Relationships between the velocities and the elastic constants of an anisotropic solid possessing orthorhombic symmetry

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ABSTRACT

This paper reviews the equations of body-wave propagation in an elastic anisotropic medium and then focuses upon the particular case of orthorhombic symmetry. This type of symmetry is believed to be appropriate for describing the wave-propagation behaviour of an industrial laminate, known as phenolic CE, which promises to be a useful material in physical seismic modelling experiments to be conducted within the CREWES Project.

Relationships are derived which enable the determination of the nine elastic stiffnesses of a material of orthorhombic symmetry from nine or more observed body-wave velocities.

INTRODUCTION

As an anisotropic material to be used in physical modelling experiments, the industrial laminate phenolic CE is very promising. This material, described by Cheadle and Lawton (1989) in this volume, exhibits three different velocities (for a given wave type) along three principal orthogonal directions. From its construction it is clear that, ideally at least (assuming flawless construction), it should possess three mutually orthogonal axes of two-fold symmetry. That is, after a rotation of 180 degrees about any of these axes, the material should behave in the same way with regard to elastic-wave propagation as before rotation. This type of symmetry is identical to that exhibited by the orthorhombic class of crystals, whose elastic properties have been studied extensively (e.g. Musgrave, 1970; Nye, 1985). In this paper, the theory of body-wave propagation in anisotropic media is reviewed and, for the case of orthorhombic symmetry, the relationships among the elastic stiffnesses and the body-wave velocities are elaborated. This will allow "calibration" of the material by observing velocities for a limited number of cases.

THE KELVIN-CHRISTOFFEL EQUATIONS

The equations of motion governing wave propagation in a generally isotropic elastic medium are given by many authors (e.g. Bullen, 1963; Fedorov, 1968; Musgrave, 1970; Aki and Richards, 1980; Crampin, 1981, 1984; Nye, 1985). For infinitesimal displacements u_i, Cartesian coordinates x_i, density ρ, stress tensor σ_{ij} and body forces per unit mass g_i:

$$\rho \ddot{u}_i = \sigma_{ij,j} + \rho g_i \tag{1}$$

where ", j" denotes the partial derivative with respect to x_j and where the Einstein summation convention (for repeated indices) applies.

The stress tensor, in terms of the strain tensor ε_{kl} and the stiffness tensor c_{ijkl}, is given in accordance with Hooke's law by:

$$\sigma_{ij} = c_{ijkl}\ \varepsilon_{kl} \tag{2}$$

where

$$\varepsilon_{kl} = \frac{1}{2}(u_{l,k} + u_{k,l})\ . \tag{3}$$

Substituting (2) and (3) into (1), neglecting any body forces, yields:

$$c_{ijkl}u_{k,lj} - \rho\ \ddot{u}_i\ = 0\ . \tag{4}$$

These equations of motion, and their solution for monochromatic plane-wave motion, are considered by many authors (e.g. Fedorov, 1968; Musgrave, 1970; Keith and Crampin, 1977; Aki and Richards 1980; Crampin, 1981, 1984) but here I follow Musgrave's treatment most closely.

The harmonic plane-wave displacement is expressed as:

$$u_k = Ap_k \exp\left[i\ \omega\left(s_r x_r - t\right)\right] \tag{5}$$

where A is the amplitude factor, p_k is the unit polarization (or displacement) vector, ω is angular frequency, s_r is the slowness vector, and in this equation only, i is the square root of -1. The slowness vector gives the direction of wave propagation and may further be written:

$$s_r = v^{-1}n_r \tag{6}$$

where v is phase velocity and n_r is the unit slowness (or wavefront-normal) vector. Substitution of (5) and (6) into (4) yields

$$\left(c_{ijkl}n_j n_l - \rho v^2 \delta_{ik}\right)p_k = 0 \; . \tag{7}$$

Thus, the determination of the details of the wave motion has been cast as an eigenvalue problem in which, having specified n_r and c_{ijkl}, one can solve for ρv^2 and p_k.

Due to the well known symmetries involved (see e.g. Musgrave, 1970; Nye, 1985)

$$c_{ijkl} = c_{ijlk} = c_{jikl} = c_{klji} \tag{8}$$

and therefore the matrix $\left(c_{ijkl}n_j n_l - \rho v^2 \sigma_{ik}\right)$ is symmetric. This implies in turn that the three eigenvalues obtained for ρv^2 by setting

$$\left|c_{ijkl}\, n_j n_l - \rho v^2 \sigma_{ik}\right| = 0 \tag{9}$$

will be real. (Throughout this paper vertical bars denote *determinant*). Using each of these three eigenvalues in turn, the three mutually orthogonal polarization vectors, p_k, may be found from (7).

A further consequence of the symmetries embodied in (8) is that there are only 21 independent stiffnesses, c_{ijkl}. Using the Voigt notation (Musgrave, 1970; Nye, 1985; Thomsen, 1986), the fourth-order stiffness tensor may be written as a second-order (6x6) matrix:

$$c_{ijkl} \rightarrow c_{mn}$$

where

$$\left.\begin{array}{ll} m = i & \text{if } i = j \,, \\ m = 9 - (i + j) & \text{if } i \neq j \end{array}\right\} \tag{10}$$

and n and kl are analogous to m and ij .

By introducing the so called Kelvin-Christoffel stiffnesses, given by Musgrave (1970) as:

$$\Gamma_{ik} = c_{ijkl} n_j n_l$$

equations (7) and (9) may be rewritten as:

$$\begin{bmatrix} \Gamma_{11} - \rho v^2 & \Gamma_{12} & \Gamma_{13} \\ \Gamma_{21} & \Gamma_{22} - \rho v^2 & \Gamma_{23} \\ \Gamma_{31} & \Gamma_{32} & \Gamma_{33} - \rho v^2 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = 0 . \tag{11}$$

This system of three equations is known as the Kelvin-Christoffel (or just Christoffel) equations. If a nontrivial solution exists, then

$$\begin{vmatrix} \Gamma_{11} - \rho v^2 & \Gamma_{12} & \Gamma_{13} \\ \Gamma_{21} & \Gamma_{22} - \rho v^2 & \Gamma_{23} \\ \Gamma_{31} & \Gamma_{32} & \Gamma_{33} - \rho v^2 \end{vmatrix} = 0 . \tag{12}$$

BODY-WAVE VELOCITIES AND PARTICLE MOTIONS IN ANISOTROPIC MEDIA OF ORTHORHOMBIC SYMMETRY

Although each Kelvin-Christoffel stiffness is, in general, a sum of nine terms [equation (7)], in the case of orthorhombic symmetry only 9 of the 21 independent stiffnesses, c_{mn}, are nonzero. These are (Musgrave, 1970; Crampin, 1981; Nye, 1985;): $c_{11}, c_{22}, c_{33}, c_{44}, c_{55}, c_{66}, c_{23}, c_{31}$ and c_{12}. The 6 independent Γ_{ik} are then:

$$\left.\begin{aligned}
\Gamma_{11} &= n_1^2 c_{11} + n_2^2 c_{66} + n_3^2 c_{55} \\
\Gamma_{22} &= n_1^2 c_{66} + n_2^2 c_{22} + n_3^2 c_{44} \\
\Gamma_{33} &= n_1^2 c_{55} + n_2^2 c_{44} + n_3^2 c_{33} \\
\Gamma_{23} &= n_2 n_3 \left(c_{23} + c_{44}\right) \\
\Gamma_{31} &= n_3 n_1 \left(c_{31} + c_{55}\right) \\
\Gamma_{12} &= n_1 n_2 \left(c_{12} + c_{66}\right) .
\end{aligned}\right\} \qquad (13)$$

Propagation along a principal direction

For a slowness vector in the 1-direction,

$$n_j = (1, 0, 0) \qquad (14)$$

and equations (13) reduce to:

$$\left.\begin{aligned}
\Gamma_{11} &= c_{11} \\
\Gamma_{22} &= c_{66} \\
\Gamma_{33} &= c_{55} \\
\Gamma_{23} = \Gamma_{31} = \Gamma_{12} &= 0 .
\end{aligned}\right\} \qquad (15)$$

Equation (11) then becomes:

$$\begin{bmatrix} \Gamma_{11} - \rho v^2 & 0 & 0 \\ 0 & \Gamma_{22} - \rho v^2 & 0 \\ 0 & 0 & \Gamma_{33} - \rho v^2 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = 0 . \qquad (16)$$

For this rather simple case, that of propagation along a principal direction, there are three obvious eigenvalues which will zero the determinant of the 3x3 matrix. For each of these, the associated eigenvector p_k is the polarization (or unit-particle-displacement) vector.

The *P* wave. - Choosing the eigenvalue solution:

$$\Gamma_{11} - \rho v^2 = 0 \tag{17}$$

reduces the three equations of (16) to two, namely:

$$\begin{bmatrix} c_{66} - c_{11} & 0 \\ 0 & c_{55} - c_{11} \end{bmatrix} \begin{bmatrix} p_2 \\ p_3 \end{bmatrix} = 0\,. \tag{18}$$

The only permissible solution to (18) is:

$$p_2 = p_3 = 0 \tag{19}$$

since otherwise at least two of the six independent stiffnesses would have to be equal, violating the assumption of orthorhombic symmetry. It follows from equations (15), (17) and (19) that

$$p_k = (1,0,0) \qquad \text{and } v_{11} = (c_{11}/\rho)^{1/2} \tag{20}$$

where v_{11} denotes that v which applies for propagation (slowness) in the 1-direction and with particle motion (polarization) in the 1-direction, that is, the P -wave velocity.

The *S* waves. - Choosing each of the other two eigenvalue solutions leads to the two solutions:

$$p_k = (0,1,0) \qquad \text{and } v_{12} = (c_{66}/\rho)^{1/2} \tag{21}$$

and

$$p_k = (0,0,1) \qquad \text{and } v_{13} = (c_{55}/\rho)^{1/2}\,, \tag{22}$$

these representing S waves polarized in the 2- and 3-directions, respectively.

The corresponding velocities and polarizations for propagation in the 2- and 3-directions are obtained from equations (20), (21) and (22) by cyclic variation of the indices (1, 2, 3) and of the indices (4, 5, 6).

Propagation at 45° to two principal axes or "edge to edge"

For a slowness vector in the 23-plane at 45° to the 2- and 3-axes

$$n_j = \left(0, 1/\sqrt{2},\ 1/\sqrt{2}\right) \tag{23}$$

and equations (13) reduce to

$$\begin{aligned}
\Gamma_{11} &= 1/2(c_{55} + c_{66}) \\
\Gamma_{22} &= 1/2(c_{22} + c_{44}) \\
\Gamma_{33} &= 1/2(c_{33} + c_{44}) \\
\Gamma_{23} &= 1/2(c_{23} + c_{44})\,.
\end{aligned} \tag{24}$$

Equation (11) then becomes:

$$\begin{bmatrix} 1/2(c_{55} + c_{66}) - \rho v^2 & 0 & 0 \\ 0 & 1/2(c_{22} + c_{44}) - \rho v^2 & 1/2(c_{23} + c_{44}) \\ 0 & 1/2(c_{23} + c_{44}) & 1/2(c_{33} + c_{44}) - \rho v^2 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = 0\,. \tag{25}$$

The quasi-*P* and quasi-*SV* waves. - To consider possible solutions with polarization in the 23-plane, assume

$$p_1 = 0. \tag{26}$$

Equation (25) then reduces to

$$\begin{bmatrix} 1/2(c_{22} + c_{44}) - \rho v^2 & 1/2(c_{23} + c_{44}) \\ 1/2(c_{23} + c_{44}) & 1/2(c_{33} + c_{44}) - \rho v^2 \end{bmatrix} \begin{bmatrix} p_2 \\ p_3 \end{bmatrix} = 0\,. \tag{27}$$

Since p_2 and p_3 cannot both vanish now, the determinant of the 2x2 matrix must do so. This leads to a quadratic in ρv^2 whose solutions are given by:

$$4\rho v^2 = c_{22} + c_{33} + 2c_{44} \pm \left[(c_{33} - c_{22})^2 + 4(c_{23} + c_{44})^2\right]^{1/2} . \tag{28}$$

One may also solve either of the two equations represented by (27) for p_2/p_3 obtaining:

$$p_2/p_3 = \left\{\pm \left[(c_{33} - c_{22})^2 + 4(c_{23} + c_{44})^2\right]^{1/2} - (c_{33} - c_{22})\right\} / \left[2(c_{23} + c_{44})\right] . \tag{29}$$

Choosing the plus sign in both equation (28) and (29) gives the quasi-*P* wave. Its polarization (or particle motion) is seen from (29) to be "nearly" longitudinal, thus the prefix *quasi*. (It would be exactly longitudinal, that is $p_2/p_3 = 1$, only for $c_{33} = c_{22}$.)

Choosing the minus sign in (28) and (29) gives the quasi-*SV* wave, with a lower v than for the quasi-*P* and with "nearly" transverse polarization. (Again, it would be exactly transverse, that is $p_2/p_3 = -1$, only for $c_{33} = c_{22}$.) This is labelled *SV* only because it is the quasi-shear wave that is coupled to the quasi-compressional wave, and not because of any significance of the vertical.

For these particular cases of propagation in the 23-plane at 45° to each of the 2- and 3-axes, we use the special symbols v_{44} and $v_{4\bar{4}}$ to denote the quasi-*P* and quasi -*S* velocities, respectively. (Recall that the single index 4 is a contraction of the double index 23.)

The *SH* wave. - Clearly, an eigenvalue of equation (25) is given by:

$$1/2(c_{55} + c_{66}) - \rho v^2 = 0 \tag{30a}$$

with the eigenvector components

$$p_2 = p_3 = 0. \tag{30b}$$

From these one has directly:

$$p_k = (1, 0, 0) \quad \text{and} \quad v_{41} = [(c_{55} + c_{66})/(2\rho)]^{1/2} \, . \tag{31}$$

The corresponding velocities and polarizations for waves traveling "edge to edge" in the 31- and 12-planes are obtained from equations (28), (29) and (31) by cyclic variation of the indices (1, 2, 3) and of the indices (4, 5, 6).

STIFFNESSES IN TERMS OF VELOCITIES

From equation (20) and by cyclic variation of indices:

$$\left.\begin{aligned} c_{11} &= \rho v_{11}^2 \\ c_{22} &= \rho v_{22}^2 \\ c_{33} &= \rho v_{33}^2 \, . \end{aligned}\right\} \tag{32}$$

From equations (21) and (22), and by cyclic variation of indices:

$$\left.\begin{aligned} c_{44} &= \rho v_{23}^2 = \rho v_{32}^2 \\ c_{55} &= \rho v_{31}^2 = \rho v_{13}^2 \\ c_{66} &= \rho v_{12}^2 = \rho v_{21}^2 \, . \end{aligned}\right\} \tag{33}$$

Solving equation (28) for c_{23}, choosing either sign on the square root, one obtains:

$$\begin{aligned} c_{23} = \big[& c_{44}^2 + c_{22}c_{33} + c_{44}(c_{22} + c_{33}) + 4\rho^2 v_{44}^4 \\ & -2\rho v_{44}^2 (c_{22} + c_{33} + 2c_{44}) \big]^{1/2} - c_{44} \end{aligned} \tag{34}$$

where $v_{4\bar{4}}$ may be substituted for v_{44}. Similarly, by cyclic variation:

$$c_{31} = \left[c_{55}^2 + c_{33}c_{11} + c_{55}(c_{33} + c_{11}) + 4\rho^2 v_{55}^4 - 2\rho v_{55}^2 (c_{33} + c_{11} + 2c_{55})\right]^{1/2} - c_{55} \tag{35}$$

and

$$c_{12} = \left[c_{66}^2 + c_{11}c_{22} + c_{66}(c_{11} + c_{22}) + 4\rho^2 v_{66}^4 - 2\rho v_{66}^2 (c_{11} + c_{22} + 2c_{66})\right]^{1/2} - c_{66}\,. \tag{36}$$

It is clear then that measurement of nine suitably chosen velocities, at least three of which must be off the principal axial directions, permits determination of the nine stiffnesses for the case of orthorhombic symmetry. Measuring more velocities than nine simply overdetermines the problem and allows one to estimate errors in the measurements and/or to judge quantitatively how appropriate the orthorhombic model actually is.

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