Strong form

- The \( i^{th} \) component of the strong (differential) form of the full elastic wave equation, in an isotropic medium \( \Omega \subset \mathbb{R}^d \), is

\[
\rho \ddot{u}_i = \partial_t \sigma_{ij} + f_i, \quad x \in \Omega, \ t > 0.
\]

(1)

- \( u = (u_1, \ldots, u_d) \) is the displacement vector.
- \( x = (x_1, \ldots, x_d) \in \mathbb{R}^d \)
- \( \rho \) is the density.
- \( t \) denotes time differentiation.
- \( \sigma_{ij} \) are the stresses.
- \( \sigma_{ij} = \lambda \left( \nabla \cdot u \right) \delta_{ij} + 2\mu u_{ij} \)
- \( u_{ij} \) is the \( i^{th} \) component of the applied force.
- sum over repeated indices.
- \( \sigma_{ij} = \frac{1}{2} (\partial_i u_j + \partial_j u_i) \)
- \( \lambda \) and \( \mu \) are the Lamé parameters.

Weak form

- Multiply both sides of equation (1) by an arbitrary function \( v(x) \) and integrate by parts to obtain the weak (integral) form,

\[
\int_{\Omega} \rho \ddot{u} v \, d\Omega + \int_{\Omega} \sigma_{ij} \partial_i u_j v \, d\Omega = \int_{\Omega} f_i v \, d\Omega + \int_{\partial\Omega} \sigma_{ij} n_j \left( \partial_i u_i \right) v \, dS
\]

(2)

- Pseudospectral methods choose a set of points \( \{x_0, \ldots, x_N\} \) in \( \Omega \) and a set of functions \( \{\phi_0, \ldots, \phi_N\} \) in \( L^2(\Omega) \) with the property

\[
\phi_{m}(x_0) = \delta_{mn}
\]

- Write the displacements as linear combinations of the basis vectors

\[
u_i(x, t) = \sum_{m=0}^{N} u_i(x_0, t) \phi_m(x)
\]

- Equation (2) is enforced for \( v = \phi_m(x) \), for all \( m = 0, \ldots, N \)

Boundary conditions in 2D

- Split the surface integral over the \( \alpha = N, S, E \) and \( W \) boundaries

\[
\int_{\partial\Omega} \sigma_{ij} n_j \left( \partial_i u_i \right) v \, dS = \sum_{\alpha} \int_{\partial\Omega_{\alpha}} \sigma_{ij} n_j \left( \partial_i u_i \right) v \, dS
\]

- The free surface condition \( \sigma_{ij} \cdot n = 0 \) implies

\[
\int_{\partial N} \sigma_{ij} n_j \left( \partial_i u_i \right) v \, dS = 0.
\]

- Second order absorbing boundary conditions along a vertical boundary at \( x = x_{\text{max}} \) can be enforced by substituting into the stresses

\[
\dot{\partial}_1 u = -\frac{1}{V_p} V_p - V_s \dot{\partial}_2 w, \quad \dot{\partial}_2 w = -\frac{1}{V_s} V_p - V_s \dot{\partial}_2 u.
\]

- Similarly, at \( z = z_{\text{max}} \) the substitution is

\[
\dot{\partial}_1 u = -\frac{1}{V_s} V_p - V_s \dot{\partial}_1 w, \quad \dot{\partial}_2 w = -\frac{1}{V_s} V_p - V_s \dot{\partial}_1 u.
\]

- At \( x = 0 \) the signs are switched.

Time-integration

- Substituting the boundary conditions into equation (2) produces a system of ordinary differential equations

\[
\mathbf{M} \ddot{\mathbf{U}}(t) + \mathbf{A} \mathbf{U}(t) + \mathbf{K} \mathbf{U}(t) = \mathbf{F}(t).
\]

which can be time-stepped numerically.

Domain decomposition

- In domain decomposition the model parameters are split up into smaller constant regions. This can be done by averaging the parameters at the 4 corners, or fitting a polynomial to the original model and evaluating at the cell centers.

- Example.

- Consider a simple two-layer medium with a free-surface and absorbing sides and bottom. The source is a Ricker wavelet applied at a single node.

- At the interface between elements we enforce the conditions

  - Continuity of displacement: \( u_i|_{\Omega_1} = u_i|_{\Omega_2} \)
  - Continuity of traction: \( \sigma_{ij} \cdot n|_{\Omega_1} = \sigma_{ij} \cdot n|_{\Omega_2} \)

- The first is done by making the functions \( \phi_m(x) \) piecewise continuous at the interfaces.

- The second we get for free by deleting all interior surface integrals.

\[
\sum_{\alpha} \int_{\partial\Omega_{\alpha}} \sigma_{ij} n_j \left( \partial_i u_i \right) v \, dS = 0.
\]

- The resulting system is very sparse.

- More complicated models can be built by using many smaller elements, akin to building an image from pixels.

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