A scattering diagram derivation of the eikonal approximation
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Summary
Scattering diagrams are a way of classifying and manipulating the non-linear terms of forward and inverse scattering, ultimately aiding in the derivation of new seismic processing algorithms. The eikonal approximation, a partial solution of the wave equation relevant to seismic processing, may be derived by manipulation of these diagrams. In fact, the diagram derivation achieves its goal in an arguably less roundabout way than other, superficially more elegant, perturbation methods.

Introduction
The eikonal approximation is an expression for modeling scalar wave propagation (Morse & Feshbach, 1953), which, together with its relative the WKBJ approximation, is relevant to seismic exploration (e.g., Clayton & Stolt, 1981; Amundsen et al., 2005). Non-linear scattering theory too is highly relevant to seismic exploration—inverse scattering diagram analysis has led to powerful algorithms for the removal of multiple reflections from seismic data, and is the subject of current research into processing and inversion of primaries (Weglein et al., 2003). Here we link the two, providing a simple derivation of the eikonal approximation from a scattering-diagram analysis of the Born series. We compare it with the derivation of Morse and Feshbach, which is based on a truncation of the integral in the Lippmann-Schwinger equation.

Consider two simple 1D media, an actual medium \( c(z) \) and a reference medium \( c_0 \), in which actual and reference wavefields, \( G \) and \( G_0 \) respectively, propagate, from a source depth \( z_s \) to an observation depth \( z_g \). \( G \) satisfies

\[
\frac{d^2}{dz_g^2} + \frac{\omega^2}{c^2(z_g)} G(z_g, z_s) = \delta(z_g - z_s),
\]

and \( G_0 \) satisfies the same equation but with \( c_0 \) replacing \( c(z_g) \). Scattering theory is an expression of actual media and fields as series expansions about reference media and fields. Defining \( \alpha(z) = 1 - c_0^2/c^2(z) \), the Lippmann-Schwinger equation is

\[
G(z_g, z_s) = G_0(z_g, z_s) + \int_{-\infty}^{\infty} G_0(z_g, z') k^2 \alpha(z') G(z', z_s) dz',
\]

where \( k = \omega/c_0 \). The Born series then arises through back-substitution of this equation into itself:

\[
G_n(z_g, z_s) = G_0(z_g, z_s) + G_1(z_g, z_s) + G_2(z_g, z_s) + \ldots
\]

where \( G_n \) is \( n \)'th order in \( \alpha \), for instance, at second order,

\[
G_2(z_g, z_s) = \int_{-\infty}^{\infty} dz' G_0(z_g, z') k^2 \alpha(z') \int_{-\infty}^{\infty} dz'' G_0(z', z'') k^2 \alpha(z'') G_0(z'', z_s),
\]

etc. Assuming convergence, summation of a large number of these terms produces an expression for the full wave field. Each term contains propagations and interactions strung together in a chain. For instance, \( G_2 \) involves (reading right to left) reference propagation from \( z_s \) to \( z'' \), where an interaction of strength \( k^2 \alpha \) occurs, then a further propagation from \( z'' \) to \( z' \),

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another interaction, and a final propagation from \( z' \) to \( z_g \). The term \( G_n \) involves \( n \) interactions and \( n+1 \) propagations in the reference medium. To discuss “scattering geometry” is to discuss the characteristic path in \( z \) that all or part of \( G_n \) takes during its \( n \) interactions.

Scattering diagram derivation of the eikonal approximation

Consider a source plane at \( z_s \) embedded in a homogeneous 1D reference medium, above a measurement plane at depth \( z_g \), and assume the reference and actual media to agree at and above \( z_s \), but differ above and below \( z_g \). In Fig. 1 some of the events of the resulting wave field are illustrated. The eikonal approximation is an expression for component (A) in Fig. 1, which dominates when the medium is smooth. The equations in the introduction reflect this arrangement if \( z_s \) is above the depth support of \( \alpha \), and \( z_g \) is below or within the perturbation.

Scattering diagrams arise because of the absolute value operation within the reference Green’s function (De Santo, 1992):

\[
G^0(z^0; z^g) = \left( \frac{1}{2\pi \mu} \right)_{z_s} T_{1g} \left( \frac{z^0 - z_g}{z^0 - z_s} \right)
\]

When this is substituted into the Born series terms, and each term is broken up into cases based on the absolute values, each broken up bit has a characteristic scattering geometry. For instance, \( G_2 \) decomposes into four cases: (A) \( z_g > z', z'>z'' \); (B) \( z_g > z', z'<z'' \); (C) \( z_g < z', z'>z'' \); and (D) \( z_g < z', z'<z'' \). These are represented by scattering diagrams (Fig. 2).

Let us allow the geometry of these diagrams to suggest an approach for deriving certain types of wave solution. We know that the eikonal approximation corresponds to the direct part of the wave; no reflections, or changes in direction with respect to the \( z \) axis occur as this part of the wave propagates. So, let us see what happens if, instead of summing together all terms in the Born series, we reject from the summation any contribution whose diagram involves a change in direction (Fig. 2 B-D). At first order, rejecting scattering interactions taking place below \( z_g \) leaves a portion of the full wave field we call \( T_1 \):

\[
T_1(z_g, z_s) = \frac{e^{ik(z_g - z_s)}}{i2k} \left( -\frac{ik}{2} \int_{z_s}^{z_g} \alpha(z')dz' \right), \quad z_g > z_s.
\]
At second order (Fig. 2), 3 of the 4 contributing terms involve a change in reference propagation direction. Our program retains only the remaining term, which we will call $T_2$:

$$T_2(z_g, z_s) = -\frac{e^{ik(z_g - z_s)}}{i2k} \frac{k^2}{4} \int_{z_s}^{z_g} \alpha(z') \int_{z_s}^{z'} \alpha(z'')dz''dz' = \frac{e^{ik(z_g - z_s)}}{i2k} \frac{1}{2} \left( -\frac{ik}{2} \int_{z_s}^{z_g} \alpha(z')dz' \right)^2, \quad z_g > z_s.$$

Repeating this retention/rejection of scattering diagrams over several orders, a pattern is discerned; we use this pattern to collect such terms at all orders. Calling the result $T$, we have

$$T(z_g, z_s) = \sum_{n=0}^{\infty} T_n(z_g, z_s) = \frac{e^{ik(z_g - z_s)}}{i2k} \sum_{n=0}^{\infty} \frac{1}{n!} \left( -\frac{ik}{2} \int_{z_s}^{z_g} \alpha(z')dz' \right)^n, \quad z_g > z_s.$$  

This is recognizable as a Maclaurin series. Summing it, we obtain

$$T(z_g, z_s) = (i2k)^{-1} e^{ik[z_g - z_s - \frac{1}{2} \int_{z_s}^{z_g} \alpha(z')dz']}, \quad z_g > z_s.$$  

This is the eikonal approximation.

**Direct integration of the Lippmann-Schwinger equation**

We next consider an alternative derivation, that of Morse & Feshbach (1953). Consider again the Lippmann-Schwinger equation, but this time altered, with the integration limit set to the measurement depth $z_g$. Again calling the field variable satisfying this new equation $T$, we have

$$T(z_g, z_s) = G_0(z_g, z_s) + \int_{-\infty}^{z_g} G_0(z_g, z') k^2 \alpha(z') T(z', z_s) dz'.$$

By substituting $G_0$ into this equation, multiplying through by $\exp(-ikz_g)$, and taking the derivative with respect to $z_g$, a differential equation is obtained:

$$-\frac{q\pi^3}{q_{\Pi} (x^3 \cdot x^3)} e^{-ikz_g} - \frac{q_{\Pi} (x^3 \cdot x^3)}{q_{\Pi} (x^3 \cdot x^3)} e^{-ikz_g} = -\frac{q}{q_{\Pi}} e^{-ikz_g} \alpha(x^3) \frac{\partial}{\partial z_g} \int_{z_s}^{z_g} \alpha(z')dz'.$$

This may be directly integrated:

$$T(z_g, z_s) = C e^{ik\left(z_g - \frac{1}{2} \int_{z_s}^{z_g} \alpha(z')dz'\right)},$$

and—importantly for our later discussion—the constant $C$ may be determined as follows. Setting $z_g=0$, and considering $z_s$ to be slightly negative, so that $\alpha=0$ between the integral limits, we have that $T(z_g, z_s)_{z_g=0}=C$. If we further stipulate that the medium is too smooth to reflect any observable wave energy, such that the only contribution to $T$ for this $z_g, z_s$ pair is the direct wave, we may equate this to $G_0$, and $C$ becomes, under our choices,

$$C = (i2k)^{-1} e^{-ikz_s},$$

at which point the eikonal approximation of the previous section is recovered.

**A comparison of the two derivations**

At first blush, since the diagram derivation appears to be “throwing away” much more of the wave field than do Morse & Feshbach, the equivalence of the above methods may seem strange. But in fact, the alteration of the integral limit in the first equation of the previous section rejects more of the field than one might expect. In fact, that integral limiting step is alone equivalent to the alterations associated with the diagram approach; the determination of $C$ is
completely superfluous, in that it brings no new information to the approximation. Let us see that this is true. Beginning again with the altered Lippmann-Schwinger equation, rather than manipulating it as an integral equation, instead we expand it in series through back-substitution, just as if we were deriving the full Born series. Taking care with the variable \( z_g \), we have:

\[
T(z_g, z_a) = G_0(z_g, z_a) + \int^{z_g}_{-\infty} dz' G_0(z_g, z')k^2\alpha(z')
\]

\[
\times \left\{ G_0(z', z_a) + \int^{z'}_{-\infty} dz'' G_0(z', z'')k^2\alpha(z'')
\right.
\]

\[
\times \left[ G_0(z'', z_a) + \int^{z''}_{-\infty} dz''' G_0(z'', z''')k^2\alpha(z''') \left[ G_0(z''', z_a) + \ldots \right] \right\}.
\]

Examination of this expansion in light of Fig. 2 clarifies that the alteration of the Lippmann-Schwinger equation (as introduced by Morse and Feshbach) interrupts contributions to the wave field precisely where they would begin to incorporate scattering interactions that involve a change of propagation direction: the integral limitation is equivalent to the retention of direct diagrams. Substituting \( G_0 \) into the above expansion confirms that the \( T_i \) therein are identical to those in the sum on pg. 3.

Conclusions
The point of these derivations and their comparison is to highlight a benefit of the directness of the diagram approach. Since their end results are the same, the two approaches appear to differ only procedurally. However, because in the truncated Lippmann-Schwinger integral approach the equation was differentiated, we were forced to additionally argue for the form of \( C \). Since it is based on a series, the scattering diagram derivation lacks a certain expediency, but it clarifies that the original integral limitation alone is sufficient to obtain the eikonal approximation in that form. In carrying out the older approach, we now see, it was necessary to at first throw away information critical to the solution, and later return it again by imposing what is in the grand scheme of things a redundant boundary condition. With diagrams we were able to avoid this extra set of arguments. It is worth also emphasizing that making direct waves with a scattering-diagram approach can and has been generalized to multidimensional fields and perturbations (Innanen, 2009), while the truncated Lippmann-Schwinger approach seems to be fundamentally restricted to 1D media. This beneficial directness is common to all diagram-based scattering methods, in forward modeling, and also in inverse scattering seismic processing algorithms.

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References


Morse, P. M., and Feshbach, 1953, Methods of theoretical physics, McGraw-Hill.