053$, and $C = 1.688$. These results are expected to be reasonably accurate for several lithologies. As with the original Gardner relation, these expressions are not expected to possess any physical meaning, but simply serve as empirical correlators of experimental data. As such, these results are expected to be useful both for predicting shear velocities and for substituting velocity contrasts for density contrasts in AVO inversion approximations.

References


