Improved modelling of spherical-wave AVO

Charles P. Ursenbach, Arnim B. Haase
CREWES, University of Calgary

Jonathan E. Downton
CGGVeritas

Summary

Spherical-wave reflection coefficients, which are vital in modelling supercritical reflections, are considerably more difficult to calculate than their plane-wave analogues. One approach to narrowing this gap is to assume that the incident spherical wave possesses a Rayleigh wavelet. This wavelet, defined by only two parameters in the zero-phase case, allows much of the numerical integration burden to be carried out analytically. We present a set of convenient explicit expressions for this method, which are general enough to use for any values of the wavelet parameters. We further show that spherical-wave reflection coefficient curves for more common wavelets, such as the Ricker and Ormsby wavelets, may be reasonably represented by curves obtained efficiently using the Rayleigh wavelet. For this to occur, the wavelet parameters must first of all be chosen so that both the Rayleigh wavelet and the target wavelet share the same average frequency. After this, a secondary empirical condition is established to uniquely fix the choice of the two Rayleigh wavelet parameters. The precise definition of this second condition depends on the identity of the target wavelet. These results thus make possible the economical calculation of spherical-wave reflection coefficients for Rayleigh, Ricker and Ormsby wavelets.
Introduction

The modelling of reflected spherical waves is well known [e.g. Aki and Richards (1980)] but has not been extensively applied in exploration seismology due to computational demands. For instance, the calculation of spherical-wave reflection coefficients has been realized as a practical scheme by Haase (2004), but for deep reflectors this still requires time-consuming numerical integration.

We have previously proposed a method (Ursenbach et al., 2005) designed to substantially reduce the time requirements for such calculations. This method assumes that the wavelet is of a form which facilitates analytical integration. Here we present a new expression for this method which is more efficient and more general than our earlier result. We also demonstrate the feasibility of using the reflectivity curve calculated with one wavelet to represent the reflectivity curve for another wavelet.

Theoretical expressions

Unlike plane waves, a spherical wave’s reflectivity is influenced by its wavelet. Of particular interest in this study is the Rayleigh wavelet (Hubral and Tygel, 1989) defined as

\[ W_{\text{Rayleigh}}(f) = |f|^n \exp\left(-nf/f_0\right), \quad (1) \]

where \( f \) is the frequency, \( n \) is a positive integer, and \( f_0 \) is the frequency corresponding to the wavelet’s maximum amplitude. Previous work (Ursenbach et al., 2005) has shown that this wavelet leads to a simple expression for a spherical-wave reflection coefficient, \( R_{PP}^{\text{sph}} \), as a function of angle of incidence, \( \theta \):

\[ R_{PP}^{\text{sph}}(\theta) = \left[ \int_{-\infty}^{\infty} \int_{0}^{\infty} R_{PP}(\theta) W_n(\theta, \theta, S) d(\cos \theta) \right]. \quad (2) \]

Here \( \int_{-\infty}^{\infty} \int_{0}^{\infty} \cdots d(\cos \theta) \) expresses a path integration in the complex plane for values of \( \cos \theta \) from 0 to 1 on the real axis and from \( \infty \) to 0 on the imaginary axis. The integrand is a product of the plane-wave reflection coefficient and a weighting function which depends on both \( \theta \) and \( \theta \), as well as \( S \), a parameter controlling the magnitude of spherical effects. It is defined as \( S = \alpha_i / (R\omega_i) \), where \( \alpha_i \) is the P-wave velocity of the layer above the interface, \( R \) is the ray distance travelled between source and receiver, and \( \omega_i \) is the angular frequency, \( 2\pi f_0 \). Here we present a new explicit expression for \( W_n \):

\[ W_n = -\frac{(nS)^{n+2}}{\tau^{n+4}} \frac{BP_n(T/\tau) + CP_{n+1}(T/\tau)}{1 + iSn/(n+1)}, \quad (3) \]

\[ B = (n+1)(i+nS)\tau, \quad (4) \]

\[ C = -(1+n)(nS)^2 - iSn \left[ 2(n+1) + \cos \theta \cos \theta \right] \]

\[ + n(\sin^2 \theta + \sin^2 \theta) + 3(1 - \cos \theta \cos \theta) - 2(\cos \theta - \cos \theta)^2, \quad (5) \]

\[ \tau = \sqrt{T^2 + \sin^2 \theta \sin^2 \theta}, \quad (6) \]

\[ T = nS + i(1 - \cos \theta \cos \theta), \quad (7) \]

where \( P_n(x) \) is a Legendre polynomial. Our previous work (Ursenbach et al., 2005) developed expressions for individual values of \( n \), but equations 2-7 may be readily executed for any value of \( n \).
Representing reflection coefficients for other wavelets

Employing a Rayleigh wavelet (equation 1) yields very efficient calculations of spherical-wave reflectivity via equations 2-7. However, the zero-phase Ormsby and Ricker wavelets are at present more widely used in modelling, and it is of interest to know how well Rayleigh reflection coefficients will represent the reflection behaviour of these more common wavelets. We show that there are simple prescriptions for $f_0$ and $n$ in equation 1 that allow the zero-phase Rayleigh wavelet to mimic the reflection behaviour of Ormsby and Ricker wavelets.

Key to this is the observation that the reflectivity behaviour of wavelets can be reduced to a relatively small number of parameters. From equation 2 it is clear that, for a given earth model, the spherical-wave reflection coefficient curve for Rayleigh wavelets is defined entirely by $n$ and $S$. Equivalently, and more conveniently, we can say they are defined by $n$ and $\bar{S}_z \equiv \alpha_1/(2zf_0^2) = 2\pi S n / [(n+1) \cos \theta z]$, where $z$ is the depth, and $f = f_0(n+1)/n$ is the average frequency of a Rayleigh wavelet. Similar results have been shown for Ricker and Ormsby wavelets (Ursenbach and Haase, 2006). Ricker wavelets are defined by 

$$w^{\text{Ricker}}(f) = f^2 \exp \left[ -\left( \frac{f}{f_0} \right)^2 \right], \quad (8)$$

and the corresponding reflection coefficient curves are defined by the value of $S$ or $\bar{S}_z$ alone. Ormsby wavelets are trapezoidal filters defined by the corner frequencies, $f_1$, $f_2$, $f_3$, and $f_4$. Their reflection coefficient curves are effectively determined by only two parameters. The first is $\bar{S}_z$, and the second is $B_z = (f_1 + f_2)/(f_3 + f_4)$, the ratio of lower and upper band edges. These results simplify the process of representing Ricker and Ormsby reflectivity behaviours by more convenient Rayleigh wavelet calculations.

Table 1 contains earth parameters specifying a Class 1 AVO system with a critical point at $\sim 43^\circ$. Following Haase (2004) we also employ depths of $z = 500$ m, and an overburden P-wave velocity of $\alpha_1 = 2000$ m/s (see Table 1). This information, together with $n$ and $f_0$, completely specifies $\bar{S}_z$, which then determines the deviation from plane wave behaviour for a Rayleigh wavelet of order $n$.

Table 1: Layer parameters for Class 1 AVO modelling in Figures 1 and 2.

<table>
<thead>
<tr>
<th>$\alpha_1/\text{[m/s]}$</th>
<th>$\beta_1/\text{[m/s]}$</th>
<th>$\rho_1/\text{[kg/m}^3\text{]}$</th>
<th>$\alpha_2/\text{[m/s]}$</th>
<th>$\beta_2/\text{[m/s]}$</th>
<th>$\rho_2/\text{[kg/m}^3\text{]}$</th>
</tr>
</thead>
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<td>2000</td>
<td>879.88</td>
<td>2400</td>
<td>2933.33</td>
<td>1882.29</td>
<td>2000</td>
</tr>
</tbody>
</table>

Figure 1a illustrates three possibilities for choosing $f_0$ for a Rayleigh wavelet in order to represent a 5/15-80/100 Hz Ormsby wavelet. First, it is approximated by a 23 Hz Rayleigh wavelet. This attempts to reproduce the lower frequencies, as these might be assumed to be more important in reproducing spherical-wave effects. Second, it is approximated by a 40 Hz wavelet, which matches the average frequency of the Ormsby wavelet. This choice also matches values of $\bar{S}_z$ as $\alpha_1$ and $z$ are unchanged. Finally, it is represented by a 70 Hz wavelet. This represents the upper frequencies of the Ormsby wavelet, and also provides a closer match of bandwidth. All three Rayleigh wavelets share the parameter value $n = 4$.

Figure 1b compares the reflection coefficient curves obtained using $f_0 = 23$, 40, or 70 Hz with that obtained from a fully numerical calculation using the Ormsby wavelet. This illustrates that adjusting $f_0$ so that the average frequency equals that of the target wavelet provides a good qualitative representation of the reflectivity, even though the wavelets themselves appear substantially different.

To explore the effect of $n$ on the fit between different curves, Figure 2a shows a Ricker wavelet reflectivity curve, and several Rayleigh wavelet curves which have the same average
frequency, but which have varying \( n \) values. The plane-wave Zoeppritz curve is also included for comparison. It is seen that all the Rayleigh wavelet curves are reasonable fits to the Ricker curve for most of the graph, so the role of \( n \) is clearly secondary to the role of \( f_0 \). To assign a goodness of fit for each value of \( n \), the L1 norm misfit is calculated between the Ricker curve and each Rayleigh curve. The results are displayed in Figure 2b, showing that \( n = 5 \) is an optimal choice. However it could be argued that the optimal value of \( n \) depends on other parameters, such as the value of \( f_0 \), or the earth parameters, for instance. To test this, two other earth models are constructed (shown in Figure 3), and figures similar to Figure 2 are constructed for various combinations of model, \( f_0 \) value, wavelet, and \( B_r \) value. The \( n \) values determined for these various cases are collected in Table 2. From this data it appears that the optimal \( n \) value is roughly independent of earth model and \( S_z \), and is determined primarily by wavelet type and, in the case of the Ormsby wavelet, \( B_r \). From these and other calculations a reasonable rule of thumb appears to be \( n = 5 \) for Ricker wavelets and \( n = 27B_r \) (rounded) for Ormsby wavelets.

<table>
<thead>
<tr>
<th>Wavelet Name</th>
<th>Optimal ( n ) value for Rayleigh Wavelet</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ricker</td>
<td>( - ) 0.01 5 5 5</td>
</tr>
<tr>
<td>Ormsby 1/9</td>
<td>3 4 3 3</td>
</tr>
<tr>
<td>Ormsby 1/4</td>
<td>7 &gt;7 7</td>
</tr>
<tr>
<td>Ormsby .1</td>
<td>6 6 6</td>
</tr>
<tr>
<td>Ormsby 1</td>
<td>6 6 &gt;7</td>
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</table>

**Table 2 Optimal \( n \) values.**

**Conclusions**

A convenient, programmable expression has been given in equations 2-7 for efficiently calculating spherical-wave reflection coefficients for Rayleigh wavelets. Simple prescriptions have also been given for choosing the Rayleigh parameters \( n \) and \( f_0 \) so that the resulting reflection coefficient curve mimics a curve for Ricker or Ormsby wavelets. In both cases \( f_0 \) should be chosen so that the average frequency of the Rayleigh wavelet matches that of the Ricker or Ormsby wavelet. The value of \( n \) is less critical, but for Ricker wavelets we recommend choosing \( n = 5 \), and for Ormsby wavelets \( n = 27B_r \) (rounded).

These results should allow spherical-wave effects in Class 1 AVO analysis to be accounted for more routinely than at present.

**Acknowledgements**

Support from CREWES and its industrial sponsorship is gratefully acknowledged.

**References**


Figure 1 Comparison of a 5/15-80/100 Hz Ormsby wavelet and three different $n = 4$ Rayleigh wavelets. (a) The wavelets in frequency space. (b) The spherical-wave reflection coefficient curves resulting from the four wavelets. The interface elastic properties are given in Table 1 and the depth of the interface is 500 m. The figure shows that reflection coefficient curves are best matched when the corresponding wavelets have the same average frequency.

Figure 2 Comparison of Ricker and Rayleigh wavelet curves. (a) The reflection coefficient curves for wavelets with the same average frequency, but varying $n$. (b) The L1 norm misfit between the Ricker wavelet curve and the various Rayleigh curves. This figure shows that $n = 5$ is optimal for representing Ricker wavelet curves with Rayleigh wavelet curves.

Figure 3 The plane-wave reflection coefficient curves for three different earth models. Model I (solid) is defined in Table 1. Model II (dotted) is the same as Model I except that $\rho_2 = 2900$ kg/m$^3$. Model III (dashed) is also the same as Model I except that $\rho_2 = 2900$ kg/m$^3$ and $\alpha_2 = 2550$ m/s.