One-dimensional scalar wave scattering

The scalar wave equation in one-dimension

$$\phi''(x) + \omega^2 c^2 \phi(x) = 0.$$  

The essential assumption in perturbation theory is definition of reference medium with a constant velocity $c_0$ and a actual medium with spatial dependent velocity $c(x)$. The relationship between the velocity in actual and reference medium is expressed by

$$c(x) = c_0 + \delta c(x) = c_0 + \delta(x),$$  

where $\delta(x)$ is the fractional velocity. Wavefield in the reference medium is

$$\phi_0(x) = \sin(kx) = \frac{1}{2i} \left( e^{ikx} - e^{-ikx} \right).$$

In perturbed medium

$$\phi(x) = \phi_0(x) - e^{\delta x} \phi_0(x),$$

where $\delta$ is called phase shift and is related to the perturbation in the medium, and scattered wave

$$\phi_s(x) = \phi(x) - \phi_0(x) = e^{\delta x} \phi_0(x).$$

For the anelastic scattering the total wave field is given by $[1]

$$\phi_s(x) = \frac{1}{2i} \left( \gamma a e^{ikx} - e^{\delta x} \right),$$

where $\gamma$ is a real number such that $0 < \gamma < 1$ and scattered wave field

$$\phi_s(x) = \frac{1}{2i} \left( \gamma a + 1 \right) \sin(kx + \delta_k) + \frac{i}{2i} \left( \gamma a - 1 \right) \cos(kx + \delta_k).$$

In asymptotic region the wave field has the following form

$$U = \frac{1}{kp_f} \sum_{i} i \left( -P_i \hat{r} \right) \cos \left( \frac{kp_f}{2} \right) + \frac{1}{kp_f} \sum_{i} i \left( P_i \hat{r} \right) + \frac{1}{kp_f} \sum_{i} i \left( P_i \hat{r} \right) \sin \left( \frac{kp_f}{2} \right) \hat{r} + \frac{1}{kp_f} \sum_{i} i \left( P_i \hat{r} \right) \sin \left( \frac{kp_f}{2} \right) \hat{r}.$$