Full-Newton Multi-parameter Full Waveform Inversion: Elastic Constants Estimation in HTI Media

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Summary

In this research, we use the multi-parameter full waveform inversion method to inverse the elastic constants in HTI media. We explain the physical meanings of the first-order and second-order terms in the multi-parameter Hessian and propose to calculate the second-order term using backpropagation technique with multi-parameter second-order virtual source. The Full-Newton method has a better convergence rate than Gauss-Newton method by suppressing the artifacts caused by the second-order scattered energy in the data residuals using the second-order preconditioner, which is verified using a 2D HTI numerical example.

Introduction

In recent years, full waveform inversion (FWI) method becomes popular for estimating the subsurface parameters (Tarantola, 1984; Virieux and Operto, 2009; Warner et al., 2013; Margrave et al., 2011; Margrave et al., 2010). It employs full wavefields information for inversion by minimizing the difference between the modelled data and observed data iteratively. FWI can also be used for fracture properties inversion when considering the fractured reservoirs as equivalent anisotropic media. In this research, we focus on inversing the elastic constants in HTI media using Full-Newton multi-parameter FWI.

In mono-parameter FWI, the steep-descent methods ignore the Hessian preconditioner resulting in a blurred and poorly-scaled update. The Hessian operator acts as a deconvolution operator to compensate for geometrical spreading at each image point, de-blur the gradient and suppress the second-order scattering effects (Pratt et al., 1998; Pan et al., 2014; Pan et al., 2014a, 2014b). Inverting multiple parameters using multi-parameter FWI is more challenging. The gradient-based updates in multi-parameter FWI also suffer from the cross-talk problem caused by the coupling effects of the different physical parameters (Operto et al., 2013; Innanen, 2014a, 2014b). The off-diagonal blocks of multi-parameter approximate Hessian indicate the correlation of partial derivative wavefields with respect to two different physical parameters, which can mitigate the cross-talk phenomena for multi-parameter FWI.

The gradient is also contaminated by the second-order scattered energy in the data residuals caused by the second-order partial derivative wavefields. Pratt et al. (1998) discussed and analyzed the second-order term in mono-parameter full Hessian, which accounts for the second-order scattering effects and can be calculated by correlation between the second-order partial derivative wavefields with the data residuals. The second-order term in mono-parameter Hessian can also be calculated using backpropagation technique (Fichtner and Trampert, 2011). The second-order term in multi-parameter Hessian become more complex and the second-order partial derivative wavefields may be caused by the perturbations of two different physical parameters. Incorporating these second-order terms for preconditioning the gradient can eliminate the second-order scattering effects in the gradient. What’s more, the adjoint state method can also be employed for calculating the second-order term. In this paper, we first, analyze the roles of the first-order and second-order terms in multi-parameter FWI with a 2D HTI case. Then, we focus on calculating the second-order term in multi-parameter full Hessian and apply the Full-Newton multi-parameter FWI for elastic constants inversion in HTI media.
Theory and Method

In this section, we will illustrate the multi-parameter update using a 2D HTI model, which can be described by 4 elastic constants (\(c_{33}, c_{55}, c_{11}, c_{13}\)). Thus, the model perturbation vector \(\delta \mathbf{m}(\mathbf{r})\) in multi-parameter Full-Newton FWI can be expressed as:

\[
\delta \mathbf{m}(\mathbf{r}) = \sum_{\mathbf{r}} H^{-1}(\mathbf{r}, \mathbf{r}') g(\mathbf{r}'),
\]

where \(\mathbf{r}=(x, y, z)\) indicates the position of the model parameter, the model perturbation vector \(\delta \mathbf{m}\) and gradient vector \(g\) consist of 4 elements corresponding to the 4 elastic constants:

\[
\delta \mathbf{m}(\mathbf{r}) = \left[\delta c_{33}(\mathbf{r}), \delta c_{55}(\mathbf{r}), \delta c_{11}(\mathbf{r}), \delta c_{13}(\mathbf{r})\right]^T, \quad g(\mathbf{r}') = \left[g_{33}(\mathbf{r}'), g_{55}(\mathbf{r}'), g_{11}(\mathbf{r}'), g_{13}(\mathbf{r}')\right]^T.
\]

where the symbol "^T" means transpose. \(H(\mathbf{r}, \mathbf{r}')\) in equation (1) indicates one element in the full Hessian matrix corresponding to positions \(\mathbf{r}\) and \(\mathbf{r}'\). The full Hessian matrix \(H\) has 4 diagonal blocks corresponding to the second-order partial derivative of the misfit function with respect to the same physical parameter and 12 off-diagonal blocks corresponding to the second-order partial derivative of the misfit function with respect to two different physical parameters. And it can be expressed as:

\[
H = \begin{bmatrix}
H_{3333} & H_{3355} & H_{3311} & H_{3313} \\
H_{5533} & H_{5555} & H_{5511} & H_{5513} \\
H_{1133} & H_{1155} & H_{1111} & H_{1113} \\
H_{1333} & H_{1355} & H_{1311} & H_{1313}
\end{bmatrix}
\]

And it can be written as the summation of the first-order term \(\tilde{H}\) (approximate Hessian) and second-order term \(\mathbf{H}\). Each element in the approximate Hessian is the correlation of two first-order partial derivative wavefields and the each element in the second-order term \(\mathbf{H}\) is the correlation of second-order partial derivative wavefields with the data residuals. For example, the element \(\tilde{H}_{3355}(\mathbf{r}, \mathbf{r}')\) in the off-diagonal block matrix \(\tilde{H}_{3355}\) and the element \(\mathbf{H}_{3355}(\mathbf{r}, \mathbf{r}')\) in the off-diagonal block matrix \(\mathbf{H}_{3355}\) can be expressed as:

\[
\tilde{H}_{3355}(\mathbf{r}, \mathbf{r}') = \frac{\partial \mathbf{u}^+}{\partial c_{33}(\mathbf{r})} \frac{\partial \mathbf{u}^+}{\partial c_{55}(\mathbf{r}')} - \frac{\partial^2 \mathbf{u}^+}{\partial c_{33}(\mathbf{r}) \partial c_{55}(\mathbf{r}')}, \quad \mathbf{H}_{3355}(\mathbf{r}, \mathbf{r}') = \frac{\partial^2 \mathbf{u}^+}{\partial c_{33}(\mathbf{r}) \partial c_{55}(\mathbf{r}')},
\]

where the symbol "^+" means complex conjugate and \(\Delta \mathbf{d}\) is the data residuals vector. The element \(\tilde{H}_{3355}(\mathbf{r}, \mathbf{r}')\) is the correlation of the first-order partial derivative wavefields due to \(\delta c_{33}\) at \(\mathbf{r}\) with the first-order partial derivative wavefields due to \(\delta c_{55}\) at \(\mathbf{r}'\). \(\frac{\partial^2 \mathbf{u}^+}{\partial c_{33}(\mathbf{r}) \partial c_{55}(\mathbf{r}')}\) indicates the second-order partial derivative wavefields and it is formed when the first-order partial derivative wavefields due to \(\delta c_{33}\) at \(\mathbf{r}\) is secondly scattered due to \(\delta c_{55}\) at \(\mathbf{r}'\).

To calculate the second-order term in multi-parameter Hessian, number of \((N_p N_m)^2 / 2\) forward modelling problems needs to be solved, where \(N_m\) is the number of node points and \(N_p\) is the number of physical parameters assigned to describe each node. Considering the first-order partial derivative wavefields due to \(\delta m_i(\mathbf{r})\):

\[
\frac{\partial \mathbf{u}}{\partial m_i(\mathbf{r})} = -L^{-1} \frac{\partial L}{\partial m_i(\mathbf{r})} \mathbf{u},
\]

where \(L\) is the impedance matrix, \(i\) indicates the physical parameter type, and the interaction of the background wavefields with the model perturbation serves as the first-order virtual source \(\tilde{f} = -\frac{\partial L}{\partial m_i(\mathbf{r})} \mathbf{u}\).
Taking partial derivative with respect to $m_j(r')$ on both sides of equation (5) forms the multi-parameter second-order partial derivative wavefields:

$$\frac{\partial^2 u}{\partial m_j(r) \partial m_j(r')} = L^{-1} \left( - \frac{\partial L}{\partial m_j(r)} \frac{\partial u}{\partial m_j(r')} - \frac{\partial L}{\partial m_j(r')} \frac{\partial u}{\partial m_j(r)} - \frac{\partial^2 L}{\partial m_j(r) \partial m_j(r')} u \right),$$

(6)

where $m_j$ means physical parameter different from $m_l$. We can then substitute equation (6) into the second-order term $\mathbf{H}$. For example, element $H_{3355}(r, r')$ in equation (4) becomes:

$$H_{3355}(r, r') = L^{-1} \tilde{f}_{3355} \Delta d^*,$$

(7)

where $\tilde{f}_{3355}$ means the second-order multi-parameter virtual source due to $\delta c_{33}$ and $\delta c_{55}$. Thus, the second-order term in the multi-parameter Hessian can also be constructed by adjoint state technique and the computational cost is reduced to $2N_pN_m$.

**Examples**

In this section, we first give a numerical example to understand the multi-parameter Hessian. The 2D HTI model is a 30x30 model with grid size of 5 m in both horizontal and vertical dimensions and 4 elastic constants $(c_{33}, c_{55}, c_{11}$ and $c_{13})$ are used to describe each node. The initial model is elastic and isotropic with elastic constants $c_{33} = 14.06\text{GPa}$, $c_{55} = 6.32\text{GPa}$, $c_{11} = 14.06\text{GPa}$ and $c_{13} = 1.42\text{GPa}$ (density $\rho = 2.0\text{g/cm}^3$), which is equivalent to P-wave velocity $\alpha = 2651.4\text{m/s}$ and S-wave velocity $\beta = 1777.6\text{m/s}$.

![Figure 1](image)

**Figure 1.** The multi-parameter approximate Hessian $\tilde{\mathbf{H}}$ for elastic constants $c_{33}, c_{55}, c_{11}$ and $c_{13}$.

Figure 1 shows the multi-parameter approximate Hessian $\tilde{\mathbf{H}}$ for elastic constants $c_{33}, c_{55}, c_{11}$ and $c_{13}$. We can see that the multi-parameter approximate Hessian is a 3600x3600 square and symmetric matrix. It has 4 diagonal blocks and 12 off-diagonal blocks and each block is a 900x900 square matrix. We can see that the multi-parameter approximate Hessian is banded due to finite frequency effects and the diagonal block $\tilde{\mathbf{H}}_{3333}$ dominates the whole matrix. This is because $c_{33}$ directly relates to P-wave velocity ($c_{33} = \rho \alpha^2$) and the first-order partial derivative wavefields recorded at the top surface caused by $\delta c_{33}$ is much stronger than those due to other elastic constants. The stronger the amplitudes in the off-diagonal blocks mean the stronger of the cross-talk between different physical parameters.

Figure 2 shows the multi-parameter full Hessian plotted in mode space. Figures 2a, b, c and d show the 555th row in the diagonal blocks $\tilde{\mathbf{H}}_{3333}$, $\tilde{\mathbf{H}}_{3555}$, $\tilde{\mathbf{H}}_{1111}$ and $\tilde{\mathbf{H}}_{1313}$ of the multi-parameter approximate Hessian. Figures 2e, f and g show the 556th row in the off-diagonal blocks $\tilde{\mathbf{H}}_{3355}$, $\tilde{\mathbf{H}}_{3311}$ and $\tilde{\mathbf{H}}_{3313}$. Figures 2h, i, j and k show the 555th row in the diagonal blocks $\tilde{\mathbf{H}}_{3333}$, $\tilde{\mathbf{H}}_{3555}$, $\tilde{\mathbf{H}}_{1111}$ and $\tilde{\mathbf{H}}_{1313}$ of the second-order term. Figures 2l, m and n show the 555th row in the off-diagonal blocks $\tilde{\mathbf{H}}_{3355}$, $\tilde{\mathbf{H}}_{3311}$ and
\( \mathbf{H}_{313} \) of the second-order term. Figures 3a, b and c show the model perturbations for elastic constants for \( c_{33} \) with first-order term preconditioning when increasing the model perturbations from 10% to 20% and 30%. Figures 3d, e and f show the model perturbations for elastic constants for \( c_{55} \) with first-order term preconditioning when increasing the model perturbations from 10% to 20% and 30%. We can see that with increasing the model errors the artifacts caused by second-order scattered energy become more obvious.

![Figures 2, 3, 4](image)

**Figure 2.** The first-order and second-order terms in the multi-parameter full Hessian plotted in model space.

**Figure 3.** The estimated model perturbations contaminated by second-order scattering effects.

**Figure 4.** The estimated elastic constants perturbations using Gauss and Full-Newton multi-parameter FWI. We then apply the Gauss-Newton and Full-Newton multi-parameter FWI on a more complex model. Figures 4a, b, c and d show the true perturbations for elastic constants \( c_{33}, c_{55}, c_{11} \) and \( c_{13} \). Figures 4e, f, g and h show the estimated model perturbations for different elastic constants using Gauss-Newton multi-parameter FWI after 10 iterations. Figures 4i, j, k and l show the estimated model perturbations using Full-Newton multi-parameter FWI iterations. It can be seen that the Full-Newton method can estimate the model perturbations better for incorporating the second-order term in the multi-parameter Hessian.

**Conclusions**

In this research, the second-order term in multi-parameter full Hessian is incorporated for inverting the elastic constants in HTI media. Compared to Gauss-Newton method, Full-Newton method can estimate the model perturbations more efficiently by suppressing the multi-parameter second-order scattering effects.

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References


