SUMMARY

In standard seismic full waveform inversion updates (e.g., of Gauss-Newton type) small angle, backscattered amplitudes are incorporated linearly. Making an effort to include nonlinearity in each update may be useful, however, both for estimation of difficult-to-discriminate parameters such as density, and for improvement of convergence rates.

We consider, in a theoretical scalar environment, one possible approach to including nonlinearity, wherein sensitivities at iteration \( n \) are computed by varying the field associated with the \( n+1 \)th, rather than \( n \)th, model iterate. This produces an extended, series form, sensitivity expression. To understand the basic character of updates based on these revised sensitivities, the expression is truncated at second order, and the resulting Gauss-Newton-like updates are analyzed to expose their use of 1st and 2nd order reflectivity information. Differences between standard and nonlinear updates suggest that the latter may hold promise for the effective incorporation of high angle and high contrast reflectivity information in FWI.

INTRODUCTION

Seismic full waveform inversion updates (Lailly, 1983; Tarantola, 1984; Virieux and Operto, 2009) can be constructed so as to respond to small-angle backscattered data, e.g., pre-critical specular reflections, in a manner consistent with linearized inverse scattering and AVO inversion (Innanen, 2014). This means a multi-parameter reflection FWI updating scheme can be protected against parameter cross-talk to the same degree as AVO and linear inverse scattering. However, it also means that linearization error will be present to the same degree as AVO and linear inverse scattering. However, it also means that linearization error will be present to the same degree as it is in those other methods, and concern registered in those domains (e.g., Weglein et al., 1986) is equally applicable to FWI.

Backscattered wave amplitudes are generally nonlinear in medium property variations. In the special case of two elastic half-spaces, for instance, the Zoeppritz equations define a highly nonlinear relationship between reflection coefficients and relative changes in elastic properties across a reflecting boundary. The relationship is often linearized; in the two half-space example, the Aki-Richards approximation, which is linear in the relative changes (Aki and Richards, 2002; Castagna and Virieux, 1986), is equally applicable to FWI. Sensitivity formulas arising from this involve a range of scattering processes, each involving one instance of \( \delta s \) and multiple instances of \( \Delta s \). These formulas are not immediately ready for use, because \( \Delta s \) is an unknown. However, this particular unknown is closely related to the data residuals \( \delta P \), and the former can be replaced by the latter via inverse scattering methods. An example nonlinear update (of otherwise Gauss-Newton form, based on these new sensitivities, and containing first and second order terms in \( \delta P \)) is then analyzed to verify that the formula correctly incorporates nonlinearity. The first iteration in the reconstruction of a single scalar boundary from a shot record of data is considered, and the improved step-length towards the true model when backscattered data at high angles and/or due to high contrasts are used, is shown.

FWI TERMS AND EQUATIONS

The wave field \( P \) giving rise to seismic data will be assumed to satisfy the scalar space-frequency domain equation

\[
\left[ V^2 + \omega^2 s(r) \right] P(r, r') = \delta(r - r'),
\]

for an observation point at \( r \) and a source at \( r' \). The velocity appears as the squared slowness parameter \( s(r) = c^{-2}(r) \); un-indexed it represents the actual distribution of wave velocities in the subsurface. The medium \( s_n(r) \), different from \( s(r) \), represents the \( n \)th FWI iterate, through which the wave field \( G \) propagates according to

\[
\left[ V^2 + \omega^2 s_n(r) \right] G(r, r', s_n) = \delta(r - r'),
\]

Data are measurements of \( P \) at points \( r = r_s \) on a measurement surface, and these \( P(r_s, r) \) can be compared to the \( G(r_s, r, s_n) \). The objective function, which involves the residuals \( \delta P(r_s, r, s_n) = P(r_s, r) - G(r_s, r, s_n) \):

\[
\phi(s_n) = \frac{1}{2} \sum_{s_n} \int d\omega |\delta P(r_s, r, s_n)|^2,
\]

is minimized iteratively, during which a model iterate \( s_n(r) \) is modified by the update \( \delta s_n(r) \) in order to determine \( s_{n+1}(r) = s_n(r) + \delta s_n(r) \); unindexed \( \delta s(r) \) represents a general variation in the medium. A Gauss-Newton update involving \( \phi(s_n) \) has the form

\[
\delta s_n(r) = - \int dr' H_n^{-1}(r, r') \delta s_n(r'),
\]
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where $g_n$ is the gradient and $H^{-1}$ is an approximation to the inverse Hessian, respectively

$$g_n(r) = -\sum_{j=1}^{N} \int d\omega \frac{\partial G(r, r_j|s_n)}{\partial s_n(r)} \delta P'(r, r_j|s_n), \quad \text{and} \quad H_n(r, r') = \sum_{j=1}^{N} \int d\omega \frac{\partial G(r, r_j|s_n)}{\partial s_n(r)} \frac{\partial G'(r, r_j|s_n)}{\partial s_n(r')} \delta s_n,$$

(5)

the former being the complex conjugate of the residuals in a product with the sensitivities.

Sensitivities involving the $n$th and the $n+1$ st iterates

The standard FWI sensitivity at iteration $n$ is the coefficient $A$ in the expansion $\delta G(s_n) = A \delta s + \ldots$. Here we will investigate the consequences of varying not $G(s_n)$ but rather the field $G(s_{n+1})$, where $s_{n+1}(r)$ is the model iterate we are in the process of determining, and setting as our revised sensitivities $A'$ where $\delta G(s_{n+1}) = A' \delta s + \ldots$. To do this, we will construct two series. The first series is the field in the $n+1$ medium expanded about the field in the nth medium:

$$G(r, r_j|s_{n+1} + \delta s) = G(r, r_j|s_n) + \omega^2 \int d\omega' G(r, r'|s_n) \delta s(r') G(r', r_j|s_n)$$

$$+ \omega^2 \int d\omega' G(r, r'|s_n) [\delta s(r') + \delta s(r) \delta (r - r')] G(r', r_j|s_n)$$

$$\delta s(r) = s_{n+1}(r) - s_n(r)$$

where $\delta s(r) = s_{n+1}(r) - s_n(r)$ is the difference between the $n+1$ and the $n$th media. To construct the second series, we add to $s_n(r)$ on the left a variation $\delta s$ localized at the position $r$, balancing the right side by adding the same quantity to each instance of $s_n(r)$:

$$G(r, r_j|s_{n+1} + \delta s) = \delta G(r, r_j|s_n)$$

$$\frac{\partial G(r, r_j|s_n)}{\partial s_n(r)} \delta s(r) + \ldots$$

(6)

where

$$G_0 = -\omega^2 \int d\omega' G(r, r'|s_n) G(r', r_j|s_n)$$

$$G_{11} = \omega^2 \int d\omega' G(r, r'|s_n) \delta s(r') G(r', r_j|s_n) \Delta s(r')$$

$$G_{12} = \omega^2 \int d\omega' G(r, r'|s_n) G'(r', r_j|s_n) \Delta s(r')$$

(7)

These terms are interpretable in terms of the fields as illustrated in Figure 1. $\Delta s_n$ and $\delta s$ both represent deviations from $s_n$, and so both act as scatterers. In the full series for $\delta G(s_{n+1})$, scattering processes, like $G_0$, $G_{11}$, and $G_{12}$, which involve one interaction with $\delta s$ (see Figures 1a-c) and multiple interactions with $s_n$, contribute to the sensitivities. At iteration $n$, $\Delta s_n(r)$ is an unknown, and so such nonlinear sensitivities appear to have limited practical importance. But, the $\Delta s_n(r)$ are related to the $n$th residuals, and by making this replacement the sensitivities become computable.

Replacements of $\Delta s_n(r)$ with the $n$th residuals

Inverse scattering theory can be used to develop a series relationship between $\delta P'(s_n)$ and $\Delta s_n(r)$. The complex conjugate of the residuals can be expressed as

$$\delta P'(r, r_j|s_n) = \int d\omega' G(r, r'|s_n) \Delta s_n(r') G'(r', r_j|s_n) + \ldots$$

$$\Delta s_n(r) = \delta s_n + \ldots$$

(8)

where in the second line the integral and Green’s functions have been collected into the operator $\delta s$. This series is reverted using standard inverse scattering series technique (Weglein et al., 2003), producing a series expression for $\Delta s_n$ in orders of the residuals:

$$\Delta s_n(r) = \delta s + \Delta s_2 + \ldots$$

(9)

Substituting equation (9) for $\Delta s_n$ in equation (6) generates sensitivities of the form:

$$\frac{\partial G(s_{n+1})}{\partial s(r)} = \frac{\partial G(s_n)}{\partial s(r)} + \frac{\partial G(s_{n+1})}{\partial s(r)}$$

(10)

where the index refers to the order of the term in the residuals $\delta P'(s_n)$.

In the zeroth order term standard FWI sensitivities are recovered:

$$\frac{\partial G(s_{n+1})}{\partial s(r)} = \omega^2 G(r, r_j|s_n) G(r, r_j|s_n) + \ldots$$

The first order term derives from $G_{11}$ in equation (7):

$$\frac{\partial G(s_{n+1})}{\partial s(r)} = \omega^2 \int d\omega' G(r, r'|s_n) G(r, r'|s_n) G'(r, r_j|s_n) + \ldots$$

Approximations are arrived at by truncating the series in equation (10).

Example: 2nd order, collocated $\Delta s_n(r)$ and $\delta s(r)$

Let us next examine a special case of these nonlinear sensitivities, wherein the resulting gradient (1) is second order in the residuals, and (2) simply incorporates nonlinear reflectivity information. This is achieved by truncating equation (10) at order 1, and considering only a portion of the $\delta$ full difference $\Delta s_n(r)$ between the $n$th and the $n+1$ th iterate, as illustrated in Figure 2. In it, the general scattering picture (see Figures 1a-c) is replaced with a $\Delta s_n(r)$ that is localized and collocated with the variation point. This is obtained by setting $\Delta s_n(r') = \delta s(r - r')$ in equations (7), such that second term in equation (10) becomes

$$\frac{\partial G(s_{n+1})}{\partial s(r)} = \omega^2 \int d\omega' G(r, r'|s_n) G(r, r_j|s_n) G(r, r_j|s_n) G'(r, r_j|s_n) \delta P'(s_n),$$

and the quantity $\delta P'(s_n)$ reduces to

$$\delta P'(s_n) = -\frac{1}{\omega^2 G(r, r_j|s_n) G'(r, r_j|s_n)} \delta s_n(r_j|s_n).$$

Putting the lowest two orders of the sensitivity together the following case of nonlinear sensitivities is obtained:

$$\frac{\partial G(s_{n+1})}{\partial s(r)} \approx \omega^2 J_{0D}(r|s_n) \left[ 1 + \frac{\delta P'(s_n)}{J_{0D}(r|s_n)} \right].$$

(11)
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Figure 2: Specular scattering processes illustrated in Figure 1, in which \( \Delta n_s(r) \) and the variation \( \delta n(r) \) are local and collocated.

\[
I_{ID}(r, |s_n|) = \frac{G^*(r_s, r, s_n)G^*(r, r, s_n)}{G(r, r, s_n)}, \quad \text{and} \quad J_{ID}(r, |s_n|) = \frac{G(r_s, r, r, s_n)}{G(r, r, s_n)}.
\]

where

Also with \( n = 0 \). If the initial medium is homogeneous, we can analyze this update using exact forms for the Green’s functions (Clayton and Stolt, 1981): \( G(k_g, r, z, s_n) = (i2q_t)^{-1}e^{-ik_g r + i\omega t(r - z)} \) and \( G(x, z, s_n) = (2\pi)^{-1} \int d\phi (\sqrt{q_t} - ie^{i\omega t(r - z)}|\phi|). \)\]

1.5D ANALYSIS OF A HIGH REFLECTIVITY CASE

In order to verify that they meaningfully incorporate nonlinear information, we consider a 1.5D version of the sensitivity formula in equation (11). In 1.5D the medium varies in depth only, and thus sensitivities are singular, implying that in 3D a principle value for the gradient integral will be required; in the 1.5D cases no poles appear.

\[
J_{ID}(z|s_n) = \int dz' G(k_g, 0, x', z, s_n)G(x', z, 0, 0|s_n),
\]

and

\[
J_{ID}^1(z|s_n) = \int dz' \int dz'' G(k_g, 0, x', z, s_n)G(x', z, x, s_n)G(x, 0, 0, 0|s_n).
\]

Nonlinear sensitivities for the first 1.5D FWI update

For the purposes of analysis we will consider FWI updates derived from residuals in the \((k_g, \omega)\) domain (i.e., one shot record of data Fourier transformed over geophone position and time), holding \( k_g \) fixed. This will allow us to distinguish between updating with high angle (large \( k_g \)) vs. low angle data. The objective function is modified to

\[
\phi(s_n) = \frac{1}{2} \int d\omega |\delta P(k_g, \omega)|^2,
\]

and it is minimized with updates of Gauss-Newton form:

\[
\delta s_n = -\int d\omega H^{-1}(\omega)\delta P(k_g, \omega)|\omega|, \quad \text{and} \quad H_0(\omega) = \int d\omega \left( \frac{\delta G(s_n+1)}{\delta \omega} \right) \delta P_0(k_g, \omega),
\]

with \( n = 0 \); the Hessian is based on standard sensitivities:

\[
H_0(z', \omega) = \int d\omega \left( \frac{\delta G(s_n)}{\delta \omega} \right) \frac{\delta G^*(s_n)}{\delta \omega},
\]

Response of 2nd-order sensitivities to backscattered data

With all the ingredients for the sensitivities now available in analytic form, we may analyze the first iteration in the reconstruction of Figure 3a. The gradient now has two terms, one 1st order in \( \delta P^* \) and the other 2nd, that is, \( g_0(z) = g_0^{(1)}(z) + g_0^{(2)}(z) \), where

\[
g_0^{(1)}(z) = \frac{c_3}{4} R(\omega) \int d\omega \left( \frac{\omega^2/\omega^2_g}{q_t^2} \right) e^{i\omega t(z - z)}|\phi|, \quad \text{and} \quad R(\omega) = \frac{1}{8} (1 + 2i\sin^2 \theta) \left( \frac{\Delta \omega}{\omega} \right)^2,
\]

etc. In Figure 4a, we note by comparing exact, first order \( R \approx R_1 \) and second order \( R \approx R_1 + R_2 \) coefficient calculations that basic linearization error can arise at low angles when contrasts are large, and (in Figure 4b) at high angle even when contrasts are low. We also note that corrections out to as low as second order can lead to significant error reduction, however. With analyzable formulas for \( R \) in hand, we can then analytically express the complex conjugate of the \((k_g, \omega)\) domain residuals at the first iteration (Innanen, 2014):
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Figure 4: (a) $R(\theta)$ for low angles but large contrasts; (b) $R(\theta)$ for small contrasts but large angles.

Noting that (Innanen, 2014) $d\omega = d\omega_0 (c_0/2 \cos \theta)$ and $d\omega_0 = (\omega_0/c_0) \cos \theta$, we can evaluate these integrals and reassemble the gradient, obtaining

$$g_0(z) = \frac{c_0^2}{4 \cos^3 \theta} [R(\theta) - 2R^2(\theta)] |S(z - z_1)|. \quad (23)$$

The Hessian, which we have given a standard Gauss-Newton approximate form (e.g., Virieux and Operto, 2009), evaluates in this simple environment (Innanen, 2014) to

$$H_0(z, z') = c_0^2 \pi (16 \sin^2 \theta)^{-1} \delta(z - z'), \quad (24)$$

and so, via equation (15), the update is of the form

$$\delta s_0(z) = -\frac{4 \cos^2 \theta}{c_0^2} [R(\theta) - 2R^2(\theta)] S(z - z_1). \quad (25)$$

Comparison of 2nd order vs. standard Gauss-Newton update

We characterized the ideal update as $\Delta s(z) = \Delta S(z - z_1)$ and related it to the reflection coefficient through equation (9). To analyze the relative accuracy of the first order and second order FWI iterations, we will substitute two truncations of the series for $R(\theta)$ into equation (25). The standard Gauss-Newton update is recovered by neglecting $R^2$; noting also that to leading order in $\sin^2 \theta$ we may replace $1/\cos^2 \theta$ with $1 + \sin^2 \theta$, we obtain

$$\left(\frac{\delta s_0(z)}{s_0}\right)_{1}\approx-\left(\frac{4}{1+\sin^2\theta}\right)\left[R(\theta)-2R^2(\theta)\right]S(z-z_1). \quad (26)$$

The nonlinear Gauss-Newton-like update, based on second order collocated sensitivities, is

$$\left(\frac{\delta s_0(z)}{s_0}\right)_{2}=\left(\frac{4}{1+\sin^2\theta}\right)\left[R(\theta)-2R^2(\theta)\right]S(z-z_1). \quad (27)$$

Let us first verify that a standard linear Gauss-Newton update is equivalent to the ideal update to 1st order. If contrasts and angles are low, 2nd order contributions to $R$ are negligible, and the reflection coefficient is

$$R(\theta) \approx R_1(\theta) = -\frac{1}{4} (1 + \sin^2 \theta) \frac{\Delta s}{s_0}, \quad (28)$$

substituting this into equation (26) we obtain

$$\left(\frac{\delta s_0(z)}{s_0}\right)_{1} \approx \left(\frac{\Delta s}{s_0}\right) S(z-z_1), \quad (29)$$

deemonstrating the equivalence of $\delta s_0(z)$ and the ideal $\Delta s(z)$. However, if the angle or contrast is such that second order contributions to $R(\theta)$ are non-negligible, referring to equation (9) we must instead substitute

$$R(\theta) \approx -\frac{1}{4} (1 + \sin^2 \theta) \frac{\Delta s}{s_0} + \frac{1}{8} (1 + 2 \sin^2 \theta) \left(\frac{\Delta s}{s_0}\right)^2, \quad (30)$$

and this produces a discrepancy at second order between the Gauss-Newton update $\delta s_0(z)$ and the ideal update $\Delta s(z)$:

$$\left(\frac{\delta s_0(z)}{s_0}\right)_{2} \approx \left(\frac{\Delta s}{s_0}\right) S(z-z_1). \quad (31)$$

Let us compare this with the update generated using the second order sensitivity expression. Substituting the reflection coefficient in equation (30) into equation (27), a corrective term is introduced at second order, exactly suppressing the second order discrepancy corrupting equation (31), such that the resulting update lapses back to

$$\left(\frac{\delta s_0(z)}{s_0}\right)_{2} \approx \left(\frac{\Delta s}{s_0}\right) S(z-z_1). \quad (32)$$

here the consistency of the candidate and ideal updates extends to second order, rather than just first. In Figure 5 the difference between 2nd order sensitivities and (standard) 1st order sensitivities is illustrated. Because we consider a fixed $k_y$, we can examine the accuracy angle by angle; a full inversion would sum over $k_y$ and thus average over these angles. In Figure 5a the interface is large contrast, going from $c_0 = (s_0)\sfrac{1}{2} = 1500\text{m/s}$ in the upper halfspace to $c_1 = (s_1)\sfrac{1}{2} = 1800\text{m/s}$ in the lower; especially in the range $\theta = 0^\circ - 30^\circ$ the difference between the standard Gauss-Newton update and that based on second order sensitivities is significant. Meanwhile in Figure 5b the interface represents a small contrast, with $c_0 = 1500\text{m/s}$, $c_1 = 1600\text{m/s}$, but is examined over a wider range of angles. Here the second order update “sticks to” the exact update to roughly $\theta = 60^\circ$, in contrast to the standard update which deviates significantly at $\theta = 30^\circ$.

CONCLUSIONS

In the application of multiparameter FWI methods to backscattered seismic amplitudes, it may be expedient to incorporate the nonlinearity of the update/residuals relationship in each iteration, enhancing our ability to distinguish difficult parameters, such as density and $V_P$, for which high angle data is typically required. Nonlinearity can be included in FWI updates through the Hessian, or through an alteration of the sensitivities. We consider the latter possibility, and show that if the sensitivities are defined via variation of the medium we are in the process of constructing, as opposed to the current medium, reflection amplitudes are incorporated such that each individual update is correct to second order in the idealized update. The next step is to move this theoretical analysis into a true multiparameter (rather than scalar) environment and continue testing.

ACKNOWLEDGEMENTS

We thank the sponsors of CREWES for continued support. This work was funded by CREWES and the NSERC (Natural Science and Engineering Research Council of Canada) grant CRDPJ 379744-08.