Nonstationary filtering: review and update

Gary F. Margrave

ABSTRACT

A nonstationary generalization of the convolution integral is presented. Called nonstationary convolution, it retains the interpretation of forming the scaled superposition of impulse responses while allowing those impulse responses to become arbitrary functions of time or position. A similar, alternate formulation is also given which also has stationary convolution as a limiting form but does not have the immediate interpretation of forming the superposition of impulse responses. Called nonstationary combination, this alternate form is closely approximated by the common practice of forming a nonstationary result by interpolation between a set of stationary filtered results. Both filter forms can be re-expressed in a dual time-frequency domain where nonstationary convolution becomes a generalized forward Fourier integral and combination is a generalized inverse Fourier integral. It is shown that pseudodifferential operators can be considered as a nonstationary combination filter whose filter form is a spectral polynomial. It is then argued that nonstationary convolution can be inverted by inverting the dual-domain filter function and applying it as a nonstationary combination and vice-versa. Finally both nonstationary filter forms are re-expressed in the full Fourier domain in a result which generalizes the convolution theorem. The possible applications of this methodology are illustrated with examples from wave propagation and deconvolution.

INTRODUCTORY CONCEPTS

The digital filtering of sampled data is arguably one of the most important processes in geophysical data processing. Filtering refers to the use of the convolution integral to convolve or filter one signal with another. Usually one signal is called the filter and the other is the input though the symmetry of the convolution integral makes this designation arbitrary. Conceptually, convolution is most intuitively described *by replacement*. This refers to the process of replacing each point on the input signal by a scaled copy of the filter. The scalar is the input point itself and the output is the superposition of all such scaled filter copies.

In a fundamental result called the *convolution theorem*, the convolution integral can be computed by Fourier transforming the two signals, multiplying their spectra, and inverse Fourier transforming. This direct link between the theory of Fourier transforms and convolution, together with the numerical advantages of the fast Fourier transform (FFT) algorithm, has lead to very efficient convolution methodologies.

There are at least two reasons for the overriding importance of convolution, one very practical and one quite theoretical. First, the practical reason is the suppression of random or coherent noise (or, alternatively, the enhancement of signal). Typically signal is bandlimited by the source characteristics while random noise may be broadband. Also coherent noise may contaminate only a portion of the signal band. In both cases, filters may be designed to reject noisy portions of the spectrum while retaining a usable portion of the signal.

The theoretical reason lies at the heart of mathematical physics. Physical phenomena are held to be well modeled by a few linear partial differential equations (PDE's). For example, elastic wave propagation is described by the elastic wave equation and this can usually be reduced to a set of scalar wave equations for the elastic potentials. Similarly, the gravitational potential is modeled by Laplace's or Poisson's equations. Also of interest is the diffusion equation for the description of heat flow and the scalar wave equation for acoustic waves. A powerful solution technique for any linear PDE, known as Green's function theory, turns out to be a convolution or filtering process. Essentially, the theory states that if the solution to the PDE for an impulsive source can be found, then the solution for a more general, distributed, source is obtained by filtering the impulsive solution. The impulsive solution is called a Green's function (or impulse response) and the distributed source becomes the filter.

An intuitive example of this Green's function theory is the propagation of waves through the application of Huygen's principle (Figure 1a). This refers to the fact that a wavefront can be stepped forward in time (i.e. extrapolated) by considering each point on the wavefront to be an impulsive source and the new wavefront is synthesized from the superposition of all such sources. Thus the wave is stepped forward by filtering the input wavefield with an appropriate Green's function, which may be called a Huygen's wavelet. (A complete mathematical description of Huygen's principle may be found in Morse and Feshbach, 1953, p847). The extrapolated wave is constructed as the linear superposition of all possible Huygen's wavelets, each one representing an expanding spherical wavefront about a point on the input wavefield. Clearly, the radius of the Huygen's wavelet must depend on the local wave propagation velocity and can only be constant if velocity is constant. In the constant velocity case, this process is a multi-dimensional convolution by replacement. In the variable velocity case, it is still a linear superposition of Huygen's wavelets but each varies its radius according to the local wavespeed (Figure 1b).

TIME DOMAIN FORMS OF NONSTATIONARY FILTERING

The foregoing discussion leads quite naturally to the need for nonstationary filtering. In order to understand what a nonstationary filter is, we must first clearly understand a stationary filter. Most filtering processes use stationary filters because that is all that the ordinary convolution integral allows. In the case of the convolution of a stationary filter a(t) with a function h(t) to yield g(t), the convolution integral is

$$g(t) = \int_{-\infty}^{\infty} a(t-\tau)h(\tau)d\tau \quad . \tag{1}$$

Note that the filter appears dependent only on "lag time" $t-\tau$. If we consider the case when $h(\tau) = c\delta(\tau-\tau_o)$ (where c is a constant and δ is the Dirac delta function), then $g(t)=ca(t-\tau_o)$. This deceptively simple result has far reaching effects and is the basis for most numerical convolution algorithms. In words, if the input to a convolution is an impulse of magnitude c at time τ_o, then the output is the filter function scaled by c and centered at τ_o. Thus the filter function is scaled and translated (shifted to τ_o) but is otherwise unchanged. For this reason, a(t) is often called the "impulse response" of the filter since it is the result when $h(\tau) = \delta(\tau)$. This result generalizes to arbitrary input functions by considering them to be the superposition of a set of impulses and captures the essence of a stationary filter. Simply put, a stationary filter keeps the filter's form invariant and replaces each point on the input function with a scaled copy of the filter. The output is the superposition (summation) of all such scaled and shifted filter functions.

Returning to the Huygen's principle discussion from the introduction, the Green's function (or filter or Huygen's wavelet) represents an expanding wavefront of radius $v\Delta t$ where v is the velocity and Δt is the extrapolation time step. If v is a constant, then the spherical wavefront will have a constant radius no matter where on the input wavefront it is placed. Thus, in this case, stationary convolution is an appropriate mathematical tool to form the required scaled superposition of Huygen's wavelets. However; when velocity is allowed to vary from point to point, as it obviously does in the earth, then the radius of the wavefront must vary and stationary convolution cannot provide the needed flexibility. A mathematical process which can form a scaled superposition of Huygen's wavelets (or any other filter) while allowing the wavelet radius to depend on the local velocity at the point of replacement is called a nonstationary convolution (Figure 1b).

A nonstationary convolution integral is needed which symbolizes the nonstationary filtering processes just discussed. It must allow the filter to depend on both input and output time and not merely their difference. Additionally, it must retain the meaning of replacing each input point with an impulse response and should approach equation (1) in the "stationary limit". To deduce a more general form, we can expand the role of the impulse response discussed previously. Now, when an impulse $h(\tau) = c\delta(\tau-\tau_o)$ arrives, we would like to have a response something like $g(t) = ca(t-\tau_o, \tau_o)$. That is, the response changes according to both the lag time and the impulse time. A simple generalization of equation (1) which achieves this is

$$g(t) = \int_{-\infty}^{\infty} a(t-\tau, \tau)h(\tau)d\tau \quad . \tag{2}$$

Thus we have generalized the concept of the impulse response function in equation (1) to a two dimensional function a(u,v), where u and v are generalized time variables. The interpretation of a(u,v) is that it prescribes the response of the linear system, without the causal delay, to an impulse arriving at time v. This is convenient because it allows the impulse responses at different times to be directly compared (Figure 2) and we incorporate the causal delay into the integration form of equation (2). The stationary limit is found by letting the v dependence become constant. Figure 3 illustrates the discrete approximations to equations (1) and (2) being computed as matrix-vector multiplications. In the Figure 3a, the stationary impulse response matrix of Figure 2a has had each column delayed to place the filter start-time on the main diagonal (creating a Toeplitz matrix). The resulting matrix multiplies the input signal to complete the convolution of equation (1). In Figure 3b, the nonstationary convolution is also a matrix operation but the nonstationary convolution matrix is formed by delaying each column of the impulse response matrix of Figure 2b and no longer has the Toeplitz symmetry.

Equation (2) is called nonstationary convolution because it meets all of the criteria which were required: it is linear, it allows the filter to depend on both input time and lag time, it has equation (1) as its stationary limit, and it forms the scaled linear superposition of impulse responses. However, there is another similar linear form which is of considerable interest because it appears in many data processing algorithms. Called nonstationary combination, it is given by

$$\underline{g}(t) = \int_{-\infty}^{\infty} a(t-\tau, t)h(\tau)d\tau \quad . \tag{3}$$

Nonstationary combination differs from convolution in that it maps the v dependence of a(u,v) into output time rather than input time. Though this difference vanishes in the stationary limit, it can be dramatic for highly nonstationary filters. The response of equation (3) to $h(\tau) = c\delta(\tau-\tau_o)$ is $\underline{g}(t)= ca(t-\tau_o, t)$ rather than $g(t) = ca(t-\tau_o, \tau_o)$ for equation (2). Given the interpretation of a(u,v) mentioned above, equation (3) cannot be regarded as forming the superposition of impulse responses since the v dependence is not mapped to the arrival time of the impulse. As will be demonstrated in the next section, nonstationary combination is related to the common practice of approximating a nonstationary filter by computing a few stationary "filter panels", using different filter parameters for each panel, and then interpolating a combined result from the panels. In fact, nonstationary combination is exactly formed by slicing through an exhaustive set of filter panels as shown in Figure 4.

Though applying the same filter form as a combination or a convolution can give very different results, it is also true that a convolution filter form can always be constructed which will give identical results to a combination filter and vice-versa. That is, if h(τ) is filtered with a(u,v) using equation (3) to get g(t) it is always possible to find a filter, $\underline{a}$(u,v), which can be applied with equation (2) to yield the identical result g(t). Therefore, though combination does not form the linear superposition of impulse responses of the filter a(u,v) in equation (3), it does form the linear superposition of the impulse responses of a related quantity $\underline{a}$(u,v).

DUAL-DOMAIN FORMS OF NONSTATIONARY FILTERING

The two forms of nonstationary filtering introduced above are very closely related. Both are linear and both have stationary convolution as a limiting form. However; when a nonstationary impulse response function, a(u,v), is applied with equation (2) or, alternatively, with equation (3) the results can differ strongly. To understand this better, we will now derive *dual-domain* forms for both operations. The name is meant to imply that the domain of the signal changes during the application of the filter, either from time to frequency or the reverse. First we define forward and inverse Fourier transforms:

$$H(\eta) = \int_{-\infty}^{\infty} h(\tau)\exp\left(-i2\pi\eta\tau\right)d\tau \tag{4a}$$

$$h(\tau) = \int_{-\infty}^{\infty} H(\eta)\exp\left(i2\pi\eta\tau\right)d\eta \tag{4b}$$

$$G(f) = \int_{-\infty}^{\infty} g(t)\exp\left(-i2\pi ft\right)dt \tag{5a}$$

$$g(t) = \int_{-\infty}^{\infty} G(f)\exp\left(i2\pi ft\right)df \quad . \tag{5b}$$

Note that the spectra of g and h are G and H and that, corresponding to the input time τ is the input frequency η while the output time, t, has frequency f. It is also emphasized that these expressions are all ordinary Fourier transforms. Most other filtering techniques which are capable of nonstationary effects use a nonstationary transform such as the wavelet transform, the Gabor transform, or the short-time Fourier transform (see Margrave, 1996 or 1998, for a discussion). The theory presented here differs from these approaches in that it uses ordinary Fourier transforms throughout. That is, the signal to be filtered is never decomposed on a two-dimensional time-frequency (or shift-dilation) grid.

The dual-domain form of nonstationary convolution follows by using equation 5a to take the forward Fourier transform of equation (2)

$$G(f) = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} a(t-\tau, \tau) h(\tau) d\tau \right] \exp\left(-i2\pi ft\right) dt \quad . \tag{6}$$

The order of integration can be reversed in equation (6) under quite general conditions which are discussed in Margrave (1996 or 1998). This gives

$$G(f) = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} a(t-\tau, \tau) \exp\left(-i2\pi ft\right) dt \right] h(\tau) d\tau \quad . \tag{7}$$

Now, the inner integral can be evaluated by letting $u = t\text{-}\tau$ to get

$$G(f) = \int_{-\infty}^{\infty} \alpha(f,\tau) h(\tau) \exp\left(-i2\pi f\tau\right) d\tau \tag{8}$$

where

$$\alpha(p,v) = \int_{-\infty}^{\infty} a(u,v) \exp\left(-i2\pi pu\right) du \quad . \tag{9}$$

In equation (9), u and v are the generalized times mentioned before and p is a frequency corresponding to u. At first, it may seem odd to write this result in terms of such generalized quantities instead of the f and τ required in equation (8) but this will allow the same expression to be used later for nonstationary combination much as a(u,v) was used in both equations (2) and (3).

Equation (8) is the desired dual form for nonstationary convolution and it uses $\alpha(p,v)$ with p mapped to f and v to τ. First note that if the v dependence of $\alpha(p,v)$ becomes constant (i.e. the stationary limit) then $\alpha(f)$ can be taken outside the integral and we have the ordinary forward Fourier transform of $h(\tau)$ times a filter function $\alpha(f)$. That is, we get the expected stationary result that the filter is applied by multiplying the spectra. More generally, $\alpha(p,v)$ depends strongly on both p and v and equation (8) shows that it is applied simultaneously with the forward Fourier transform of the input function $h(\tau)$. the final output signal g(t) is obtained as the ordinary inverse Fourier transform of G(f) as in equation (5b).

As defined by equation (9), $\alpha(p,v)$ is the ordinary Fourier transform over the first time coordinate of the impulse response function and is called the nonstationary transfer function. That is, $\alpha(p,v)$ gives the Fourier spectrum of the impulse response for each impulse arrival time v. There are few restrictions on the nature of $\alpha(p,v)$ so that nearly arbitrarily complex spectral functions can be applied and they may vary arbitrarily with time. (The only restrictions required are those sufficient to make the integrals converge and to allow their interchange.) Since equation (9) is an ordinary Fourier transform, it is easily inverted to give a(u,v) in terms of $\alpha(p,v)$.

Figure 5 illustrates the application of stationary and nonstationary filters through the discrete equivalent matrix operations for equation (8).

The dual-domain form of nonstationary combination is derived by substituting for $h(\tau)$ in equation (3) its expression as an inverse Fourier transform of $H(\eta)$ as given by equation (4b). The derivation is similar to that already given and involves a reversal of the order of integration and a change of variables. The details may be found in Margrave (1996 or 1998) and the result is

$$\underline{g}(t) = \int_{-\infty}^{\infty} \alpha(\eta,t)H(\eta)\exp\left(i2\pi\eta t\right)d\eta \quad . \tag{10}$$

This result contrasts strongly with equation (8). In both cases the same nonstationary transfer function is involved; however in equation (8) it is applied simultaneously with the forward Fourier transform while in equation (10) it is with the inverse Fourier transform. Furthermore, note that the integration in equation (8) involves the v dependence of $\alpha(p,v)$ while equation (10) integrates over the p dependence.

These dual-domain forms are conceptually rich and offer great flexibility in the design of filtering algorithms. Considerable insight can be gained into the distinction between the convolution and combination forms by analyzing the simple case

$$\alpha(p,v) = \begin{cases} \alpha_1(p), & v < 0 \\ \alpha_2(p), & v \geq 0 \end{cases} \quad . \tag{11}$$

Here $\alpha(p,v)$ is composed of two stationary filters, discontinuously juxtaposed at $v = 0$. First, compute nonstationary convolution from equation (8)

$$G(f) = \alpha_1(f)\int_{-\infty}^{0} h(\tau)\exp\left(-i2\pi f\tau\right)d\tau + \alpha_2(f)\int_{0}^{\infty} h(\tau)\exp\left(-i2\pi f\tau\right)d\tau \quad . \tag{12}$$

Now define two window functions

$$\Omega_1(\tau) = \begin{cases} 1, & \tau < 0 \\ 0, & \tau \geq 0 \end{cases} \text{ and } \Omega_2(\tau) = \begin{cases} 0, & \tau < 0 \\ 1, & \tau \geq 0; \end{cases} \quad . \tag{13}$$

Using these window functions, equation (12) can be rewritten as

$$G(f) = \alpha_1(f)\int_{-\infty}^{\infty} \Omega_1(\tau)h(\tau)\exp\left(-i2\pi f\tau\right)d\tau + \alpha_2(f)\int_{-\infty}^{\infty} \Omega_2(\tau)h(\tau)\exp\left(-i2\pi f\tau\right)d\tau \tag{14}$$

in which the integrals are now ordinary forward Fourier transforms. If we let FT denote a forward Fourier transform and IFT an inverse Fourier transform, then the final filtered result in the time domain is

$$g(t) = \underset{f \to t}{\mathrm{IFT}}\left(\sum_{k=1}^{2} \alpha_k(f)\underset{\tau \to f}{\mathrm{FT}}\left(\Omega_k(\tau)h(\tau)\right)\right) \quad . \tag{15}$$

This result has an obvious generalization to the case where $\alpha(p,v)$ consists of any number of piecewise constant segments. In this case, it shows that nonstationary convolution can be formed from a set of stationary operations by windowing the input dataset to select that portion corresponding to each stationary filter segment, filtering the windowed segment, and superimposing the results. (Since the inverse Fourier transform is linear, the summation and the IFT can be interchanged in equation (15).)

Now consider the computation of nonstationary combination with equation (10) for the filter form of equation (11). In this case, equation (10) becomes

$$\underline{g}(t) = \begin{cases} \int_{-\infty}^{\infty} \alpha_1(\eta)H(\eta)\exp\left(i2\pi\eta t\right)d\eta, \ t < 0 \\ \int_{-\infty}^{\infty} \alpha_2(\eta)H(\eta)\exp\left(i2\pi\eta t\right)d\eta, \ t \geq 0; \end{cases} \quad (16)$$

Using the window functions of equation (13), equation (16) can be written as a single specification

$$\underline{g}(t) = \Omega_1(t)\int_{-\infty}^{\infty} \alpha_1(\eta)H(\eta)\exp\left(i2\pi\eta t\right)d\eta + \Omega_2(t)\int_{-\infty}^{\infty} \alpha_2(\eta)H(\eta)\exp\left(i2\pi\eta t\right)d\eta \quad (17)$$

and this result can be written in a form similar to equation (15) as

$$\underline{g}(t) = \sum_{k=1}^{2} \Omega_k(t)\left(\underset{\eta \to t}{\mathrm{IFT}}\, \alpha_k(\eta) \underset{\tau \to \eta}{\mathrm{FT}}\left(h(\tau)\right)\right) . \quad (18)$$

As before, this result can easily be extended to an arbitrary number of piecewise constant filter segments.

Equations (15) and (18) offer an intuitive understanding of the essential difference between nonstationary convolution and combination. Suppose that the nonstationary filter specification can be written as piecewise constant (in time), that is, as a countable number stationary filter spectra defined over time zones. Then define a set of window functions, one for each stationary filter such that the window is unity over the specification zone of the filter and zero elsewhere. Then, a nonstationary convolution is computed by windowing the input dataset, filtering each windowed result, and superimposing them. In contrast, a nonstationary combination proceeds by filtering the input dataset (without windowing) with each filter specification, windowing the filtered results, and superimposing. Imagine an input dataset that contains a single live sample (an impulse) which falls in the middle of the jth filter specification zone. Suppose further that the impulse responses of each stationary filter are very long in time such that they easily span the specification zones of many other filters. The nonstationary convolution will simply be the impulse response of the jth stationary filter since only the jth window applied to the input data will contain nonzero energy. However; nonstationary combination gives a much different answer. Since the stationary filters in equation (18) are applied before windowing, each will produce its own impulse response of the single live input sample. The application of windows after-the-fact will result in a composite “impulse response” which changes discontinuously at each filter specification boundary. This result is even more general than this analysis indicates. Nonstationary convolution offers a powerful method of solution to physics problems which is superior to nonstationary combination because the former creates a scaled superposition of impulse responses while the latter does not.

A general property of a combination filter, suggested by the previous paragraph, is that if $\alpha(p,v)$ has any discontinuities in the time direction (v), these will result in discontinuities in the filtered output. Thus it is possible to produce a completely abrupt change in the time domain output. Nonstationary convolution cannot show this behavior since the continuous superposition of impulse responses will smooth over any filter discontinuities.

As an example, the method of wavefield extrapolation by phase shift (Gazdag, 1978) can be straight forwardly extended to nonstationary phase shift (NSPS) (Margrave and Ferguson, 1997) by simply replacing v by v(x) (v is velocity) in the constant velocity expression and formulating it correctly as convolutional expression. The same nonstationary wavefield extrapolator, when applied as a nonstationary combination leads to the method of PSPI (Gazdag and Squazerro, 1984). Figure 6 shows a comparison between these approaches. In Figure 6a is a velocity field which varies rapidly with lateral position but is constant vertically. Figure 6b is a row of impulses which will be extrapolated through a single 50m vertical step with both methods. The NSPS result (Figure 6c) can be seen to be a superposition of hyperbolic impulse responses while the PSPI result (Figure 6d) is relatively chaotic.

RELATIONSHIP WITH PSEUDODIFFERENTIAL OPERATORS

One of the most enduring reasons for the utility of Fourier analysis is its ability to reduce constant-coefficient PDE's to ordinary differential equations or even simple algebraic equations which are then solved directly. The use of Fourier theory has therefore established a great many elegant solutions to physical problems but with the limitation that they must be described by a constant coefficient PDE. Well known examples in the geophysical literature are the F-K migration theory of Stolt (1978) and the phase shift extrapolation method of Gazdag (1978).

The theory of pseudodifferential operators was developed over the last several decades as an extension of the concepts of Fourier analysis to the solution of variable coefficient PDE's. (For summaries see Stein, 1993, or Taylor, 1996.) As an example of a pseudodifferential operator, consider one of relevance to wave propagation such as

$$v^2(x)\frac{\partial^2}{\partial x^2}\psi(x) = v^2(x)\frac{\partial^2}{\partial x^2}\int_{-\infty}^{\infty}\phi(k)\exp\left(i2\pi kx\right)dk = \int_{-\infty}^{\infty}\alpha(x,k)\phi(k)\exp\left(i2\pi kx\right)dk \qquad (19)$$

where

$$\alpha(x,k) = \left(i2\pi kv(x)\right)^2 \qquad (20)$$

In equation (19), the first step applies the differential operator $v^2(x)\partial^2/\partial x^2$ to a wavefield, $\psi(x)$; the second step expresses the wavefield as an inverse Fourier transform of its spectrum, $\phi(k)$; and the last step moves the differential operator inside the Fourier integral. The last term is called a pseudodifferential operator and is a prescription for the operator application in the Fourier domain. $\alpha(x,k)$ as given by equation (20) is called the symbol of the differential operator.

Pseudodifferential operators are easily defined for differential operators which are arbitrary polynomials of partial derivatives with variable coefficients in any number of dimensions. The resulting forms are all mathematically similar to equation (19) and the symbols of such operators generally become algebraic polynomials in k with coefficients which depend on x. Comparison of equation (19) with equation (10) shows that a pseudodifferential operator is applied with a nonstationary combination filter whose form is an algebraic polynomial in the spectral variables. This observation establishes a direct link between nonstationary filter theory and the theory of pseudodifferential operators. A forward pseudodifferential operator is a nonstationary combination filter of polynomial form in the spectral variables. Indeed, it seems that all pseudodifferential operators and the related Fourier integral operators can be viewed as nonstationary filters.

On the contrary, it is not always fruitful to consider nonstationary filters as pseudodifferential operators. Many nonstationary filters are designed for noise reduction, signal shaping, or other purely data processing tasks. Even if such filters could be expressed as spectral polynomials, the corresponding differential operators would have little meaning. Furthermore, though the combination form of nonstationary filters emerges directly in pseudodifferential operator theory, the convolution form does not. Though, as mentioned above, any combination operator has an equivalent convolutional form, the recognition of the two distinct forms is a strength of nonstationary filter theory. It allows Green’s function or Fourier results from stationary theory to be extended to the nonstationary case with relatively little effort compared to the pseudodifferential operator approach.

Pseudodifferential operator theory can generate more accurate solutions to complex physical phenomena than the nonstationary filter approach. The essential assumption in the latter case is that a solution to a variable coefficient PDE can be constructed from an appropriate superposition of solutions from constant coefficient PDE’s, whose coefficients are “frozen” at local values. This implicitly assumes that local gradients of coefficients do not affect the solution. In principle, the pseudodifferential operator approach does not have this limitation though this is purchased at the expense of very complex mathematics. The aforementioned extension of the phase shift method on wavefield extrapolation to nonstationary phase shift is a case in point. This advance is a straight forward application of nonstationary filtering and offers great promise. A similar formulation, reached by pseudodifferential operator methods has yet to be published. On the other hand, de Hoop (1996) uses pseudodifferential operators to reach a very general solution to the 3-D acoustic scattering problem assuming only that material parameters are smoothly variable (infinitely differentiable) functions of position. Though a major advance, the mathematical complexity of this work has slowed its acceptance. Note also, the nonstationary filtering approach does not require smoothly variable media.

NONSTATIONARY INVERSE FILTERS

Just as nonstationary filter theory is a direct generalization of stationary concepts, it appears that inverse filters can be formulated through suitable generalizations of stationary ideas. This promising avenue has just begun to be explored. As an example, it seems reasonable to expect that if a forward filter is applied by a convolutional form then the inverse will be a combinational form and vice-versa. Consider the possibility that the inverse to equation (8) might be found by

$$\hat{h}(\tau) = \int_{-\infty}^{\infty} \alpha^{-1}(f,\tau) G(f) \exp\left(i2\pi f\tau\right) df \quad . \tag{21}$$

Substitution of equation (8) for G(f) in this expression gives

$$\hat{h}(\tau) = \int_{-\infty}^{\infty} \alpha^{-1}(f,\tau) \left[\int_{-\infty}^{\infty} \alpha(f,u) h(u) \exp\left(-i2\pi fu\right) du \right] \exp\left(i2\pi f\tau\right) df \quad . \tag{22}$$

Interchanging the order of integration leads to

$$\hat{h}(\tau) = \int_{-\infty}^{\infty} h(u)\Delta(u,\tau)du \tag{23}$$

where

$$\Delta(u,\tau) = \int_{-\infty}^{\infty} \alpha(f,u)\alpha^{-1}(f,\tau)\exp\left(i2\pi f(\tau-u)\right)df \tag{24}$$

is the resolution kernel of the possible inversion. Clearly, we would like to have $\Delta(u,\tau)=\delta(\tau\text{-}u)$ for a perfect inversion and it is appropriate to inquire about whether this is possible. If $\alpha(f,u)\alpha^{-1}(f,\tau)=1$ for all u and τ then Δ does yield the desired Dirac delta function; however, we cannot expect this condition to be satisfied for all times with a nonstationary filter. It seems apparent that as long as $\alpha(f,u)\alpha^{-1}(f,\tau)$ changes much more slowly with f than does $\exp(i2\pi f(\tau\text{-}u))$ then $\Delta(u,\tau)$ can be expected to be very delta-like. Note that, for $u=\tau$, $\alpha\alpha^{-1}$ is unity provided only that α^{-1} exists from which it follows that equation (24) has a singularity at $u=\tau$. Thus it is likely that the resolution kernel is very sharply peaked, for quite general $\alpha(f,u)$, and that equation (23) is a useful inversion formula.

More study on the issue of inverse filters is required. Among the many questions to be answered are: What conditions are required to guarantee that equation (23) is a valid (or at least useful) inverse? What is the relation between the approach here and the established methods for Fredholm integral equations? Is nonstationary combination sufficiently stable to be used in a robust inversion of noisy data? Would a convolution-based inversion of a convolutional expression be more stable and more useful?

As a first step, Schoepp and Margrave (1997) have demonstrated that robust and useful deconvolution routines can be built based on these ideas. This approach merges the ideas of stationary frequency domain deconvolution and inverse Q filtering. The result is an algorithm which can deconvolve the source waveform, compensate for both the amplitude and phase effects of absorption, and address a wider class of multiples than conventional deconvolution. Moreover, it does not require the precise knowledge of Q that inverse Q filters typically need. Such a nonstationary deconvolution approach promises higher resolution seismic images with a stationary embedded wavelet and more meaningful reflection amplitudes.

FOURIER DOMAIN EXPRESSIONS

These nonstationary filter forms, convolution and combination, can be moved entirely into the Fourier domain. The derivations are similar to that used above in the dual-domain expressions and are given completely in Margrave (1996 and 1998). Simply put, nonstationary convolution can be moved entirely into the Fourier domain to give

$$G(f) = \int_{-\infty}^{\infty} H(\eta)A(f,f-\eta)d\eta \tag{25}$$

where

$$A(p,q) = \int_{-\infty}^{\infty} \alpha(p,v)\exp\left(i2\pi pu\right)dv\alpha(p,v) = \int_{-\infty}^{\infty} a(u,v)\exp\left(i2\pi pu\right)du \quad . \tag{26}$$

Equation (26) defines the frequency connection function, A(p,q), so-called because it determines the connection or mapping between input and output frequency. A(p,q) may be computed from $\alpha(p,v)$ by an ordinary Fourier transform over v or, equivalently from a(u,v) by a 2-D Fourier transform over u and v.

The comparable expression for nonstationary combination in the Fourier domain is

$$\underline{G}(f) = \int_{-\infty}^{\infty} H(\eta)A(\eta,f-\eta)d\eta \quad . \tag{27}$$

The subtle difference between equations (25) and (27) is very intriguing. Both can be seen to be nonstationary filter forms which are mathematically similar to the time domain expressions of equations (2) and (3). In fact, nonstationary convolution has a combination form in the Fourier domain and vice-versa for nonstationary combination.

Figure 7 shows the discrete computations of both equation (25) and a stationary filter as matrix operations. The filter being applied is the same as in Figures 3 and 5. In the stationary limit, A(p,q) becomes $A(p)\delta(q)$ (Margrave, 1996 or 1998) and the integral in equation (25) collapses to a scalar multiply. For a discrete application, this becomes a diagonal matrix multiply as shown in Figure 7a. If the diagonal were displayed in profile, the spectrum of the stationary filter would be seen. In the nonstationary case, the filter matrix can have significant power everywhere, though it will be diagonally dominant for a large class of quasi-stationary filters.

Comparing the matrices in Figures 3 and 7, we can appreciate that as the stationary limit is attained, the convolution matrix becomes Toeplitz and the spectral matrix becomes diagonal. In a general nonstationary setting, both matrices are non-Toeplitz and potentially fully populated.

Equation (26), like equation (9) moves the nonstationary filter specification between domains using ordinary Fourier transforms. As such, these equations can easily be inverted and provides general prescriptions for moving a filter between domains. It is usually preferable to design the filter in the dual-domain and then move it to the Fourier or time domains for application.

CONCLUSIONS

The concept of convolution is important to data processing as a filter application technique; but more importantly, plays a central role in physical theory. The partial differential equations of physics can have their solutions written as convolutions of an impulse response with a source distribution. When the coefficients of the equation, which are determined by the physical parameters of the medium, are constant, the convolution is stationary and the solution is often exact. When the coefficients vary with space or time, the convolution is nonstationary and the solution is generally approximate.

Nonstationary filter theory provides a complete methodology for description and application of nonstationary filters. Nonstationary convolution filters form the scaled linear superposition of the impulse responses of the nonstationary filter. An alternate filter form, nonstationary combination, still forms a linear superposition and reduces to stationary convolution in the stationary limit; but does not form the superposition of the filter's impulse responses. The theory provides arbitrary control of the time and frequency characteristics of the nonstationary filter and applies the filter without having to decompose the signal with a nonstationary transform. Only stationary Fourier transforms enter into the theory.

When moved into the dual-domain of time-frequency, nonstationary convolution becomes a generalized forward Fourier integral. This expression applies the nonstationary filter simultaneously with the transformation of the signal from the time domain to the frequency domain. Nonstationary combination is a generalized inverse Fourier integral which moves the signal from the frequency to the time domain while applying the signal.

For a nonstationary filter whose temporal variation is piecewise constant, the dual-domain forms can be manipulated to produce an intuitive windowing technique of filter application. Given a set of boxcar windows, one per filter segment with its passband centered on the filter specification zone, nonstationary convolution windows the data, filters each windowed subset, and superimposes the results. Nonstationary combination, filters the unwindowed data with each filter segment, then windows and superimposes. Applied to wavefield extrapolation, nonstationary convolution (NSPS) produces a dramatically superior result to a combination approach (PSPI).

Pseudodifferential operators and Fourier integral operators are nonstationary combination filters whose filter form is a polynomial in the spectral variables. However, many nonstationary filters, such as those designed form purely data processing reasons, are not usefully described as pseudodifferential operators. Nonstationary filter theory can lead quickly to valuable nonstationary extensions of known stationary solutions to physical problems. This is done under the assumption that the a local solution is always well approximated by a stationary solution. Pseudodifferential operator theory provides an approach capable of developing more general solutions but at the expense of more complicated mathematics.

A useful approximate inverse filter to a nonstationary convolution is given by a nonstationary combination with the inverse of the forward filter. Such an inversion formalism can provide a sharp resolution kernel. Nonstationary deconvolution can be formulated along these lines.

Nonstationary filters can also be re-expressed in the full Fourier domain. If the time domain discrete filter matrix is Toeplitz (stationary) the Fourier domain filter matrix is diagonal and vice-versa.

Nonstationary filter theory holds great promise for many geophysical problems.

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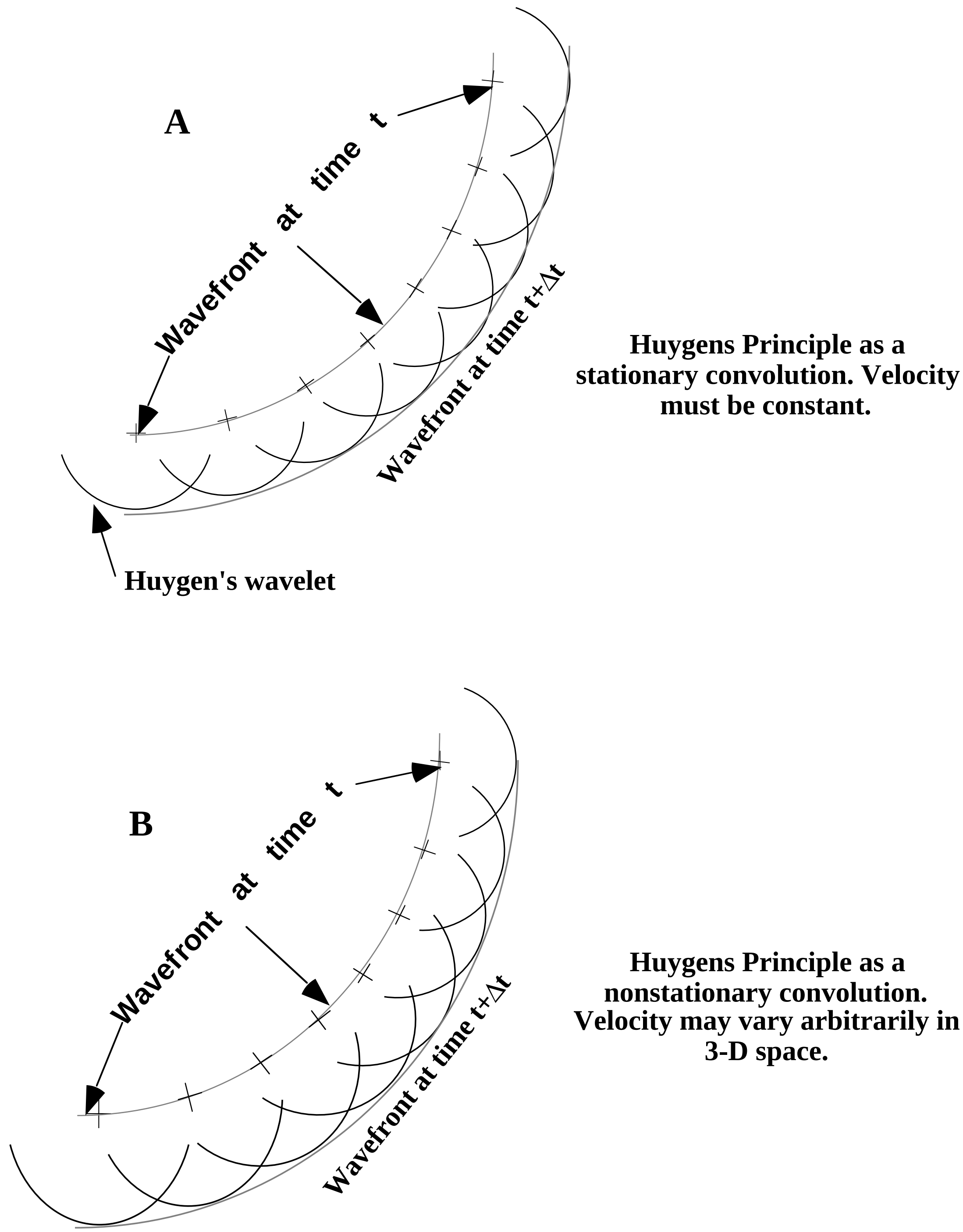

Fig. 1. A) Huygen's principle in a constant velocity medium showing an input wavefront (time t) and an output wavefront (time t+Δt). Every point on the input wavefront is replaced by a Huygen's wavelet of radius vΔt and the superposition of all such wavelets forms the output wavefront. Mathematically, this process is a convolution. B) Huygen's principle for a variable velocity medium. The Huygen's wavelets simply change their radius according to the local velocity but all else is the same. This is a nonstationary convolution.

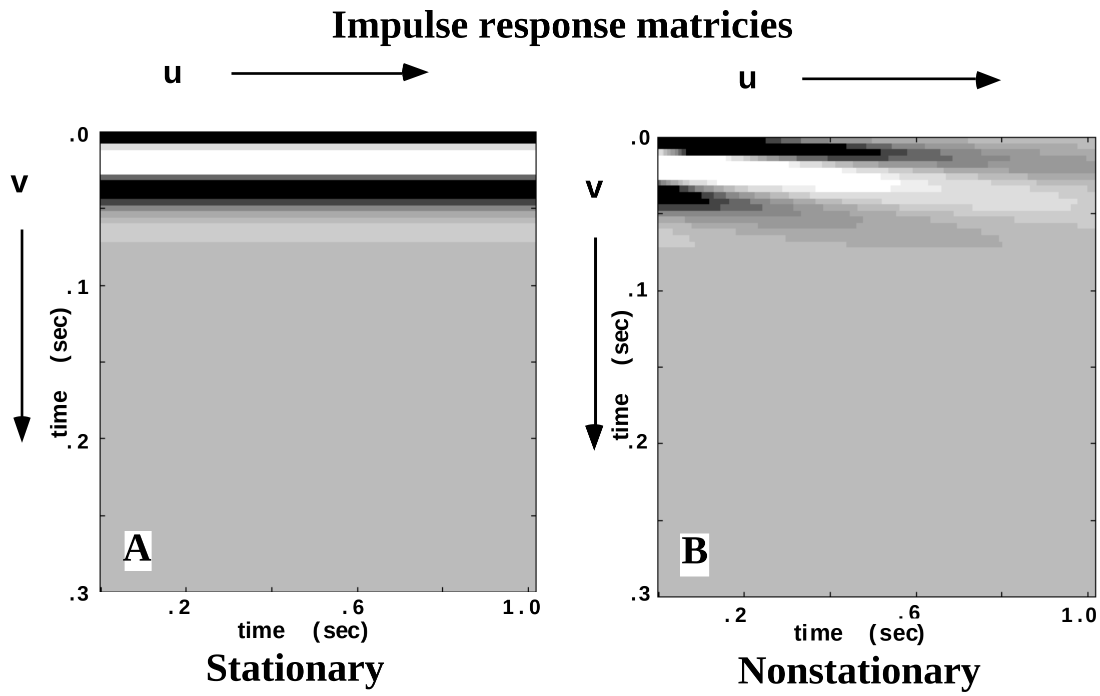

Fig. 2. A) A stationary impulse response matrix. B) A nonstationary impulse response matrix. In both matrices, each column contains the impulse response of alinear filter at the time, u, denoting the arrival time of an impulse. In the stationary case, the impulse response is constant with u while in the nonstationary case it varies, in this case according to a constant Q model.

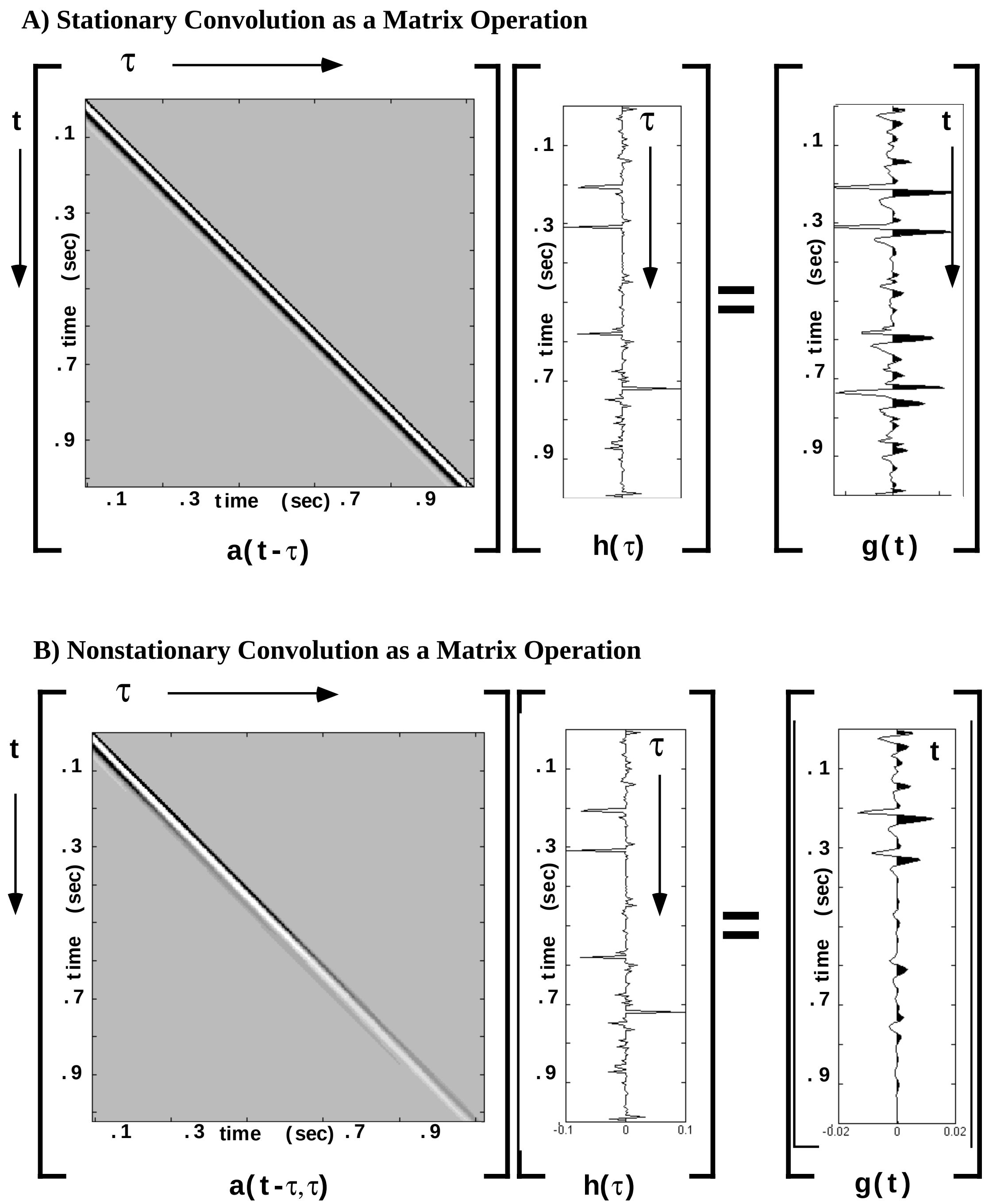

Fig. 3. A) Stationary convolution as a matrix operation. The matrix a(t-τ) multiplies the input column vector h(τ) to give g(t). B) Nonstationary convolution as a matrix operation. In both cases, the convolution matrices are formed from the impulse response matrices of Figure 2.

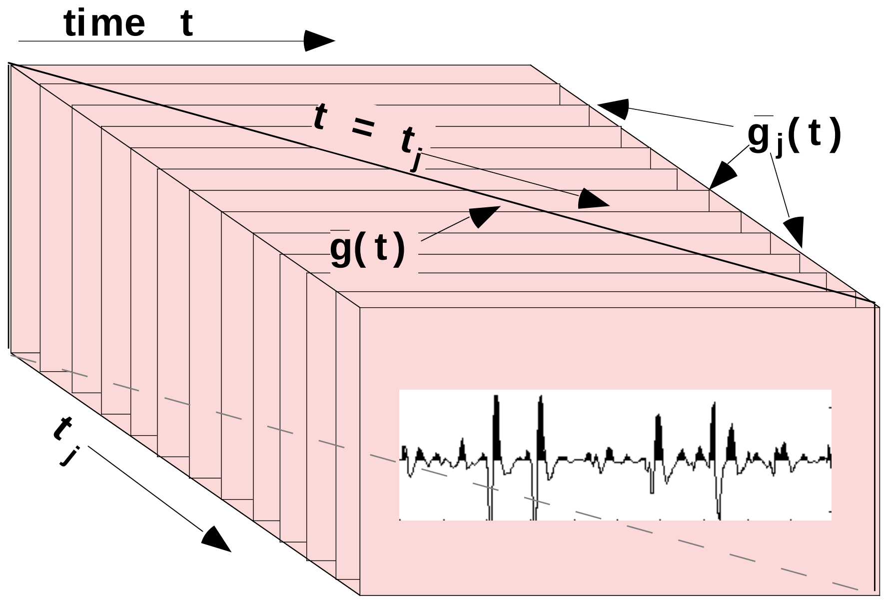

Fig. 4. A nonstationary combination does not form the linear superposition of impulse responses. Instead, it can be described as a slicing operation through an exhaustive set of stationary filters, one for each output time.

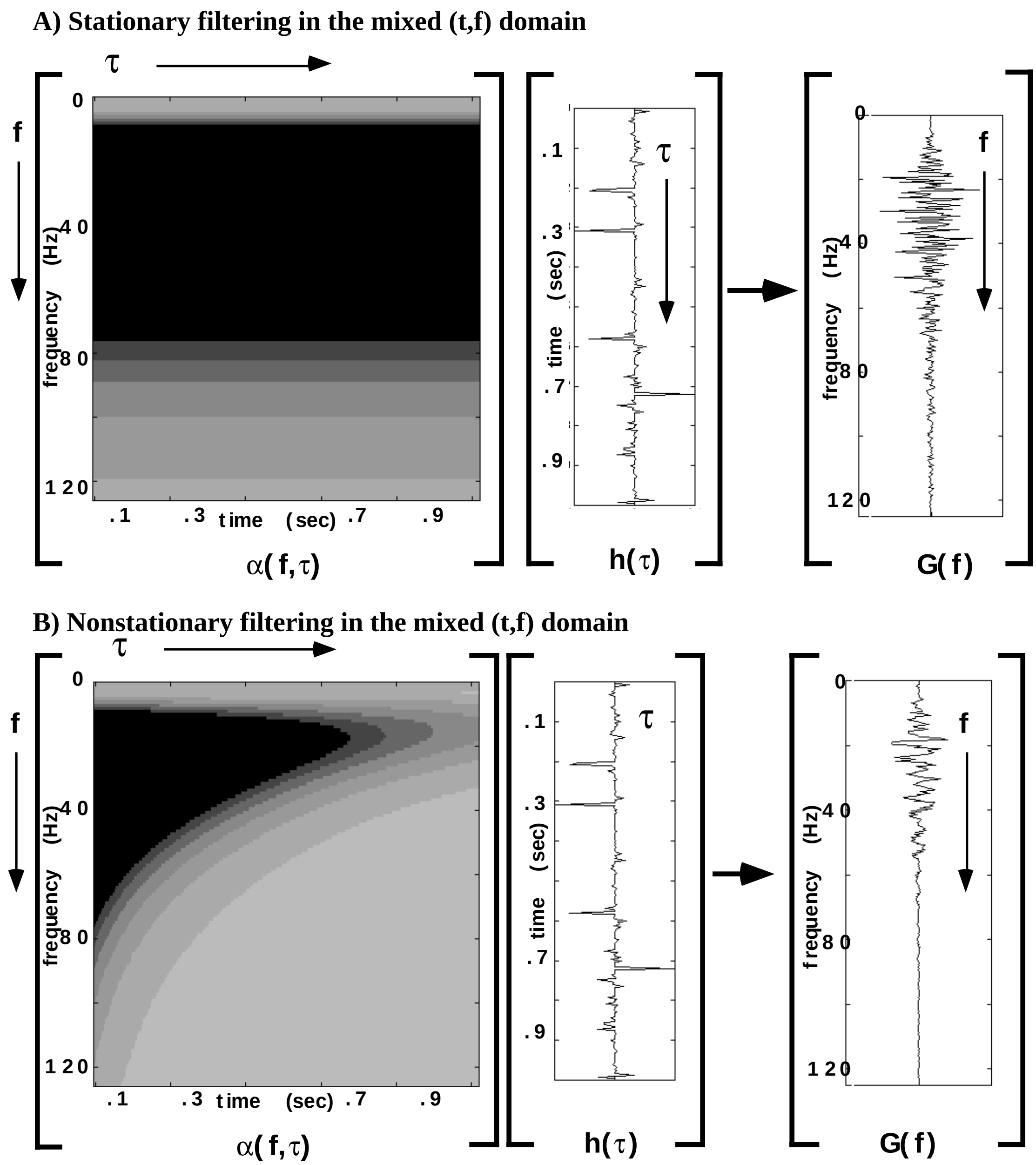

Fig. 5. A) Stationary convolution and B) nonstationary convolution represented in the dual domain (f,τ) for the same case as Figure 3. The filter matrices show the spectrum of the filter for each input time τ. In the stationary case the filter does not vary with τ, while it varies systematically in the nonstationary case. The dual domain filtering operation moves the data from time to frequency as it applies the filter. (These graphics are not true matrix equations since they omit the depiction of a Fourier transform matrix.)

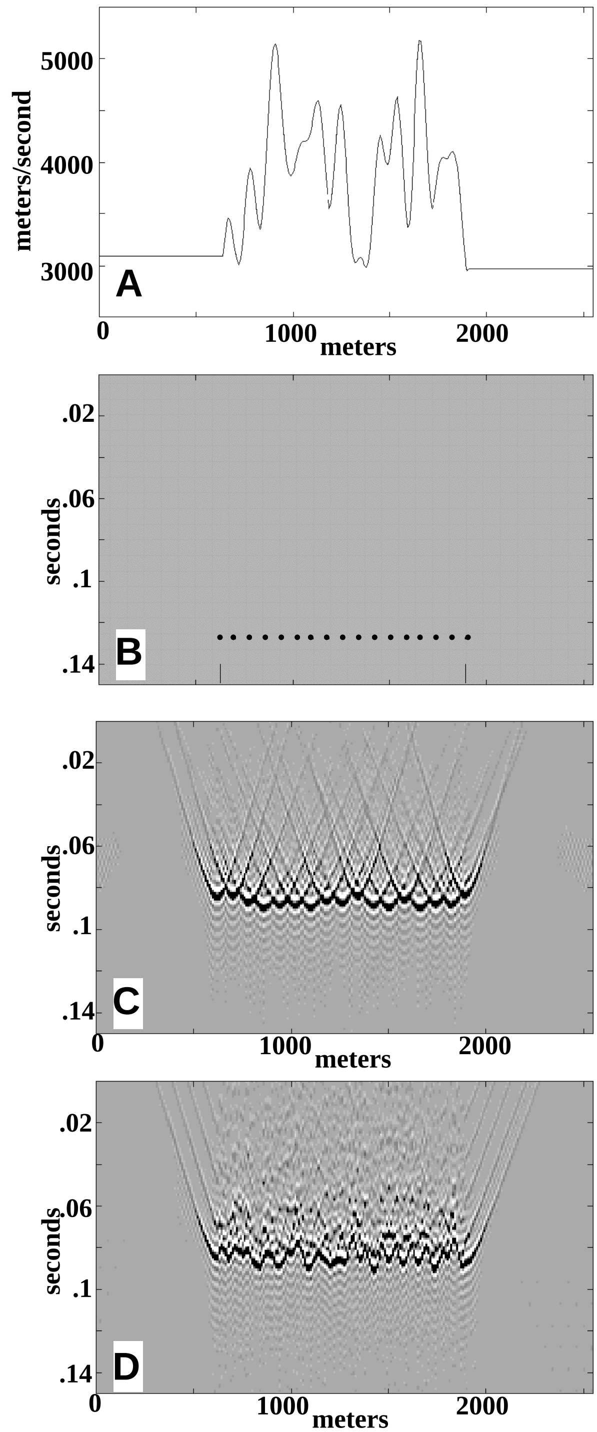

Fig. 6. Illustration of wavefield extrapolation as a nonstationary filtering operation. A) The input velocity distribution. Velocity varies rapidly laterally but is constant vertically. B) The input wavefield is a set of impulses designed to capture the impulse response of the extrapolator. C) Output wavefield after extrapolation though a 50m vertical step using NSPS (nonstationary phase shift). This method is a nonstationary convolution and has clearly formed a superposition of impulse responses. D) Output wavefield after extrapolation through though a 50m vertical step using generalized PSPI (phase shift plus interpolation). This method is a nonstationary combination and has not formed a clean superposition of impulse responses.

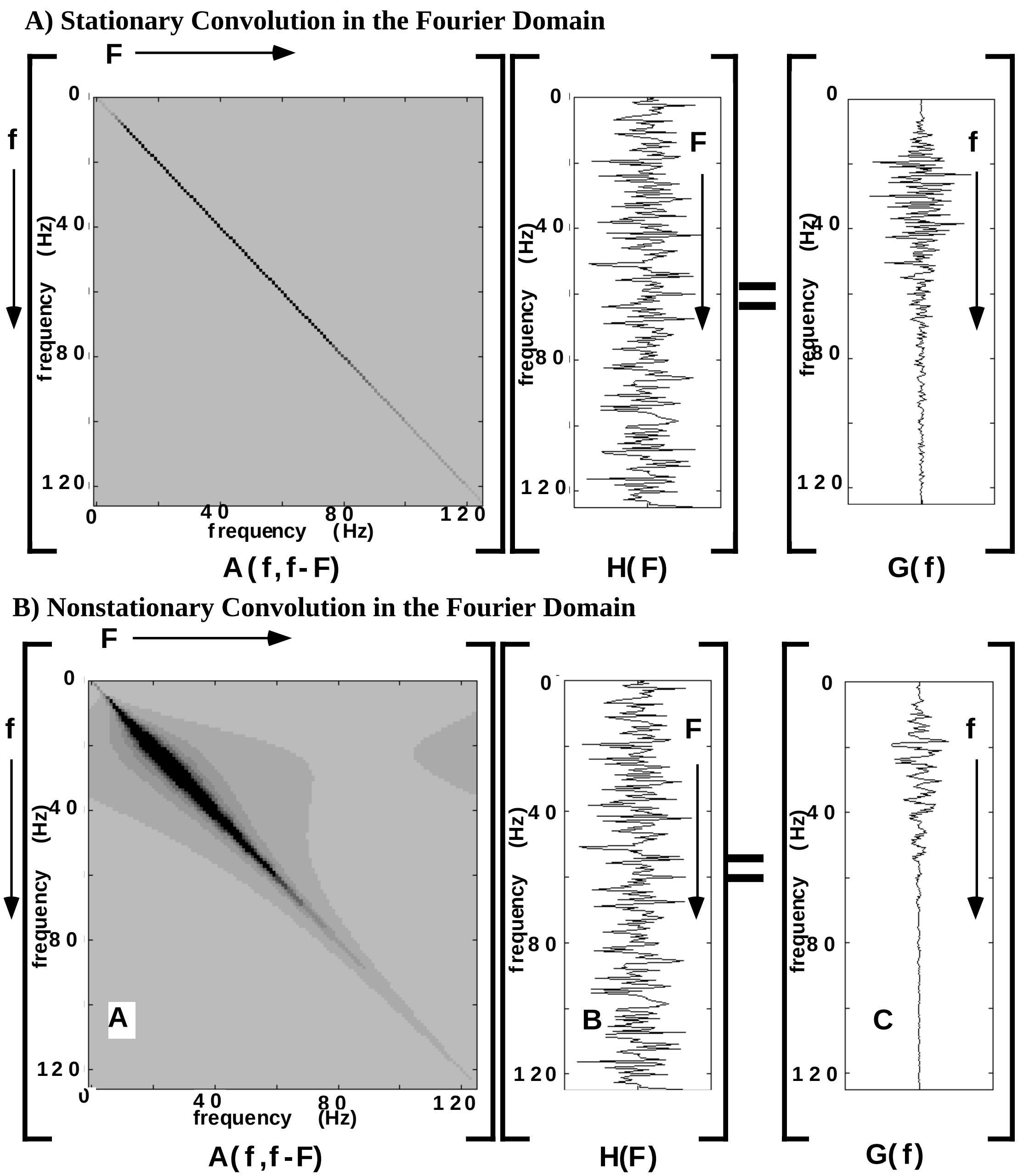

Fig. 7. Stationary (A) and nonstationary (B) convolution in the Fourier domain for the same filter as Figure 3. In both cases, the filter matrix multiplies the input spectrum to yield the output spectrum. In the stationary case the filter matrix is purely diagonal with the Fourier spectrum of the filter along the diagonal. This recreates the convolution theorem. In the nonstationary case, the filter matrix contains off-diagonal power which describes the nonstationarity of the filter. This result generalizes the convolution theorem.