Linearized AVO in viscoelastic media
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SUMMARY

Study of linearized reflectivity is very important for amplitude versus offset (AVO) analysis. Linearized reflection coefficients for interfaces of a low contrast, separating two isotropic, low-loss viscoelastic media are derived using the Zoeppritz equations. To calculate the phase and attenuation angles in each layer in terms of the perturbations, we linearized the generalized Snell’s law in viscoelastic medium and showed that perturbation in attenuation angles is a sum of the perturbations in corresponding velocity and quality factor weighted by phase and attenuation angle averages. The ray parameter and slowness vectors are introduced as a function of attenuation angle and incident angle. For incident inhomogeneous P-wave, we show that reflection coefficient is a complex function, that its imaginary part is due to anelasticity in medium.

INTRODUCTION

Reflection coefficients have a complicated dependency upon density, velocities and quality factors. Using the exact form of these functions we can not explicitly determine the dependency of the reflection coefficients upon the physical properties. For a two layer elastic medium with a low contrast in physical properties, the approximate form of the reflectivities is derived by Aki and Richards(2002). There are two main assumption in the procedure of linearization. First of all the differences in the properties along the boundary are small compared to the average of the properties above and below the interface. Second assumption is that the angle of incident should be smaller that any critical angle.

Another approach for linearizing the reflection coefficients is the Born approximation based on perturbation theory. In this method the actual medium in which the wave propagates is decomposed to a reference medium whose properties are known, plus unknown perturbations in the properties.

The approach to calculating the viscoelastic linearized reflectivity is to express the quantities like density, P- and S-wave velocities and corresponding quality factors, in terms of differences between these quantities in upper and lower media. Besides the above physical quantities, we have the incident and transmitted phase and attenuation angles that should be expressed in terms of perturbations in phase and attenuation angles. In this paper we introduce the Zoeppritz equations for viscoelastic medium and linearize them according to the assumptions we mentioned above. In this procedure we analyze Snell’s law in anelastic medium and linearized that. Finally we introduce a map that converts the linearized reflectivities to the scattering potential obtained by Born approximation.

Anelasticity is generally held to be a key contributor to seismic attenuation, or “seismic Q”, which has received several decades worth of careful attention in the literature Aki and Richards (2002); Futterman (1967). Development of methods for analysis Tomn (1991), processing Bickel and Natarajan (1985); Hargreaves and Calvert (1991); Innanen and Lira (2010); Schwaiger et al. (2007); Wang (2006); Zhang and Ulryan (2007), and inversion Dahl and Ursin (1992); Causse et al. (1999); Innanen and Weglein (2007); Hicks and Pratt (2001); Ribodetti and Virieux (1998) of wave data exhibiting the attenuation and dispersion of seismic Q remains a very active research area.

Wave propagation in linear viscoelastic media has been extensively studied, both numerically and analytically Borcherdt (973a,b, 1977, 1982, 1985a); Borcherdt et al. (1986); Borcherdt (1985b, 2009); Carcione et al. (988a); Carcione (1993); Carcione et al. (988b), results for viscoelastic anisotropic media have also been derived Behura and Tsvankin (006a); Cerveny and Psencik (2005a,b,c); Ursin and Stovas (2002). Borcherdt Borcherdt (2009) has presented a complete theory for seismic waves propagating in layered linear viscoelastic media. Borcherdt in particular provides the complete theory for the reflection and refraction of plane homogenous and inhomogeneous P, S, and SII waves at planar viscoelastic boundaries.

The main goal of reflection seismology is to estimate the density, P- and S-wave velocities and corresponding quality factors of the earth layers from recorded seismic data. Amplitude versus offset (AVO) analysis is based on the analytic expressions for reflection coefficients in elastic media. If anelasticity is present it is modified to a complex quantity whose real part is the elastic reflection coefficients. The exact form of the reflectivities are too complicated to extract intuitively much information about the physical properties of the subsurface earth. Practically for most reflecting interfaces in seismology the change in the elastic and anelastic properties are small, so that we can linearize the reflectivities in terms of perturbations of earth properties, defined as the ratio of the difference to the average of the properties of the contiguous layers.

In this paper, the linearized forms of P-to-P reflection coefficient for low contrast interfaces separating two arbitrary low-loss viscoelastic media is derived. The linearized reflection coefficient we obtained is complex. The real part is the elastic reflectivity and imaginary part includes the perturbation in both elastic and anelastic properties weighted by phase and attenuating angles. It is shown that the reflectivity not only depends upon the perturbations in elastic properties, but also on perturbations in quality factors for P- and S-waves. To derive the the linearized reflectivities, we linearized Snell’s law for a two layer viscoelastic media and show that in the linearized reflectivity only the average of attenuation angle affects the reflectivity.
VISCOELASTIC WAVES

In a viscoelastic medium, wavenumber vector is a complex vector given by \( \mathbf{K} = P - iA \), where its real part characterizes the direction of wave propagation and imaginary part characterizes the attenuation of the wave. In the case that attenuation and propagation vectors are in the same direction, wave is called homogeneous. An elastic media is represented by \( A = 0 \). P- and S-velocities are the complex numbers related to the elastic P- or S-velocity \( v_E \) as

\[
V = v_E \left( 1 + \frac{i}{2} Q^{-1} \right),
\]

where \( Q \) is the quality factor for P- or S wave. In the case of inhomogeneous waves, the attenuation and propagation vectors are not in the same direction. This makes the displacement vectors different from those of the homogenous case. The particle motion for P-waves is elliptical in the plane constructed by attenuation and propagation vectors. This elliptical motion reduces to a linear motion in the homogenous limit. Particle motion also distinguishes the two types of shear waves, SI and SII. The first, which is the generalization of the SH wave has a linear motion does not convert to P or SI waves. The second one, which is the generalization of the SI wave, has a linear particle motion perpendicular to the propagation-attenuation plane. This is the SII-type wave.

The analysis of the Zoeppritz equation and continuity of the displacement and stress tensor across the boundary are similar to the elastic case. The difference is that ray parameter and vertical slowness are complex. For example let us consider the incident P-wave

\[
\mathbf{U}^i_p = (\xi_x + \xi_z) \exp \{ i \omega (px + qpz) \},
\]

where \( \downarrow \) indicates the direction of the vertical component of the incident wave propagation vector, and we defined the \( x \) and \( z \)-components of polarization vectors as

\[
\xi_x = p v_p = \sin \theta + \frac{i}{2} Q^{-1} \tan \delta_v \cos \theta_v,
\]

\[
\xi_z = q v_p = \cos \theta_v - \frac{i}{2} Q^{-1} \tan \delta_v \sin \theta_v.
\]

where \( p \) and \( q \) are complex ray parameter and vertical slowness vectors respectively. In the process of linearization we need to obtain the perturbation in phase and attenuation angles in terms of perturbations in physical properties. Snell’s law for viscoelastic media is discussed by Wennerberg (1985); Borchardt (2009). Since the ray parameter in a viscoelastic medium is complex, it is obvious that the snell’s law has two parts, real and imaginary.

To study the contributions of the jumps in elastic and anelastic properties on reflectivities, we need to linearized the reflection amplitude. To calculate the approximate reflectivities for a low contrast model, and to write the physical quantities in medium 1, and medium 2 in terms of fractional perturbations, we need to express the phase and attenuation angles in perturbed form. To do this we need to linearized the generalized Snell’s law.

The main assumption required to extract the approximate form of the reflectivities is that the physical properties in the two layer are slightly different. Other words we can define fractional changes in properties as perturbations which are much smaller that one. This procedure is straightforward for properties of the medium like density, velocities and quality factors. For example The density of the layer above/below the reflector is given by

\[
\rho_{1/2} = \rho \left( 1 \pm \frac{\Delta \rho}{2 \rho} \right).
\]

Where \( \rho \) denotes the average in density of two layer. In the final form of the linearized reflectivity we shouldn’t have any quantities related explicitly to either the upper or lower medium. Hence we need to express the phase and attenuation angles in terms of corresponding perturbations. This can be done by linearization of Snell’s law. By applying the linearization to the real part we obtain the perturbation in phase angle in terms of perturbation in corresponding velocity weighted by the average by the phase angle. Using the linearization of the imaginary part we obtain that perturbation in attenuation angle in terms of the perturbations in corresponding velocities and quality factors. Similar expressions are valid for \( v_p, v_s, Q_p \) and \( Q_s \). In the above relations \( \Delta \) refers to the difference in the lower and upper layers and \( bar \) indicates the average of the quantities. Also we need to express the phase angle and attenuation angles in terms of differences and averages in angles. Linearization of the real part of the Snell’s law for P-wave results

\[
\Delta \theta_p \approx \Delta \frac{\rho}{v_p} \tan \theta_p.
\]

In a similar manner linearization of the imagnionary part of the Snell’s law results

\[
\Delta \delta_p = \frac{1}{2} \sin 2 \delta_p \left\{ \frac{\Delta \frac{\rho v_p}{v_{pe} \cos \theta_p}}{\frac{\rho v_p}{v_{pe} \cos \theta_p} + \left( 1 - \frac{\tan \theta_p}{\tan \delta_p} \right) \frac{\Delta Q_p}{Q_p} } \right\}.
\]

As a result, perturbation in attenuation angle can be expressed in terms of perturbation in elastic velocities and quality factors. Also perturbation in attenuation angle depends to the average angle \( \theta \).

LINEARIZATION OF REFLECTIVITY

It has been shown that waves with elliptical polarization can not be converted to waves with the linear polarizations (Borchardt, 2009; Moradi and Innanen, 2013). For example SII wave that has a linear polarization does not convert to P or SI waves. As a result we can write the reflection transmission for P- and SI-waves as a 4 \( \times \) 4 matrix is given by

\[
R = \begin{pmatrix}
\uparrow p^p & \uparrow s^p & \downarrow p^p & \downarrow s^p \\
\uparrow s^s & \uparrow s^s & \downarrow s^s & \downarrow s^s \\
\uparrow p^s & \downarrow s^p & \downarrow p^p & \uparrow s^p \\
\uparrow s^p & \downarrow s^s & \downarrow p^s & \uparrow s^s
\end{pmatrix}.
\]

The diagonal elements of the reflection-transmission matrix represents the reflections that preserve the type of the waves. For example \( \uparrow p^p \) refers to the reflected upgoing P-wave from downgoing incidence P-wave and similar explanations for other
diagonal elements. On the other hand some off-diagonal elements indicate converted waves. For instance, $^3\text{SIP}^p$ denotes a reflected upgoing P from incidence downgoing SI wave. Other off-diagonal elements refer to transmitted waves either converted modes or preserved modes. For example, $^3\text{SISI}^p$ is related to the transmitted downgoing SI wave from a downgoing incidence SI wave and $^3\text{SIP}^p$ is a downgoing transmitted P wave from a downgoing incidence SI wave. Similar to the Aki and Richards (2002) notation, after solving the Zoepritz equations, the reflection coefficients for the PP-mode are given by

$$\left(^1\text{pp}^p\right) = \frac{c_1d_2-c_3d_4}{d_1d_2+d_3d_4}, \quad (9)$$

where

$$d_1 = -2\rho^2\Delta M(q_1q_5+\rho_1q_5+\rho_2q_5),$$
$$d_2 = -2\rho^2\Delta M(q_1q_5-q_5)+2\rho_1q_5,$$
$$d_3 = -p\left[2\Delta M(q_1q_5+p^2)+\Delta \rho\right],$$
$$d_4 = -p\left[2\Delta M(q_1q_5+p^2)+\Delta \rho\right],$$
$$c_1 = -2\rho^2\Delta M(q_1q_5+q_5)-2\rho(q_1q_5+q_5),$$
$$c_3 = p\left[2\Delta M(q_1q_5-p^2+\Delta \rho\right]. \quad (15)$$

Where, $p$ is the complex ray parameter and $q$ is the complex slowness vector. We can write the differences in complex moduli as a sum of the differences in elastic shear modulus plus an imaginary part

$$\Delta M = \Delta \mu + i\Delta \mu_A, \quad (16)$$

where the real part is given by

$$\Delta \mu_E = \mu_{E2} - \mu_{E1} = \rho_2V_{SE2}^2 - \rho_1V_{SE1}^2, \quad (17)$$

and imaginary part by

$$\Delta \mu_A = Q_{S2}^{-1}\mu_{E2} - Q_{S1}^{-1}\mu_{E1} = \rho_2V_{SE2}^2Q_{S2}^{-1} - \rho_1V_{SE1}^2Q_{S1}^{-1}. \quad (18)$$

By having the exact reflection coefficients, and the linearization tools for viscoelastic properties, we are ready to calculate the approximate reflectivities. Let us consider the case that an inhomogeneous P-wave hits the boundary of a slightly different low-loss viscoelastic medium. In this case the reflected P-wave is also an inhomogeneous wave. All complex quantities and expressions we have defined yet include the first order of attenuation factor $Q^{-1}$. As a result, in the low-loss approximation any term which includes of multiplication of two imaginary parts is negligible. Finally, we arrive at the linearized P-to-P reflectivity as

$$\left(^1\text{pp}^p\right)_A = \left(^1\text{pp}^p\right)_E + i\left(^1\text{pp}^p\right)_A, \quad (19)$$

and the imaginary part

$$\left(^1\text{pp}^p\right)_E = \frac{1}{2}\left(\frac{\Delta \rho}{\rho} + \frac{\Delta V_{PE}}{V_{PE}}\right) - 2\sin^2\theta_P\left(\frac{V_{SE}}{V_{PE}}\right)^2\left(\frac{\Delta \rho}{\rho} + 2\frac{\Delta V_{SE}}{V_{SE}}\right). \quad (20)$$
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layer viscoelastic media and show that in the linearized reflectivity only the average of attenuation angle effects the reflectivity. Also we showed that the linearized reflectivities can be transformed to the scattering potential obtained using the Born approximation.

To model the anelasticity in a medium linear viscoelasticity is used. Plane waves are generally inhomogeneous, where the attenuation and propagation are not in the same direction. The elastic reflectivity can be obtained in the limit that attenuation quantities $\bar{Q}$ go to zero.

The linearized reflection coefficients we obtained are all complex. The real part is the elastic reflectivity and imaginary part includes the perturbation in both elastic and anelastic properties weighted by phase and attenuating angles.

If all parameters related to the anelasticity go to zero the viscoelastic reflectivities reduced to the linearized elastic isotropic reflection coefficients obtained by Aki and Richards.

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