Exact solutions for reflection coefficients, in 2D

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Summary

In this paper, we derive the exact formula for reflection coefficients when there is a velocity ramp present in the 2D case. We require the density and modulus satisfy the relation established in (Lamoureux et al., 2012) and (Lamoureux et al., 2013) using a fixed parameter α. We consider both the case when the plane wave hits a transition zone at normal incidence for parameter α, and when the plane wave hits the transition zone at non-normal incidence given varying density. The motivation is to extend the work of the authors in (Lamoureux et al., 2012) and (Lamoureux et al., 2013) and to demonstrate an explicit formula for reflection coefficients in a continuously varying velocity field.

Introduction

In seismic imaging, a reflection coefficient describes the relative amplitude and phase of the reflected portions of a seismic wave returned from a subsurface anomaly and is a key indicator of geological features beneath the earth's surface. In complex models, reflection coefficients are computed numerically and used to infer information about where structures reside underground. Numerical results do not offer the same precision as an exact solution, and exact formulas for reflection coefficients can offer insights into problems in reflection seismology. For instance, when studying inverse problems, we know that to some degree these problems are not well-posed. An exact solution for a reflection coefficient enables us to explore the nature of ill-posedness of such problems. As such, it allows us to study the kind of reflection obtained when a smooth transition zone is present.

In this paper, we will extend the work of (Lamoureux et al., 2012) and (Lamoureux et al., 2013) to 2D. We will begin with a discussion of the 2D problem and the methods for finding formulas for reflection coefficients when a smooth transition zone is present. We will then consider two examples: when the plane wave hits the transition zone at normal incidence and when it hits at non-normal incidence.

Exact Solutions in Reflection Seismology

Recall the 2D elastic wave equation

$$\rho(x,z)\frac{\partial^2 u}{\partial t^2} = \nabla(\mathrm{K}(\mathrm{x},\mathrm{z}) \cdot \nabla \mathrm{u}) \qquad (1)$$

where ρ represents the density and K represents the bulk modulus. As in (Lamoureux et al., 2012), we define $\rho(x,z) = c(x,z)^{\alpha-2}$ and $K(x,z) = c(x,z)^{\alpha}$ for some parameter α. This relation preserves the wave speed given via the ratio $K(x,z)/\rho(x,z) = c(x,z)^2$. We are specifically interested in the case of a velocity field which has constant velocity prior to a linear increasing ramp and constant after the ramp, as indicated in Figure 1.

The region of the velocity field with the linear increasing ramp is called the transition zone. In the normal plane wave incidence case, the wavefront is orthogonal to this transition zone. Therefore, the solution of (1) is constant in the x-direction. In the non-normal incidence case, the plane wave hits the transition zone at a non-zero angle θ. Thus, both the x- and z-directions vary in the solution of (1) unlike in the normal incidence case.

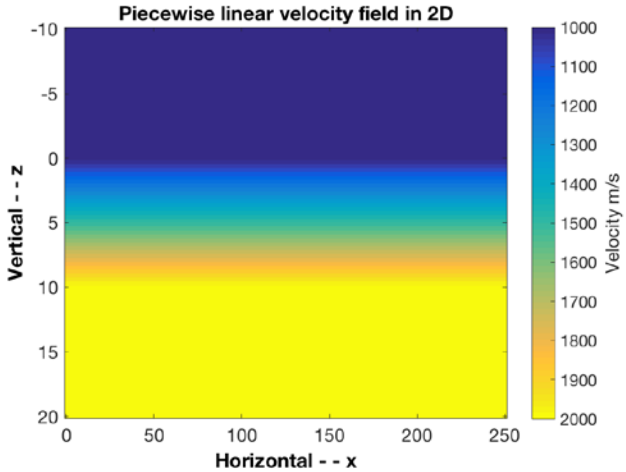

Figure 1: A velocity field which has a ramp moving from a constant velocity of 1000 m/s to 2000 m/s over 10 m.

We model these problems using the solution to equation (1) for parameter α to define regional solutions. These solutions relate what is occurring above the transition zone, in the transition zone, and below the transition zone, denoted u_{top}, u_{trans}, and u_{bottom} respectively. Within the top region, we infer a reflection coefficient R to depict the relative magnitude and phase of the wave which is reflected off the transition zone. In the regional solution for below the transition zone, there will be a transmission coefficient T to indicate the portion of the wave which is transmitted through the transition zone. Once we define the regional solutions, we establish continuity conditions to preserve displacement continuity and continuity of force. Applying these continuity conditions to the regional solutions, we get a system of equations which we can solve to find the reflection and transmission coefficients.

In the next section, we will solve for the reflection coefficient in the normal incidence case for parameter α. Afterwards, we will find the solution for the reflection coefficient in the non-normal incidence case when just the density $\rho(x,z)$ varies.

Examples

We fix the top and bottom regions to have constant velocities c_1 and c_2, respectively, and the transition region to have a depth D. The change in velocity in the ramp is given by a slope $m = (c_2 - c_1)/D$. In the normal incidence case, we note the following regional solutions:

$$\begin{aligned} u_{top}(x,z,t) &= e^{i\omega(z/c_1 - t)} + Re^{i\omega(-z/c_1 - t)} \\ u_{trans}(x,z,t) &= A\left(1 + \frac{m}{c_1}z\right)^{n_1} e^{-i\omega t} + B\left(1 + \frac{m}{c_1}z\right)^{n_2} e^{-i\omega t} \\ u_{bottom}(x,z,t) &= Te^{i\omega(z/c_2 - t)} \end{aligned} \quad (2)$$

where R is the reflection coefficient, T is the transmission coefficient, $n_1 = (1-\alpha)/2 + \sqrt{(1-\alpha)^2/4 - \omega^2/m^2}$, and $n_2 = (1-\alpha)/2 - \sqrt{(1-\alpha)^2/4 - \omega^2/m^2}$. We apply the continuity conditions described in the previous section to get a system of equations. Solving this system of equations in Matlab, we get the following exact solution for the reflection coefficient:

$$R(\omega) = \frac{im\omega(r^{n_1} - r^{n_2})(1-\alpha)}{2\omega^2(r^{n_1} - r^{n_2}) + im\omega(r^{n_1} + r^{n_2})(n_1 - n_2)} \quad (3)$$

where $r = c_2/c_1$. Note that the reflection coefficient is frequency dependent whereas when a sharp discontinuity is present the reflection coefficient is not dependent on frequency (Lamoureux et al., 2012). The non-normal incidence case below is will also depend on frequency.

Recall for the non-normal incidence case that the solution to (1) will vary in both the x- and z-directions. Hence, we will focus on the case when only the density varies for (1). Specifically, the case when $\alpha = 0$.

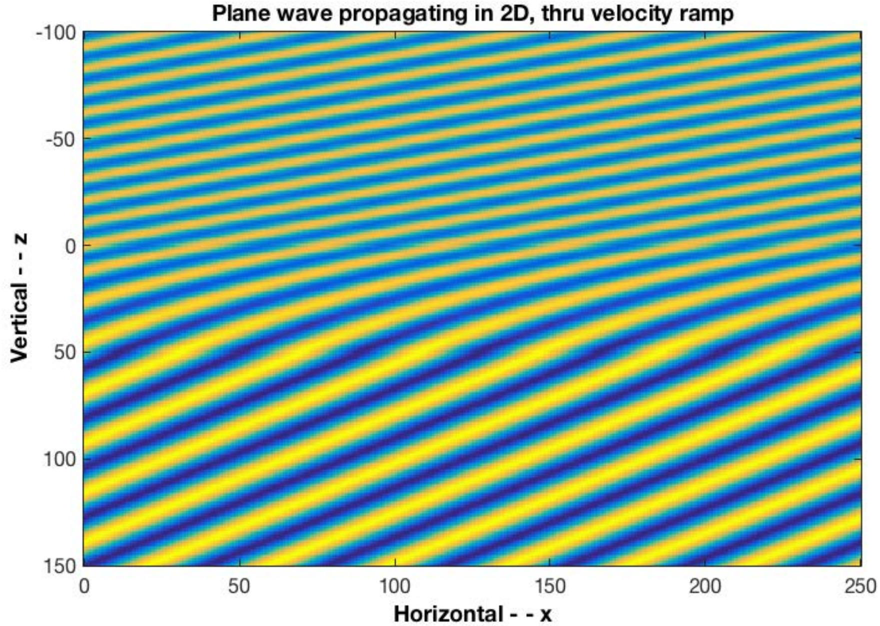

Figure 2: The plane wave hits the transition zone at an angle in this case. A portion of the wave travels through the transition zone from $z = 0$ to $z = 50$ and is transmitted on the other side while the rest is reflected.

To model Fig. 2, we use the following regional solutions:

$$\begin{aligned} u_{top}(x,z,t) &= e^{i(k_x x + k_z z - \omega t)} + Re^{i(k_x x - k_z z - \omega t)} \\ u_{trans}(x,z,t) &= e^{i(k_x x - \omega t)}(AZ_1(z) + BZ_2(z)) \\ u_{bottom}(x,z,t) &= Te^{i(k_x x + k_z' z - \omega t)} \end{aligned} \tag{4}$$

where $Z_1 = M_{0,\beta}(2c(x,z)k_x/m)$ and $Z_2 = W_{0,\beta}(2c(x,z)k_x/m)$ for Whittaker functions M and W. We also require that the parameters k_x, k_z, and k_z' satisfy the dispersion relation:

$$k_x^2 + k_z^2 = \frac{\omega^2}{c_1^2} \tag{5}$$

and

$$k_x^2 + (k_z')^2 = \frac{\omega^2}{c_2^2} \tag{6}$$

where c_1 is the velocity in the upper region. Applying the continuity conditions from the previous case, we get a system of equations which we solve to find the reflection coefficient:

$$\begin{aligned} R(\omega) = -\frac{1}{N}(&Z_1'(0)Z_2'(D) - Z_2'(0)Z_1'(D) - ik_z\big(Z_1(0)Z_2'(D) - Z_2(0)Z_1'(D)\big) \\ &- ik_z'\big(Z_1'(0)Z_2(D) - Z_2'(0)Z_1(D)\big) - k_z k_z'(Z_1(0)Z_2(D) - Z_2(0)Z_1(D))) \end{aligned} \tag{7}$$

where

$$\begin{aligned} N = &Z_1'(0)Z_2'(D) - Z_2'(0)Z_1'(D) + ik_z\big(Z_1(0)Z_2'(D) - Z_2(0)Z_1'(D)\big) \\ &- ik_z'\big(Z_1'(0)Z_2(D) - Z_2'(0)Z_1(D)\big) + k_z k_z'(Z_1(0)Z_2(D) - Z_2(0)Z_1(D))) \end{aligned} \tag{8}$$

In this case, the reflection coefficient depends on frequency also. We can use a range of ω to determine the other parameters and plot the reflection coefficients for different incidence angles. In Figure 4, we compare the results of six different values of the incident angle θ for the reflection coefficient. In particular, we consider $\theta = 5°, 10°, 15°, 20°, 25°$, and $30°$.

Conclusions

We found the exact solutions for reflection coefficients for the case when a 2D velocity ramp is present. We considered a plane wave hitting the ramp at normal incidence for some parameter α. We also looked at the non-normal incidence case when density varied for the 2D velocity ramp and found an equation for the reflection coefficient with respect to $k_x, k_z, \omega,$ and k_z'. Finally, we compared reflection coefficients for different incidence angles θ.

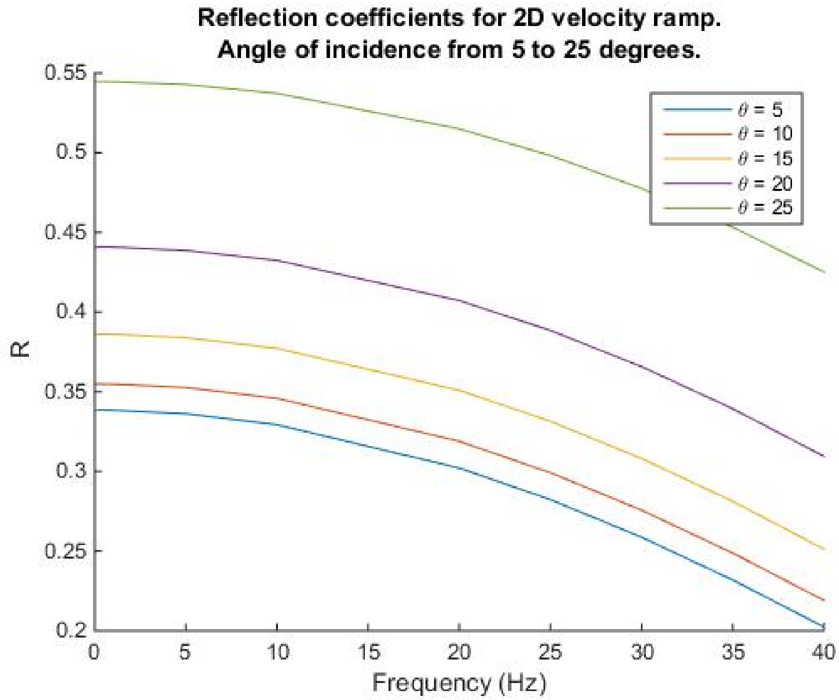

Figure 3: Reflection coefficients for non-normal incident case for the 2D velocity ramp. Each line represents the reflection coefficient for a different incidence angle θ. Specifically, $\theta = 5°,\ 10°,\ 15°,\ 20°,\ 25°$.

Acknowledgements

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