Wavefield extrapolation in laterally inhomogeneous media by generalized eigenfunction transform method

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ABSTRACT

Using separation of variables the wave equation in laterally inhomogeneous media can be spatially separated into two eigen-equations corresponding to the direction of extrapolation and the other transverse direction. The two equations are connected via eigenvalues, which leads to an eigenvalue problem represented by a second order differential equation. This equation can be solved either approximately or exactly. The solutions obtained with different approximations can lead to different methods in seismic exploration seismology.

INTRODUCTION

The study of wavefield extrapolation in laterally inhomogeneous media is important because the geological structures are often modeled as laterally inhomogeneous layered structures. The main purpose in exploration seismology is to create a seismic image is of the area of interest. Many methods for seismic imaging are built on the concept of wavefield extrapolation.

In homogenous media, the wavefield extrapolation problem can be solved analytically. However, in strongly inhomogeneous media, finding the solution is a great challenge. Many standard methods fail in this case.

The Kirchhoff approach, the most commonly used method for the problem, relies on high-frequency ray tracing. This is a problem for caustics, shadow zones, and even chaotic rays (Fei et al. 1996; Audebert et. al. 1997). More accurate ray tracing is computationally more expensive and more difficult to implement. Therefore, it is desirable to find wave-equation methods that can avoid these difficulties.

The full wave equation (two-way wave propagation) (Baysal et. al., 1985; Wapennar, 1987) is capable of solving wave solutions for arbitrary complex media. Unfortunately, such approaches are very time consuming. Moreover, the methods lacks control over multiple scattering, resulting in a strong noise background (Jin et. al., 1998). Based on the paraxial wave equation, one-way wave propagation methods are very efficient but are limited to a range of propagation angles (e.g. Claerbout, 1985).

Methods implemented in the frequency-wavenumber (F-K) domain are very efficient due to the use of fast Fourier transform. However, most of the methods developed are incapable of dealing with strong laterally inhomogeneous media. Phase shift (Gazdag, 1978) is accurate only if the velocity is constant. The screen propagators, including the split step Fourier transform method (e.g. Stoffa et. al., 1990), phase screen and pseudo screen methods (e.g. Wu, 1994; Jin et. al., 1998), have been proposed to calculate one way wave propagation for either forward modeling or seismic migration. Even though these methods work well for weak contrast media, in the case of complex structure with strong velocity contrasts, such as the salt dome related structure where velocity of the salt body can be about three times higher than the surrounding media, the methods are only accurate for waves with small angle of incidence. In order to improve the accuracy in this case, several extensions of the phase shift and split step Fourier methods have been developed, e.g. phase shift plus interpolation (PSPI) (Gazdag, 1981) and the extended split step Fourier method with multiple reference velocities (e.g. Kessinger, 1992). However, inaccuracies associated with the interpolation process may arise.

Recently, the non-stationary phase shift method was developed based on pseudo-differential operator theory for improving the efficiency and accuracy of the wavefield extrapolation (Margrave & Ferguson 1997, 1999). Comparing with original PSPI, it overcomes the shortcoming of the global nature of reference velocities selection and still keeps many advantages of the PSPI. However, in order to reach a satisfactory accuracy, a recursion has to be performed.

Based on eigen-expansion techniques, Grimbergen et. al. (1998) developed the modal expansion method for wavefield extrapolation in laterally varied velocity structure. The numerical implementation of the method given by Grimbergen et. al. (1998) to 2D wavefield extrapolation leads to a very similar result as that of nonstationary filter (Margrave & Ferguson 1997, 1999).

The methods mentioned above for wavefield extrapolation in inhomogeneous media are all approximate methods. Therefore, they have limits, such as accuracy and stability. In this paper, a new method based on a generalized eigenfunction transform is presented for wavefield extrapolation in laterally varied velocity media.

METHOD DESCRIPTION

The acoustic wave equation (2D) is often written as the Helmoholz equation

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}+\frac{\omega^2}{v^2(x)}\right)\Psi(x,z,\omega)=0 \tag{1}$$

with initial condition

$$\Psi(x,z=0,\omega)=\Psi_0 \tag{2a}$$

and radiation condition

$$\Psi(x,z=\pm\infty,\omega)=0 \tag{2b}$$

where Ψ denotes the pressure, v is the velocity field that is independent of z coordinate and ω is angular frequency, i.e. $2\pi f$.

Using the technique of separation of variables, we seek a solution for the wavefield in the form

$$\Psi(x,z,\omega) = \Psi_x(x,\omega)\Psi_z(z,\omega) \quad . \tag{3}$$

Substituting equation (3) into wave equation (1) and dividing through by $\Psi_x \Psi_z$, we have

$$\frac{\partial^2 \Psi_z}{\partial z^2}\frac{1}{\Psi_z} + (\frac{\partial^2 \Psi_x}{\partial x^2}\frac{1}{\Psi_x} + \frac{\omega^2}{v^2(x)}) = 0 \quad . \tag{4}$$

The two terms on the left side of equation (4) are the functions of z and x respectively. Therefore, the only way the equation can be satisfied is if each of them is equal to a constant that is independent of x and z, i.e.

$$\frac{\partial^2 \Psi_z}{\partial z^2}\frac{1}{\Psi_z} = -(\frac{\partial^2 \Psi_x}{\partial x^2}\frac{1}{\Psi_x} + \frac{\omega^2}{v^2(x)}) = k^2 \tag{5}$$

or

$$\frac{\partial^2 \Psi_z}{\partial z^2}\frac{1}{\Psi_z} = k^2 \quad , \qquad -(\frac{\partial^2 \Psi_x}{\partial x^2}\frac{1}{\Psi_x} + \frac{\omega^2}{v^2(x)}) = k^2 \quad . \tag{6}$$

Equation (6) can be compared to the classical Sturm-Liouville eigenvalue problem (e.g. Stakgold, 1979), which has the properties: the equation has an infinite number of solutions which are the modes of a vibrating string, the modes are characterized by a mode shape function (eigenfunction) Ψ_z or Ψ_x and a constant k (eigenvalue).

Now the problem is to find the eigenvalue k. Solving equation (6) under different conditions leads to different methods.

Constant velocity

When the velocity is constant, the second equation of (6) is

$$D_x^2 \Psi_x = (\frac{\omega^2}{v^2} - k^2)\Psi_x \tag{7}$$

where D_x^2 denotes second order differential operator. Applying the Fourier transform pair

$$F_x f(x) = \int f(x)\exp(-ik_x x)dx \qquad F_x^{-1} f(k_x) = \int f(k_x)\exp(ik_x x)dk_x$$

to D_x^2, equation (7) becomes

$$F_x^{-1} k_x^2 F_x \Psi_x = (\frac{\omega^2}{v^2} - k^2)\Psi_x \tag{8}$$

then the eigenvalue can be found by

$$k^2 = F_x \left(\frac{\omega^2}{v^2} - k_x^2\right) F_x^{-1} \tag{9}$$

Substituting equation (9) into the first equation of (6) and applying the conditions (2a) and (2b), the solution to the wave equation can be written as

$$\Psi(x,z,\omega) = F_x^{-1} \exp\left(\pm i\sqrt{\frac{\omega^2}{v^2} - k_x^2}\right) F_x \Psi_0 \tag{10}$$

where $\Psi_0 = \Psi(x, z=0, \omega)$. The analytic solution (10) is the phase shift method for the up and downward extrapolation (Gazdag, 1978).

Generally, if velocity is an arbitrary continuous function of *x*, seeking an analytic solution like (10) is very difficult. However, if the velocity model is discretized, finding the solution numerically may become easier. In the following, the wavefield and the velocity function are discretized (along the x coordinate, z coordinate is ignored for convenience) into columns vectors, i.e. $(\psi^1, \psi^2, \cdots\cdots, \psi^n)^T$ and $(v_1, v_2, \cdots\cdots, v_n)^T$, respectively.

Finite difference method

For each local spatial point, the second equation of (6) can be written as

$$D_x^2 \Psi_x^i = \left(\frac{\omega^2}{v_i^2} - k_i^2\right) \Psi_x^i \qquad (i=1,2, ...n) \tag{11}$$

Using the second order finite difference operator D_x^f to approximate the second differential operator in equation (11), an equation for all locations can be written as the matrix equation

$$D_x^f \Psi_x = (C^2 - K^2) \Psi_x \tag{12}$$

where K^2 is the diagonal matrix containing the elements of k_i,

$$D_x^f = \begin{pmatrix} -2 & 1 & 0 & \cdot & \cdot & 0 & 0 \\ 1 & -2 & 1 & \cdot & \cdot & 0 & 0 \\ 0 & 1 & -2 & \cdot & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & & & -2 & 1 \\ 0 & 0 & 0 & & & 1 & -2 \end{pmatrix} \tag{13}$$

and

$$C^2 = \begin{pmatrix} (\frac{\omega}{v_1})^2 & 0 & \cdot & \cdot & 0 \\ 0 & (\frac{\omega}{v_2})^2 & \cdot & \cdot & 0 \\ \cdot & & \cdot & & \cdot \\ \cdot & & & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & (\frac{\omega}{v_n})^2 \end{pmatrix} \tag{14}$$

From equation (12) K^2 can be found as

$$K^2 = C^2 - D_x^F \tag{15}$$

which is a real valued and symmetric matrix. Based on matrix algebra theory this matrix can be decomposed into

$$K^2 = L\Lambda L^{-1} \tag{16}$$

where L is the matrix with its columns composed of the eigenvectors of the matrix K^2, L^{-1} is the inverse of L , which is equal to the transpose of L, and Λ is a diagonal matrix with its diagonal elements composed of the eigenvalues of K^2. Substituting (16) into the first equation of (6) and applying the conditions (1b) and (1c), the solution to the wave equation can be written as

$$\Psi(x,z,\omega) = L\exp(\pm i\Lambda^{1/2} z)L^{-1}\Psi_0 \tag{17}$$

which is the modal expansion method for the up- and downward extrapolation (Grimbergen et. al., 1998).

Nostationary phase shift method

Rewriting equation (8) for laterally varying velocity

$$F_x^{-1} k_x^2 F_x \Psi_x = (\frac{\omega^2}{v^2(x)} - k^2)\Psi_x \tag{18}$$

this equation can be changed into

$$k_x^2 F_x \Psi_x = F_x\,(\frac{\omega^2}{v^2(x)} - k^2)\Psi_x = (\frac{\omega^2}{v^2(x)} - k^2)F_x\Psi_x + [F_x, \frac{\omega^2}{v^2(x)}] \tag{19}$$

where the commutator $[F_x, \frac{\omega^2}{v^2(x)}] = F_x\frac{\omega^2}{v^2(x)} - \frac{\omega^2}{v^2(x)}F_x$. If the commutator can be ignored as a approximation, then

$$k^2 = F_x^{-1}\left(\frac{\omega^2}{v^2(x)} - k^2\right)F_x \tag{20}$$

and the final solution can be written as

$$\Psi(x, \Delta z, \omega) = F_x^{-1} \exp\left(\pm i\sqrt{\frac{\omega^2}{v(x)^2} - k_x^2}\,\Delta z\right)F_x\Psi_0, \tag{21}$$

which is the limiting form of PSPI in non-stationary phase shift method(Margrave & Ferguson, 1997, 1999). We show in another paper that the nonstationary phaseshift method can be considered as a generalization of the pseudospectral method (Yao & Margrave, 1999).

It should be mentioned that the neglect of the commutator in equation (19) can be exact only if the velocity is constant. In general, the velocity changes in a small area can be considered as small as required to be ignored. In this case, the condition required for (19) is satisfied. Therefore, like the finite difference approach, equation (21) is also a localized technique and when it is applied in wavefield extrapolation, the step length of extrapolation should be carefully chosen to ensure accuracy and avoid instability (Yao & Margrave, 1999).

Comparing the finite difference method to the nonstationary phaseshift method shown above, the difference is in the spatial derivatives calculation. The calculation of the derivatives by Fourier transform can very much improve the quality or accuracy of the solution (Kosloff & Baysal, 1982). Therefore, equation (21) is superior to equation (17) .

Full spectral method

Equation (19) can be further written as a set of equations coupled over k_x

$$K_x^2 F_x \Psi_x = F_x\left(\frac{\omega^2}{v^2(x)} - k^2\right)\Psi_x = (\omega_2 S^2 - K^2)F_x\Psi_x \tag{22}$$

where the diagonal matrix K_x^2 is

$$K_x^2 = \begin{pmatrix} -k_x^2(1) & & & \\ & . & & \\ & & . & \\ & & & -k_x^2(n) \end{pmatrix} \tag{23a}$$

and the Toeplitz matrix S^2 is

$$S^2 = \begin{pmatrix} s_0 & . & . & s_{-(n-1)} \\ . & & & \\ . & & & \\ s_{(n-1)} & . & . & s_n \end{pmatrix} \tag{23b}$$

In the equation above, k_x is the horizontal wavenumber with discretization interval $\Delta k_x = 2\pi / L$, defined by the horizontal length of the model *L*. The elements s_n are the Fourier components of $v^{-2}(x)$, i.e.

$$s_n = \frac{1}{L}\int v^{-2}(x)\exp(-in\Delta k_x x)dx \tag{24}$$

Matrix *S* is a Hermitian Toeplitz matrix. If velocity is constant, then the matrix become diagonal and each equation can be solved separately, which leads to phase shift method (Gazdag, 1978). The off diagonal elements in the matrix expressed the components of Ψ coupled with each other in lateral inhomogeneous media. Solving equation (22)

$$K^2 = \omega^2 S - K_x^2 \tag{25}$$

which is a Hermitian matrix and can be decomposited via its eigenvaluses and eigenvectors, i.e.

$$K^2 = VDV^H \tag{26}$$

where V is the matrix with its columns composed of the eigenvectors of the matix K^2, V^H is the conjugate complex transpose of *V* and *D* is a diagonal matrix with its diagonal elements composed of the eigenvalues of K^2. Substituting (26*)* into the first equation of (6) and applying the conditions (2a) and (2b), the solution to the wave equation can be written as

$$\Psi(x,z,\omega) = F_x V \exp(\pm iD^{1/2} z)(F_x V)^{-1}\Psi_0 \tag{27}$$

which is the exact solution to the wave equation.

As the summary, the solutions expressed by equation (9), (17), (21) and (27) can be uniformly written in the form of

$$\Psi(x,\Delta z,\omega) = A\exp(\pm i\Omega^{1/2}\Delta z)A^{-1}\Psi_0 \tag{28}$$

where A is the matrix composed of generalized eigenfunctions and then finding the solution is equivalent to find the transform defined by eigenfunctions, i.e. $\Psi^* = A^{-1}\Psi$. Under the transform components of the wavefield can be separated and the solution can be simply obtained by analogue to the standard phase shift method, i.e.

$$\Psi^{*}(x,\Delta z,\omega)=\exp(\pm iD^{1/2}\Delta z)\Psi_{0}^{*} \tag{29}$$

A NUMERICAL EXAMPLE

The numerical examples shown in following are only regarding to the full spectrum method because the examples for the rest of the methods mentioned above can be found in corresponding literatures.

Figure 1 shows a model composed of two blocks. The velocity on the left block is 1500 m/s and on the right it is 2500 m/s. One point source is located close to the velocity boundary on the left. After extrapolation with *Δz=200 m*, the result is shown in Figure 2. The red solid line is the traveltime of first arrival calculated by solving eikonal equation with finite-difference method (Vidale, 1988) that matches our result very well. The reflection from the velocity boundary occurs in the left lower velocity block following the direct wavefront. The energy of the wavefront in the right high velocity block is week comparing to that in the left because the transmission coefficient effect. Figure 3 shows the result when the wavefield extrapolated back to Δz=0. The original spike pulse is well recovered, which means that the method itself is consistence for both forward and backward extrapolations.

DISCUSSIONS AND CONCLUSIONS

The methods presented above solve the wavefield extrapolation problem in lateral inhomogeneous media. The algorithms based on the approximations are the localized techniques, which limit the step length of extrapolation. However, if the step length of extrapolation is too small, the cost of calculation time will be expensive. Furthermore, for the sharp changes of the velocity media, these methods may have stability problem at the sharp change boundary (Etgen, 1994). Very often the sharp change boundary is of interest in seismic interpretation. Full spectral approach provides the exact solution for wavefield extrapolation that can avoid this problem. It should be expected that it can improve the seismic image when applying to seismic migration.

It is well known from time-frequency transforms by FFT that under-sampling in one domain cause aliasing (wrap-around) in the other domain (e,g, Margrave & Ferguson, 1997), This is due to the periodicity assumed by introducing the discrete Fourier transform. For the discrete Fourier transform to be correct, the field must vanish outside the range interval considered, otherwise it will be wrapped into the neighboring windows. Therefore, the evaluations of equation (9), (22) and (27) do not produce ψ(x,z) but rather $\sum_{-\infty}^{\infty}\Psi(x+nL,z)$, i.e. a sum of the signals in all range windows of width *L=Ndx* (*N* is total number of sample points). The alias can be reduced either by choosing L so large that the signal is known to die out within the range window or by aperture compensation (Margrave & Ferguson, 1997).

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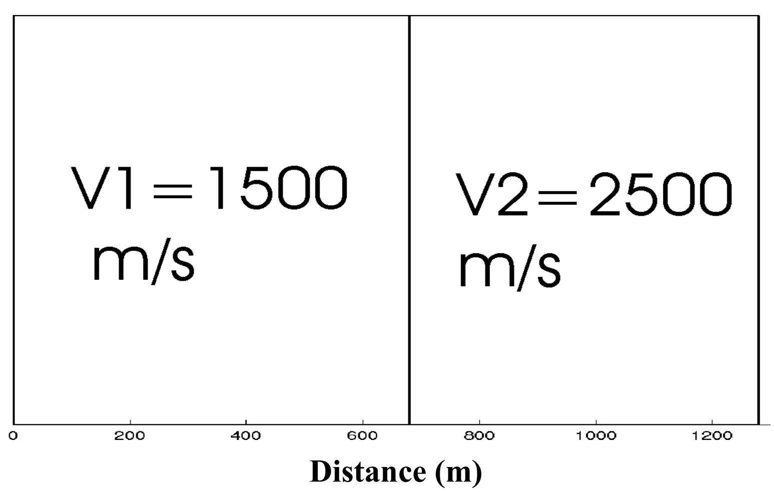

Figure 1. The velocity model.

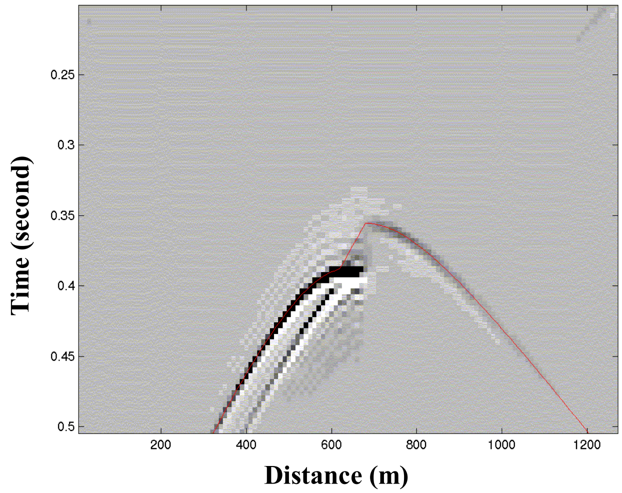

Figure 2. The result of the wavefield extrapolated with Δz=200 m of a pulse source located in the middle of the velocity model.

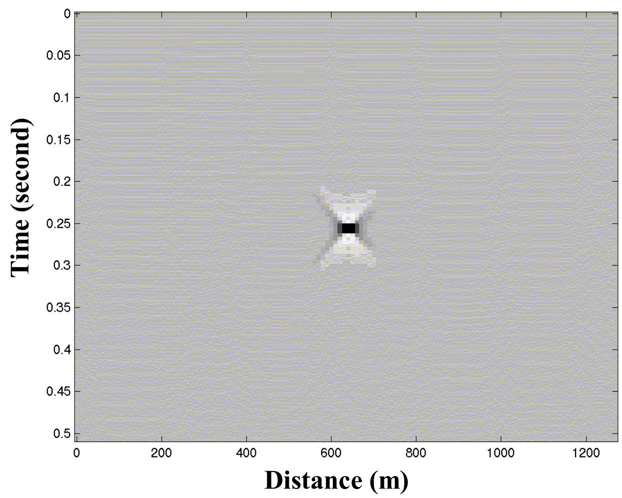

Figure 3. The result of the previous wavefield extrapolated back to Δz=0.