A recipe for stability analysis of finite-difference wave equation computations

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INTRODUCTION

Finite-difference solutions to the wave equation are pervasive in the modeling of seismic wave propagation (Kelly and Marfurt, 1990) and in seismic imaging (Bording and Lines, 1997). That is, they are useful for the forward problem (modeling) and the inverse problem (migration). In computational solutions to the wave equation, it is necessary to be aware of conditions for numerical stability. In this short note, we examine a convenient recipe for insuring stability in our finite-difference solutions to the wave equation. The stability analysis for finite-difference solutions of partial differential equations is handled using a method originally developed by Von Neumann and described by Press et al. (1986, p. 827-830).

DERIVATION OF A RECIPE FOR STABILITY

For our discussions, we consider the wave equation for a homogeneous acoustic medium, which in 3-D is given by:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2} \tag{1}$$

where $u(x, y, z, t)$ is the pressure wavefield as a function of space and time and v is the acoustic velocity. Finite-difference computations require determinations of spatial and temporal sampling criteria. As pointed out by Kelly and Marfurt (1990), Mufti (1990) and others, spatial sampling is generally chosen to avoid grid dispersion in solutions. Then, having chosen spatial sampling, the temporal sampling is chosen to avoid numerical instability.

Recent papers by Mufti (1990) and Wu et al. (1996) derived stability criteria for three-dimensional finite-difference solutions to the wave equations for second order and fourth order systems. While these papers are correct, there is apparently a general recipe for a stability criterion which is generally applicable for arbitrary orders of accuracy and spatial dimensions. For the moment, we will facetiously call this condition "Bording's conjecture". The criterion states that for a grid size of h, a time sampling interval of Δt, a seismic velocity of v, the stability formula for finite-difference computation is given by:

$$\frac{v\Delta t}{h} \le \sqrt{\frac{a_1}{a_2}} \tag{2a}$$

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Here a_1 = sum of absolute values of weights of the finite-difference operator for $\partial^2 u/\partial t^2$, and a_2 = sum of absolute values of weights for the finite difference approximations to $\nabla^2 u$. We present here a heuristic proof of this criterion for second order time differences, since for approximations to $\partial^2 u/\partial t^2$, the second order differencing operator of (1,-2,1) is almost always used, our proof will be for the stability criterion where $a_1 = 4$. That is, we shall prove the result that:

$$\frac{v\Delta t}{h} \le \frac{2}{\sqrt{a_2}} \tag{2b}$$

To convince ourselves of the correctness of this result, we examine some of the familiar stability results in Table 1 for the cases of second and fourth order spatial differencing in 1-D, 2-D, and 3-D cases. Having the correct predictions by using the formula in (2), we now proceed with a proof of this result.

Table 1. Stability limits for 2-D and 3-D models using 2nd and 4th order finite differences.

Dimension	2nd Order	4th Order
1-D	1	$\sqrt{3}/2$
2-D	$1/\sqrt{2}$	$\sqrt{3/8}$
3-D	$1/\sqrt{3}$	$1/2$

Following the stability discussions of Mufti (1990) and Wu et al. (1996), we consider a Fourier component of the computational error , ε_{ijk}^{n} at the (i,j,k) grid point and at time step n. For the grid spacings $\Delta x, \Delta y, \Delta z$, the Fourier component of the computational error is given by:

$$\varepsilon_{ijk}^{n} = \Gamma^n e^{\iota p i \Delta x} e^{\iota q j \Delta y} e^{\iota r k \Delta z} \tag{3}$$

where $\iota = \sqrt{-1}$ and spatial wavenumbers in x, y, z are denoted by p, q, r. Γ^n is the Fourier amplitude at a particular time step n.

Although strictly speaking, computational error equals the sum of complex exponentials, it is sufficient for stability analysis that we consider a specific Fourier component as given in (3). For simplicity, we will deal with a uniform grid by setting $\Delta x = \Delta y = \Delta z = h$. Therefore, our goal is to select values of h and Δt such that stability is satisfied for arbitrary spatial dimensions and orders of finite-difference approximations.

Let the computed wavefield be u, let the exact wavefield be U and let the computational error be ε, so that $u = U + \varepsilon$, and ε satisfies the numerical solutions to the wave equation (1) as well.

In order to generalize the results of Wu et al. (1996), we express the values of second derivatives such as $\partial^2 u/\partial x^2$, $\partial^2 u/\partial y^2$, and $\partial^2 u/\partial z^2$ in terms of weighted sums of values of u about a grid point (i, j, k). Let the spatial weights for the second derivatives in the x, y and z directions be $w_{m,}$ $\ddot{w}_m$, and $\tilde{w}_m$ respectively. That is, the second partial derivative in x for the wavefield is given by:

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{\Delta x^2} \sum_{m=-M}^{M} w_m u_{i+m,j,k}^{n} \tag{4}$$

For second order systems, $M = 1$ and (w_{-1}, w_0, w_1) is given by (1,-2,1). For fourth order systems, $M = 2$ and $(w_{-2,} w_{-1,} w_0, w_1, w_2)$ is given by 1/12(-1,16,-30,16,-1). (Detailed derivations of these operators are given by Kelly, 1998). Similarly for $\partial^2 u/\partial t^2$, we have the difference equation given by:

$$\frac{\partial^2 u}{\partial t^2} = \frac{1}{\Delta t^2} \sum_{l=-L}^{L} W_l u_{i,j,k}^{n+l} \tag{5}$$

For our discussions, we will consider only second order systems in time such that $L = 1$ and $(W_{-1}, W_{0,} W_1)$ is given by (1,-2,1). Given that the error in our wavefield calculations also satisfies equation (1), we can write the error wavefield equation in the following form by using ε in place of u in equations (4), (5), and (1). This produces:

$$\frac{1}{h^2} \sum_{m=-M}^{M} (w_m \varepsilon_{i+m,j,k}^{n} + \ddot{w}_m \varepsilon_{i,j+m,k}^{n} + \tilde{w}_m \varepsilon_{i,j,k+m}^{n}) = \frac{1}{v^2 \Delta t^2} \sum_{l=-L}^{L} W_l \varepsilon_{i,j,k}^{n+l} \tag{6}$$

In order to analyze the Fourier components of the error, we then substitute equation (3) into equation (6) and divide by $e^{\imath h(pi+qj+rk)}$ to obtain an expression for the amplitude of the sinusoidal component as a function of time steps.

$$\frac{v^2 \Delta t^2}{h^2} \sum_{m=-M}^{M} \Gamma^n (w_m e^{\imath mph} + \ddot{w}_m e^{\imath mqh} + \tilde{w}_m e^{\imath mrh}) = \sum_{l=-L}^{L} W_l \Gamma^{n+l} \tag{7}$$

For second order approximations to the wave equation, the right hand side of (7) would be given by:

$$\sum_{l=-1}^{1} W_l \Gamma^{n+l} = \Gamma^{n+1} - 2\Gamma^n + \Gamma^{n-1} \tag{8}$$

This allows us to rewrite equation (7) in a simpler form as:

$$\Gamma^{n+1} = -\Gamma^{n-1} + 2A\Gamma^{n} \tag{9}$$

where

$$A = 1 + \frac{v^2 \Delta t^2}{2h^2} \sum_{m=-M}^{M} \left(w_m e^{\imath mph} + \ddot{w}_m e^{\imath mqh} + \tilde{w}_m e^{\imath mrh} \right) \tag{10}$$

Equation (9) has been analyzed for stability in a couple ways. One method due to Mufti (1990) provides a sufficient condition for stability by considering the ratio of the error Fourier amplitudes as a function of time steps. That is, Mufti (1990) considers this ratio as $\gamma = \Gamma^{n+1}/\Gamma^{n} = \Gamma^{n}/\Gamma^{n-1}$ to be the ratio of successive iterations. Therefore we can insure stability by requiring that $|\gamma| \leq 1$. We can consider the stability in terms of γ by dividing equation (9) by Γ^{n-1} to obtain:

$$\gamma^2 - 2A\gamma + 1 = 0 \tag{11}$$

such that $\gamma = A \pm \sqrt{A^2 - 1}$.

Stability is assured if $-1 \leq A \leq 1$. This requirement on A yields the result that

$$4 \leq \frac{v^2 \Delta t^2}{h^2} \sum_{m=-M}^{M} \left(w_m e^{\imath mph} + \ddot{w}_m e^{\imath mqh} + \tilde{w}_m e^{\imath mrh} \right) \leq 0 \tag{12}$$

As indicated by the computational results of Wu et al. (1996) and by Smith (1965, p. 72) the useful inequality is the left hand side of (12). If we multiply equation (12) by –1, we obtain:

$$\frac{v^2 \Delta t^2}{h^2} \left| \sum_{m=-M}^{M} \left(w_m e^{\imath mph} + \ddot{w}_m e^{\imath mqh} + \tilde{w}_m e^{\imath mrh} \right) \right| \leq 4 \tag{13}$$

Now due to the triangle inequality that the absolute value of a sum is less than the sum of the absolute values, we can use the fact that:

$$\left| \sum_{m=-M}^{M} \left(w_m e^{\imath mph} + \ddot{w}_m e^{\imath mqh} + \tilde{w}_m e^{\imath mrh} \right) \right| \leq \sum_{m=-M}^{M} |w_m| + |\ddot{w}_m| + |\tilde{w}_m| \tag{14}$$

and from (13) we can see that a sufficient condition for stability is the result which we wished to prove. Stability is assured if

$$\frac{v\Delta t}{h} \le \frac{2}{\sqrt{a_2}} \tag{15}$$

where $a_2 = \sum_{m=-M}^{M} |w_m| + |\ddot{w}_m| + |\tilde{w}_m|$ is defined as the sum of the absolute values of the weights of the spatial derivatives as given by the right hand side of (14).

We will note that equation (15) produces the familiar results for second and fourth order methods in both 2-D and 3-D. Let us check out the results. For example, for second order in 2-D, the second derivative operator is (1,-2,1), the value of a_2 would be (1+2+1) + (1+2+1) or 8. Therefore, this gives $2/\sqrt{a_2} = 1/\sqrt{2}$ and this requires the value of $v\Delta t/h \le 1/\sqrt{2}$, which is a familiar result given by Mitchell and Griffiths(1980). For second order methods in general, we see that each spatial dimension adds a value of 4 in the denominator and we essentially have the stability result which is in agreement with the Courant condition that $v\Delta t/h \le 1/\sqrt{n}$ where n is the dimensionality in space (Mitchell and Griffiths, 1980).

For the fourth order methods in space, the derivative operator is a five point operator given by 1/12(-1,16,-30,16,-1), so that the bounds on $v\Delta t/h$ for 2-D and 3-D would be $\sqrt{3/8}$ and $1/2$ respectively, which are results proven by Wu et al. (1996).

CONCLUSIONS

The heuristic derivation of the useful stability formulae in equation (15) is an extension of the discussions by Mufti (1990) and Wu et al. (1996). Along with a criterion for avoiding grid dispersion, this stability recipe allows us to choose proper space and time sampling for the finite-difference computations used in seismic modeling and imaging.

ACKNOWLEDGEMENTS

We thank Wen-Jing Wu of Geo-X Systems Ltd. and Dr. Gary Margrave of the University of Calgary for their comments on our paper. We also thank the sponsors of the CREWES project for their support.

LIST OF SYMBOLS

$u(x, y, z, t)$ = pressure wavefield as a function of spatial coordinates (x, y, z) and time t.

v = acoustic velocity of medium.

Δt = temporal sample interval.

h = grid spacing of finite difference mesh for uniform grid.

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$\Delta x, \Delta y, \Delta z =$ grid spacings in the x, y, z directions

ε_{ijk}^{n} = Fourier component of error for the (i, j, k) grid point and the nth time step.

Γ^n = Fourier amplitude of computational error.

γ = ratio of successive Γ values.

$p, q, r =$ spatial wavenumbers in x, y, z directions.

$w_m, \ddot{w}_m, \tilde{w}_m =$ spatial weights for difference approximations to second derivatives in the x, y, z directions.

w_k = temporal weights for second partial derivatives in time.

a_1 = sum of absolute values of temporal weights for derivatives.

a_2 = sum of absolute values of spatial weights for derivatives.

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