Reconciling seismic AVO and precritical reflection FWI – analysis of the inverse Hessian

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SUMMARY

We analyze the role played by amplitude-variation-with-offset (AVO) information in the construction of full waveform inversion (FWI) updates from pre-critical seismic reflection data. In particular an approximate inverse Hessian operator which emphasizes correlations between different parameters collocated in space is discussed. This approximation, which we refer to as being of parameter-type, combines data across angles in a manner consistent with linearized AVO-inversion.

In this paper, the second of two communications on the subject, the details of this approximation are exposed with an analytic calculation of the first iteration in a FWI reconstruction of a single acoustic interface. In the course of the calculation parameter cross-talk is seen to be eliminated leading order.

INTRODUCTION

FWI has advanced significantly in recent years, with progress being reported in multiparameter inversion (Brossier et al., 2010; Plessix et al., 2013; Operto et al., 2013), and in inversion of reflection data (Brossier et al., 2013; Wang et al., 2013). A fully realized methodology for pre-critical reflection data has not as yet been achieved, in part because of longstanding issues such as missing wavelengths (Kelly et al., 2013). We will proceed in this paper assuming pre-critical reflection modes have a significant role to play in practical FWI.

There are some pressing theoretical questions. Reflection data amplitudes require multiple parameters to be properly accounted for (i.e., AVO; Castagna and Backus, 1993; Foster et al., 2010). Managing issues like cross-talk, therefore, will be central to incorporation of pre-critical reflection data amplitudes into FWI. AVO inversion methods (Hampson and Russell, 1990) and their inverse scattering generalizations (Raz, 1981; Clayton and Stolt, 1981), involve data mixtures which, to first order, suppress cross-talk. Does a properly assembled full-FWI update do the same thing automatically? Where in the inverse Hessian operators are to be found the mechanisms which suppress pre-critical cross-talk? In this, the second of two communications on the subject (see also Innanen, 2014), we address these questions, by formulating a version of frequency-domain, and analyzing an approximation we refer to as being of parameter-type.

We consider the reconstruction of a 1D Earth model, in which acoustic parameters $\chi$ (bulk modulus) and $\rho$ (density) vary across a single horizontal interface, by minimizing an objective function defined in terms of a discrete sum over lateral wavenumbers, or equivalently angles, and a continuous sum over temporal frequencies. This configuration isolates the AVO aspect of the problem of interest. Furthermore, every ingredient in this problem, including the data (Weglein et al., 1996), can be supplied analytically, and in terms of ideal results, which are in this case the FWI step-lengths taking us directly to the correct answer. The output of any provisional FWI scheme can thus be judged by comparing it against this ideal.

MINIMIZATION FORMULAS

In scalar minimization, a scalar functional of one function of space is invoked: $\Phi (s(\mathbf{r}))$, with $s$ representing the scalar wave velocity as a function of the position vector $\mathbf{r}$. The scalar Newton step is of the form

$$\delta s(\mathbf{r}) = -\int \mathrm{d}^3 \mathbf{r}' H^{-1}_{1}(\mathbf{r},\mathbf{r}') \Phi (\mathbf{r}') = \Phi (\mathbf{r})$$

where $g_1(\mathbf{r}) = \partial_\mathbf{r} \Phi (s)/\partial_\mathbf{r} s(\mathbf{r})$ and $H_1(\mathbf{r},\mathbf{r}') = \delta^2 g_1(s)/\delta s(\mathbf{r}) \delta s(\mathbf{r}')$, and where $H^{-1}_{1}$ is the inverse of $H_1$ in the sense that

$$\int \mathrm{d}^3 \mathbf{r} H^{-1}_{1}(\mathbf{r},\mathbf{r}') H_1(\mathbf{r}',\mathbf{r}) = \delta(\mathbf{r} - \mathbf{r}')$$

with $\delta(\mathbf{r})$ being the Dirac delta function. At the $n$th iteration, given a current model $s^{(n)}(\mathbf{r})$ and the $n$th gradient and Hessian functions, equation (1) allows us to compute the update $s^{(n+1)}(\mathbf{r}) = s^{(n)}(\mathbf{r}) + \delta s^{(n)}(\mathbf{r})$.

The 2-parameter acoustic extension of this involves two functions, $s_\chi(\mathbf{r})$ and $s_\rho (\mathbf{r})$, representing distributions of bulk modulus $\chi$ and density $\rho$, and an objective function $\Phi_2(s_\chi(\mathbf{r}), s_\rho (\mathbf{r}))$. The gradient in the acoustic problem involves a 2×2 matrix whose elements are integral operators:

$$\left[ \begin{array}{c} \delta s_\chi(\mathbf{r}) \\ \delta s_\rho (\mathbf{r}) \end{array} \right] = -\int \mathrm{d}^3 \mathbf{r}' \left[ \begin{array}{c} H_{\chi\chi}(\mathbf{r},\mathbf{r}') \\ H_{\chi\rho}(\mathbf{r},\mathbf{r}') \end{array} \right] \left[ \begin{array}{c} \delta s_\chi(\mathbf{r}') \\ \delta s_\rho (\mathbf{r}') \end{array} \right]$$

where $H_{ \alpha\beta}(\mathbf{r},\mathbf{r}') = \delta^2 \Phi_2/\delta s_\alpha \delta s_\beta$ are the four Hessian functions ($A$ and $B$ representing $\chi$ and $\rho$ as needed), extensions of the single scalar function $H(\mathbf{r},\mathbf{r}')$. Defining a generalized determinant of the form

$$\mathcal{H}_2(\mathbf{r},\mathbf{r}') = \int \mathrm{d}^3 \mathbf{r}'' [H_{\chi\chi}(\mathbf{r},\mathbf{r}'')H_{\rho\rho}(\mathbf{r}'',\mathbf{r}') - H_{\chi\rho}(\mathbf{r},\mathbf{r}'')H_{\rho\chi}(\mathbf{r}'',\mathbf{r}')]$$

the two parameter Newton step becomes, if the Hessian operators are assumed to commute,

$$\left[ \begin{array}{c} \delta s_\chi(\mathbf{r}) \\ \delta s_\rho (\mathbf{r}) \end{array} \right] = \int \mathrm{d}^3 \mathbf{r}' \mathcal{H}_2^{-1}(\mathbf{r},\mathbf{r}') \int \mathrm{d}^3 \mathbf{r}'' \left[ \begin{array}{c} -H_{\rho\chi}(\mathbf{r}',\mathbf{r}'') \\ H_{\chi\rho}(\mathbf{r}',\mathbf{r}'') \end{array} \right] \left[ \begin{array}{c} g_\chi(\mathbf{r}') \\ g_\rho (\mathbf{r}') \end{array} \right]$$

where $\mathcal{H}_2^{-1}$ is the inverse of $\mathcal{H}_2$ in the sense that

$$\int \mathrm{d}^3 \mathbf{r}'' \mathcal{H}_2^{-1}(\mathbf{r},\mathbf{r}'') \mathcal{H}_2(\mathbf{r}'',\mathbf{r}') = \delta(\mathbf{r} - \mathbf{r}')$$

Equation (3), though not a general formula for the Newton update, is applicable in the cases we will study here. After equation (3) is evaluated using the $n$th gradient and Hessian functions, it can be added to the $n$th model iteration to complete the update:

$$\left[ \begin{array}{c} s_{\chi}^{(n+1)}(\mathbf{r}) \\ s_{\rho}^{(n+1)}(\mathbf{r}) \end{array} \right] = \left[ \begin{array}{c} s_{\chi}^{(n)}(\mathbf{r}) \\ s_{\rho}^{(n)}(\mathbf{r}) \end{array} \right] + \left[ \begin{array}{c} \delta s_{\chi}^{(n)}(\mathbf{r}) \\ \delta s_{\rho}^{(n)}(\mathbf{r}) \end{array} \right]$$

The parameter-type approximation involves retaining the on- and off-diagonal block elements of the 2×2 system in equation (3), but neglecting the off-diagonal components internal to each of these operator elements. When this retention/rejection is applied to each of the Hessian functions $H_{\chi\rho}(\mathbf{r},\mathbf{r}')$ in the acoustic problem, i.e.,

$$H_{\chi\chi}(\mathbf{r},\mathbf{r}') \approx h_{\chi\chi}(\mathbf{r}) \delta(\mathbf{r} - \mathbf{r}'), \quad H_{\chi\rho}(\mathbf{r},\mathbf{r}') \approx h_{\chi\rho}(\mathbf{r}) \delta(\mathbf{r} - \mathbf{r}'), \quad H_{\rho\chi}(\mathbf{r},\mathbf{r}') \approx h_{\rho\chi}(\mathbf{r}) \delta(\mathbf{r} - \mathbf{r}'), \quad H_{\rho\rho}(\mathbf{r},\mathbf{r}') \approx h_{\rho\rho}(\mathbf{r}) \delta(\mathbf{r} - \mathbf{r}'),$$

the inverse of $\mathcal{H}_2$ becomes

$$\mathcal{H}_2^{-1}(\mathbf{r},\mathbf{r}') \approx \frac{\delta(\mathbf{r} - \mathbf{r}')} {h_{\chi\chi}(\mathbf{r})h_{\rho\rho}(\mathbf{r}) - h_{\chi\rho}(\mathbf{r})h_{\rho\chi}(\mathbf{r})}$$
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\[ \rho_1, \eta_0 \]
\[ \rho_1, \eta_1 \]
\[ z_0 = z_1 = 0 \]
\[ \theta \]
\[ R(\theta) \]
\[ z_1 \]

Figure 1: Geometry of a reflection from a single horizontal planar acoustic interface.

and the update formula in equation (3) reduces to:

\[ \frac{\delta x_k(r)}{\delta p}(r) \approx -\frac{1}{h_{kk}(r)h_{pp}(r) - h_{kp}(r)h_{pk}(r)} \left[ \begin{array}{c} -h_{pp}(r) \\ h_{kp}(r) \\ -h_{pk}(r) \\ h_{kk}(r) \end{array} \right] \left[ \begin{array}{c} g_k(r) \\ g_p(r) \end{array} \right], \]  

(8)

which can be used given the \( n \)th gradients and Hessian functions to determine the update. Equation (8), which emphasizes correlations between the different parameters (and neglects correlations between different points in space), will be referred to in the remainder of this paper as a parameter-type update.

**FULL WAVEFORM INVERSION AND AVO**

We next analyze the parameter-type update formula by applying it to the first iteration in the reconstruction of a single acoustic boundary, using analytic data and analytic modelled fields as input.

**Linearized acoustic AVO equations - review**

We invoke the Aki-Richards approximation (Aki and Richards, 2002) in the acoustic limit, expressed in terms of FWI update quantities (Innanen, 2014). A plane wave impinging at angle \( \theta \) on an interface separating properties \( \rho_0 \) and \( \kappa_0 \) from properties \( \rho_1 \) and \( \kappa_1 \) (Figure 1). Defining the reciprocal modulus and density parameters \( s_k = \kappa_k^{-1} \), \( s_0 = \kappa_0^{-1} \), \( s_p = \rho_1^{-1} \) and \( s_p = \rho_0^{-1} \), the reflection coefficient, to first order in the changes \( \delta s_k = s_k - s_0 \) and \( \delta s_p = s_p - s_{p0} \), and \( \sin^2 \theta \), is

\[ R(\theta) \approx \frac{1}{4} \frac{\delta s_k}{s_0} \cos^2 \theta - \frac{1}{4} \frac{\delta s_p}{s_{p0}} \cos^2 \theta. \]  

(9)

If the initial FWI medium model is homogeneous, and the data are due to a reflection from a single acoustic interface at depth \( z_1 \), then the “ideal update”, i.e., that which steps to the correct model, is

\[ \delta s_k(z) = \delta s_k S(z - z_1), \quad \delta s_p(z) = \delta s_p S(z - z_1), \]  

(10)

where \( S(z) \) is the Heaviside function, and where \( \delta s_k \) and \( \delta s_p \) on the right hand sides of equations (10) satisfy equation (9) to first order.

**Ingredients for a FWI update**

We will assume that the data are full bandwidth, source-wavelet deconvolved measurements of an acoustic field \( P \) satisfying

\[ \nabla \cdot \left( \frac{1}{\rho_0(r)} \nabla + \frac{\omega^2}{k(r)} \right) P(r, \omega, \theta) = \delta(r - r_s), \]  

(11)

where \( r_s \) and \( r_r \) are the observation and source points and \( \omega \) is the angular frequency. We also consider modelled fields \( G \) satisfying similar versions of this equation. Expressing the position vectors in a 2D Cartesian system \( r = (x, z) \), and taking the Fourier transform with respect to the lateral observation variable \( x_s \), it can be shown that the sensitivity for the depth-dependent bulk modulus parameter is

\[ \frac{\partial G_n(k_s, z_s - z, \omega)}{\partial s_k^{(n)}(z)}(-\omega^2) \int dx' G_n(k_s, z_s, x', \omega) G_n(x', z_s - z, \omega), \]  

and the sensitivity for a depth-dependent density parameter is

\[ \frac{\partial G_n(k_s, z_s - z, \omega)}{\partial s_p^{(n)}(z)}(-\omega^2) \int dx' \frac{\partial^2 G_n(x', z_s - z, \omega)}{\partial z^2} G_n(x', z_s, x, \omega). \]  

We now focus on the \( n = 0 \) iteration, assuming a homogeneous background medium. This allows us to make use of the analytic solution (Clayton and Stolt, 1981):

\[ G_0(k_s, z, x, z_s, \omega) = \rho_0(2q_s)^{-1} e^{-ik_s x_s} e^{i\theta(z - z_s)}. \]  

(12)

where \( q_s = (\omega/c_0)(\sqrt{1 - k_s^2/c_0^2}) \), where \( c_0 = \sqrt{\rho_0 \kappa_0} \) is the constant background acoustic wave velocity. The data \( D = P(k_s, z_s - z, \omega) \) are an evaluation of the wave \( P \) on the surface \( z = z_s \). Let us now assume these \( D \) correspond to a reflection from a single horizontal interface, and further that \( z_s = z_s - x_s = 0 \). Then (see, e.g., Weglein et al., 1986)

\[ D(\theta, \omega) = \rho_0(2q_s)^{-1} \left[ 1 + R(\theta) e^{i\theta z_s} \right]. \]  

(13)

The relationship between \( \theta \) and \( k_x, q_x \) and \( \omega \) provides the following useful relations (see Figure 2):

\[ \tan \theta = \frac{k_x}{q_x}, \quad \cos \theta = \frac{q_x}{\omega \sqrt{\rho_0 / \kappa_0}}, \quad d \omega = d(2q_x) \left( \frac{c_0}{2 \cos \theta} \right). \]  

(14)

The complex conjugates of the residuals \( \delta P = P - G_0 \) are required to calculate an FWI update. In the first iteration, the modelled field is \( G_0(k_x, 0, 0, 0, \omega) = \rho_0/(2q_x) \). The residual quantity is, therefore,

\[ \delta P_0(k, \omega) = -\rho_0(2q_x)^{-1} R(\theta) e^{-i\theta z_s}. \]  

(15)

In general an objective function penalizes data misfit and undesirable models (Oldenburg, 1984; Parker, 1994; Symes and Carazzzone, 1991; Guittion and Symes, 2003). Here we will employ a simple objective function based on data misfit:

\[ \Phi_2(s_k, s_p) = \frac{1}{2} \sum_x \int d\omega |\delta P|^2, \]  

(16)

in which we sum continuously over \( \omega \) and discretely over \( k_x \).

**Acoustic gradients for depth-varying models**

The unknown Earth properties \( s_k \) and \( s_p \), and therefore the gradient and Hessian functions also, have been restricted to be functions of \( z \) only. Gradients associated with \( \Phi_2 \) in equation (16) are of the form

\[ g_k^{(n)}(z) = -\frac{1}{2} \int d\omega \frac{\partial G_n(k_s, z_s - x_s, z, \omega)}{\partial s_k^{(n)}(z)} \delta P_n, \]  

(17)

with \( A \) representing \( \kappa \) or \( \rho \) as needed. Using the sensitivities for \( \kappa \) and \( \rho \) to analyze the derivatives of equation (16), with the field in equation
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\[ H_{xx}(z, \zeta) = \sum_{k} \frac{\rho_0^4}{(2g_k)^5} \frac{\omega^4}{(2g_k)^5} e^{2\eta_k(z-\zeta)} + \text{terms in } \delta R', \]

in which we distinguish the parts of \( H_{xx}(z, \zeta) \) depending on \( \delta P \) from those not. The remaining Hessian functions are likewise found to be

\[ H_{xp}(z, \zeta) = H_{xp}^{(0)}(z, \zeta), \]

as well as \( H_{pp}^{(0)}(z, \zeta) = H_{pp}^{(0)}(z, \zeta) \), and

\[ H_{pp}^{(0)}(z, \zeta) = \sum_{k} \frac{\rho_0^4}{(2g_k)^5} \frac{\omega^4}{(2g_k)^5} \frac{\delta^2}{\delta \rho^2} e^{2\eta_k(z-\zeta)} + \text{terms in } \delta R'. \]

Calculation of the parameter-type update

In order to form an update of parameter-type, we seek portions of the Hessian functions which can be written

\[ H_{xx}^{(0)}(z, \zeta) \approx h_{xx}^{(0)}(z) \delta(z-\zeta), \quad H_{xp}^{(0)}(z, \zeta) \approx h_{xp}^{(0)}(z) \delta(z-\zeta), \quad H_{pp}^{(0)}(z, \zeta) \approx h_{pp}^{(0)}(z) \delta(z-\zeta), \]

i.e., which are diagonal in their depth-dependence. The residual independent parts of the Hessian functions fit this bill. For instance, in equation (21), we find by neglecting the term proportional to \( \delta P' \),

\[ H_{xx}^{(0)}(z, \zeta) \approx \sum_{k} \frac{c_0^2 \rho_0^4}{52 \cos^5 \theta} \int d(2\eta_k) e^{2\eta_k(z-\zeta)} \approx \sum_{k} \frac{c_0^2 \rho_0^4}{16 \cos^3 \theta} \delta(z-\zeta). \]

The coefficient we seek is, therefore,

\[ h_{xx}^{(0)} = \sum_{k} \frac{c_0^2 \rho_0^4}{16 \cos^3 \theta}. \]

Likewise we find

\[ h_{xp}^{(0)} = h_{xp}^{(0)} = \sum_{k} \frac{c_0^2 \rho_0^4}{16 \cos \theta}. \]

We now have all the necessary ingredients to calculate the first-parameter-type update, as we would for data from a single interface:

\[ \left[ \begin{array}{c} \delta x_k^{(0)}(z) \\ \delta s_{\rho}^{(0)}(z) \end{array} \right] \approx \left[ \begin{array}{cc} 1 & -\frac{h_{xp}^{(0)}}{h_{xx}^{(0)}} \\ -\frac{h_{pp}^{(0)}}{h_{xx}^{(0)}} & h_{pp}^{(0)} \end{array} \right] \left[ \begin{array}{c} g_{x}^{(0)}(z) \\ g_{\rho}^{(0)}(z) \end{array} \right]. \]

Analysis of the parameter-type update

To leading order, the parameter-type approximation suppresses cross-talk. Consider the bulk modulus update at the first iteration, which is formed through the calculation of the first element of equation (25):

\[ \delta \rho_{xx}^{(0)}(z) = \frac{h_{pp}^{(0)} g_{x}^{(0)}(z) - h_{xp}^{(0)} g_{\rho}^{(0)}(z)}{h_{pp}^{(0)} h_{pp}^{(0)} - h_{xp}^{(0)} h_{xp}^{(0)}}. \]

The simplest well-posed inversion involves two angles of data. Evaluating equations (23)–(24) with sums over angles \( \theta_1 \) and \( \theta_2 \)
Meanwhile the denominator becomes

\[
\delta \rho = \left( \frac{s_0}{\rho_0} \right) - \frac{c_0^2}{\rho_0^2} \int \left[ \cos^2 \theta_1 + \cos^2 \theta_2 \right] \left( \frac{\delta \rho}{\rho_0} \right) S(z - \tau_1). \]

and \(H^{(0)}_{\kappa \rho}(\rho^{(0)})\), which is

\[
- \frac{c_0^2}{\rho_0^2} \left[ \cos^3 \theta_1 + \cos^3 \theta_2 \right] \left( \frac{\delta \rho}{\rho_0} \right) S(z - \tau_1). \]

In the numerator these two quantities are subtracted:

\[
\frac{\delta \rho^{(0)}}{\delta \rho}(z) = \frac{\delta \rho^{(0)}(z)}{\rho^{(0)}} = \frac{c_0^2}{\rho_0^2} \int \left[ \cos^3 \theta_1 + \cos^3 \theta_2 \right] \left( \frac{\delta \rho}{\rho_0} \right) S(z - \tau_1). \]

Meanwhile the denominator becomes

\[
\frac{256}{c_0^2} \int \left[ \cos^3 \theta_1 + \cos^3 \theta_2 \right] \left( \frac{\delta \rho}{\rho_0} \right) S(z - \tau_1) \equiv \frac{1}{a} \frac{1}{b} \left( \frac{\delta \rho}{\rho_0} \right) S(z - \tau_1). \]

Assembling these quantities as per equation (26), and recalling that \(s_0 = \left( \frac{\rho_0}{\rho_0} \right)^{-1}\), we find

\[
\delta \rho^{(0)}(z) = \delta \rho S(z - \tau_1), \quad (27)
\]

which is accurate to leading order in \(\delta \rho, \delta \rho,\) and \(\sin^2 \theta\). An identical calculation of the parameter-type update for the density leads to

\[
\delta \rho^{(0)}(z) = \delta \rho S(z - \tau_1). \]

The gradients in the parameter-type update do not act in isolation, but are weighted and subtracted from each other. The weighted subtraction produces a cancellation between terms proportional to \(\delta \rho\). The combination of the off- and on-diagonal elements of the small discrete cofactor matrix is, therefore, the mechanism for suppressing cross-talk. To first order in \(\delta \kappa, \delta \rho,\) and \(\sin^2 \theta,\) we see that the parameter-type inverse Hessian components produce an update which is indistinguishable from the ideal benchmark, and which is free of cross-talk.

**CONCLUSIONS**

We seek a sort of “map” of the Hessian, highlighting where and how important aspects of the multiparameter seismic inverse problem might be managed in some candidate FWI updating scheme. A diagram representing the general two-parameter (acoustic) Hessian is presented in Figure 3a. The approximate update of parameter-type in which we emphasize coupling between parameters is illustrated in Figure 3b. Here, the four block elements are each assumed to be internally diagonal, meaning we neglect correlations between different points in space, but all four such block elements are retained, meaning we do not neglect correlations between parameters. The essential consistency of the first iteration of a parameter-type FWI update and direct AVO inversion is seen as follows. The bulk modulus update in equation (26) evaluated for the single-interface case with two angles \(\theta_1\) and \(\theta_2\) can be written

\[
\frac{\delta \kappa}{\delta \rho} = F_{\kappa}^{\text{fwi}}(\theta_1, \theta_2) R(\theta_1) + F_{\kappa}^{\text{fwi}}(\theta_2, \theta_1) R(\theta_2), \quad (28)
\]

and is a result which has been shown (in a different form) to be a special case of linearized acoustic inverse scattering (Zhang and Weglein, 2009). A parameter-type FWI update and direct AVO inversion are equivalent to the extent that these two formulas agree. There are differences between acoustic and elastic results. The analysis in this paper highlights a quasi-independent system within the full inverse problem, the system isolated in the parameter-type approximation should be expected to have a tangible positive effect on the outcome.

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