Finite-difference modelling with correction filters in variable velocity
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Summary
The technique of correcting finite-difference models is applied in one and two dimensions. The means of deriving relatively small spatial correction filters is explained and demonstrated. A quantification of uncorrected numerical dispersion is also developed and illustrated.

Introduction
The theory of correction filters for one and two dimensional finite-difference modelling has been presented by the authors in earlier papers (Manning and Margrave 2000, 2001). In those papers we showed how a correction filter could compensate for finite-difference sampling effects (with inherently stable conditions), and thereby obtained results that matched analytic modelling. These papers considered media with a single uniform velocity.

The present paper shows how correction filters may be employed within a model with more than one velocity. We will illustrate our ideas with the one dimensional model because the points are easier to demonstrate and understand. The same considerations apply to a two dimensional model, but only one example will be shown.

It is first shown how a spatial filter of limited size may be designed (in a least squares sense), to provide the benefits of a Fourier domain filter throughout the most relevant (lower wavenumbers) part of the spatial spectrum. This filter is then applied in a realistic, two velocity setting where it is shown that a traveling wavelet preserves its form, and has almost the same amplitude as an analytic wavelet solution.

Finally, a procedure is defined which gives a measure of dispersion for uncorrected modeling, and shows some characteristic effects on a wavelet. This measure may help to select an economic choice of sample rates for a given velocity regime, time length of model, and allowable dispersion.

Theory
In Manning and Margrave 2000 it was shown that, any single wavenumber of a wavefield can be time-stepped exactly, e.g. without dispersion or instability, with the equation

\[ \phi(x,t + \Delta t, k) = 2\phi(x,t,k) - \phi(x,t - \Delta t, k) + \left( \frac{v \Delta t}{\Delta x} \right)^{-1} \text{sinc}\left(\frac{k v \Delta t}{2}\right) \text{sinc}\left(\frac{k \Delta x}{2}\right) D_x^2 \phi(x,t,k), \] (1)

where \( \Delta t \) is the time step, \( \Delta x \) is the spatial grid size, \( v \) is velocity, \( k \) is wavenumber, \( \phi(x,t,k) \) is the single wavenumber of the 1D wavefield, and \( D_x^2 \) is the centered, second order spatial difference

\[ D_x^2 \phi(x,t,k) = \phi(x + \Delta x, t, k) - 2\phi(x,t,k) + \phi(x - \Delta x, t, k), \] (2)

We call the third term on the right-hand side of Equation (1) the acceleration term and we call the ratio of the two squared sinc functions in the acceleration term the correction factor. In the usual time-stepping equation, the correction factor appears as unity, while here it takes the value necessary to make the equation exact. If the wavefield contains more than one wavenumber, the correction factor must be applied in the wavenumber domain. The value of the function \( D_x^2 \phi(x,t,k) \) at a given wavenumber must be multiplied by the correction factor for that wavenumber. An example of an ideal correction factor is shown in the blue curve of Figure (1) for a velocity of 1000 m/sec and the noted sample rates. For example, this curve indicates that the Nyquist wavenumber’s amplitude must be multiplied by 2 to match the analytic result.

Alternatively, the correction factor may be transformed into the space domain, where we call it a correction filter, and convolved with \( D_x^2 \phi(x,t,k) \). A correction filter becomes more practical if it is not too long. This is especially true when the velocity factor in the acceleration term must vary spatially. A discretely sampled, least-squares filter of particular length may be designed by starting with the matrix equation

\[ (F a) \equiv c \] (3)

where double underscores are matrices, single underscores are column vectors, \( a \) is the desired spatial filter, \( c \) is a vector of samples of the correction factor, and \( F \) is a discrete Fourier transform matrix. This matrix is not square but is limited in columns by the choice of length for \( a \). Since the length of \( a \) is assumed to be much less than the length of \( c \), this is an overdetermined system that can be solved by least squares in the standard fashion. An example of the frequency response of a reasonable length
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correction filter is given by the green curve in Figure 1. It is a least-squares fit of a 5 point zero-phase filter designed to match the ideal wavenumber response of the blue curve, but only to 50% of the Nyquist wavenumber marked by the aqua line. The coefficients of this filter are printed near the top of the plot.

Equation (1) shows that the higher frequencies of the finite-difference model must be enhanced for accuracy. This is because for \( \Delta x > v\Delta t \) (for stability), the \( \text{sinc} \) function ratio is greater than zero. An uncorrected model, then, will lose more and more high frequencies with each step.

From the same theory it can be shown that the uncorrected time-stepping equation appears to propagate waves with a wavenumber dependent velocity. Since equation (1) is an exact reformulation of the non-dispersive, constant velocity, 1D wave equation, it must propagate waves with velocity \( v \), independent of wavenumber. However, in the derivation of equation (1) we required that \( v \) be spatially constant but made no assumptions about possible wavenumber dependence. Thus, equation (1) could equally well model a 1D scalar-wave equation with velocity, \( v(k) \), being an arbitrary function of wavenumber. So, if we ask what must this wavenumber dependent velocity be to give the usual time-stepping equation, with correction factor of unity, the answer is immediate that

\[
v(k) = \frac{\text{sinc} \left( \frac{k\Delta x}{2} \right)}{\text{sinc} \left( \frac{k\nu\Delta t}{2} \right)} v.
\]  

Substitution of \( v(k) \) for \( v \) in equation (1) cancels the correction factor and gives the usual time-stepping equation. This wavenumber dependent propagation velocity is often the dominant effect seen with the earlier stages of signal degradation.

**Application of the correction filter**

A one-dimensional finite-difference model with and without the correction filter is shown in Figures 2 and 3. Figure 2 shows the initial Ormsby wavelet (frequencies 20/30 – 50/60) with a dashed line, about to propagate right, and the propagated wavelets with solid lines. The red vertical line marks a velocity boundary. To the left of the boundary the perfect sampling condition exists (\( dt = dx/v \)) because the velocity \( v \) is 2000 m/sec. (actually 1999 m/sec. to ensure stability). Under these conditions no correction filter is required, and the wavelet propagates without dispersion or instability. To the right of the boundary the velocity is 1000 m/sec., and the dispersion is apparent from the change in the waveform.

Figure 3 shows the same model, but with the 5 point correction filter from Figure 1 applied in the area to the right of the boundary. Near the boundary, when the correction filter extends across the boundary, we used the velocity at the center point of the filter. The preservation of the zero phase character of the wavelet in this zone is now consistent with analytic theory.

Comparison of the transmitted and reflected event amplitudes is also very close to what analytic theory predicts. The transmitted wavelet is within 1% of expected, and the reflected wavelet is within 8% of expected.

Figure 4 shows another set of parameters used on a model of the same physical dimensions. As with Figure 2, the processing is uncorrected, but with the sample rates in space and time halved. The character of the transmitted wavelet is much better than the previous uncorrected version, but deviates from the proper zero phase shape as it propagates further from the boundary. The relative amplitudes before moving far from the boundary are quite satisfactory, with the transmitted wave within 1% of expected, and the reflected wave within 3% of expected.

**Numerical dispersion measurement**

An example of the use of equation (4) is shown in Figure 5. The two lines give the lag of a single frequency wavelet behind a wavelet moving at the actual velocity of the material, in this case for 25 and 55 Hertz. The difference between the two curves is a measure of the dispersion within a wavelet of this frequency range. Also shown is the length of a quarter wavelength for the high frequency component. A quarter wavelength lag across a wavelet will appear very similar to a 90 degree phase shift of the wavelet. In this case the critical length corresponds to about 110 steps.

An uncorrected wavelet propagated with these parameters for 110 steps is shown in Figure 6. The roughly 90 degree phase shift appearance is apparent. Also apparent here are the lagging high frequencies and leading low frequencies at the wavelet ends.

A further 110 step propagation is shown in Figure 7. The model here has been designed to wrap around from the right back into the left to save time and space. This has caused an additional 90 degree phase shift to give a 180 degree (reversed) appearance. A total propagation of 440 steps is shown in Figure 8, which has the appearance of a 360 degree phase shift. The growing separation of the high and low frequencies is also becoming more apparent.
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Two dimensional modeling example
The correction filters shown here may also be extended for use on two dimensional models. An example of uncorrected and corrected models are shown in Figures (9) and (10). The filters used to correct the model in (10) consist of two sets of 5, each two dimensional of size 7 by 7. The Manning and Margrave (2001) paper defines these.

Conclusions
We have shown the correction theory given in earlier papers may be used in the form of limited size spatial operators. These operators can then be used in the appropriate sections of a variable velocity medium to provide much more accurate wavelet propagation.

The developed dispersion equation has been used to predict and analyse the results of uncorrected wavelet propagation. This can lead to more efficient design of finite-difference modeling.

References

Figures

Figure 1, A comparison of the ideal and the practical corrections, 1000 m/sec medium.

Figure 2, Uncorrected wavelet propagation. Right side of model has half the velocity of the left side.

Figure 3, Propagation with the right side of the model corrected at each step with a 5 point spatial operator.

Figure 4, Uncorrected wavelet propagation with half the sample rates of Figure 2.
Figure 5, Total dispersion of 25 and 55 Herz wavelets after a given number of steps, 1000 m/sec medium.

Figure 6, Wavelet propagated 110 steps. Note the apparent 90 degree phase shift.

Figure 7, Wavelet propagated 220 steps, wrapped around to the left. Note the apparent 180 degree phase shift (reversal).

Figure 8, Wavelet propagated 440 steps. Shift appears zero phase. The lag of the higher frequencies is more apparent.

Figure 9, Uncorrected two dimensional model. The source is at 1500, just above the velocity contrast.

Figure 10, Corrected model. Reflected P and S, and transmitted P and faint transmitted S waves can be seen.