An error and stability analysis of four nonstationary wavefield extrapolators

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ABSTRACT

An error and stability analysis is presented for the elementary nonstationary wavefield extrapolators $L_N^+, L_p^+$ and their symmetric hybrids $L_A^+$ and $L_{PN}^+$. The analysis is based on analytic expressions that describe the inversion of wavefields extrapolated by the four operators. Our analysis shows that $L_A^+$ and $L_{PN}^+$ are more accurate and more stable than elementary extrapolators $L_N^+$ and $L_p^+$.

The Marmousi synthetic data is used to provide a comparison of depth imaging using the different extrapolators. The largest mean absolute amplitudes of the resulting depth images corresponding to $L_N^+ (~1000)$ and $L_p^+ (~1000)$ indicate that recursive application of these extrapolators caused growth in the extrapolated wavefield. The mean absolute amplitudes of $L_A^+ (~800)$ and $L_{PN}^+ (~800)$ were an order of magnitude less indicating greater stability. The best image of the model was returned by the $L_A^+$ method.

INTRODUCTION

Margrave and Ferguson (1999) present a comparison of four wavefield extrapolators that are useful in recursive explicit depth imaging methods. Two of them, $L_N^+$ and $L_p^+$ (NSPS and PSPI as introduced by Margrave and Ferguson (1997, 1999)), are derived from the Taylor series representation of extrapolated wavefields (Margrave and Ferguson (1999)). The remaining extrapolators, $L_A^+$ and $L_{PN}^+$, are formed by averaging ($L_A^+$) or cascading ($L_{PN}^+$) the elementary ones ($L_N^+$ and $L_p^+$).

Margrave and Ferguson (1999) demonstrate that, when velocity varies laterally, inversion of wavefields extrapolated by $L_A^+$ or $L_{PN}^+$ more closely return the input than do $L_N^+$ or $L_p^+$. Through analysis of the asymptotic formulae of the symbols of the depth derivatives corresponding to $L_N^+$ and $L_p^+$ they suggest that $L_A^+$ and $L_{PN}^+$ benefit from the averaging of these symbols. The symbols are complex valued and opposite in sign in the odd orders. Their average cancels these terms and returns a symbol that is real valued with error terms of higher order than the elementary symbols. The $L_A^+$ extrapolator averages the symbols at the Taylor series level through a summing of depth derivatives (Margrave and Ferguson, 1999). The $L_{PN}^+$ extrapolator may have the effect of averaging them through the cascade process (Margrave and Ferguson, 1999).
Margrave and Ferguson (1999) also demonstrate, by an analysis of the singular values of the extrapolators, that all four have some values larger than unity in the nonevanescent region. Thus, when used recursively, e.g., as in depth imaging they are prone to uncontrolled growth in the amplitude of the extrapolated wavefield, and extrapolation is unstable. Margrave and Ferguson (1999) find that $L^+_A$ and $L^+_PNL$ have singular values closer to unity than $L^+_N$ and $L^+_PL$, with $L^+_A$ being closest, suggesting that $L^+_A$ and $L^+_PNL$ are more stable than the other two, and that $L^+_A$ is the most stable.

In this paper, wavefield extrapolators $L^+_P$, $L^+_N$, $L^+_PNL$ and $L^+_AL$ are evaluated for accuracy and stability by deriving the mathematical analogues of the inverse extrapolations presented by Margrave and Ferguson (1999). The resulting equations represent propagation from $z = 0$ to $z$ through a laterally variable medium, followed by propagation from $z$ back to $z = 0$. Assuming smooth variation of the extrapolation symbol $\alpha$ in lateral coordinate $x$ we prove that $L^+_PNL$ and $L^+_AL$ are invertable and that inversion of $L^+_PL$ and $L^+_NL$ results in complex valued error terms.

The Marmousi synthetic data set (Bourgeois et al., 1990) is used to compare the accuracy and stability of depth imaging methods based on the different extrapolators.

**INVERSE OPERATORS, ACCURACY AND STABILITY**

Inversion of a wavefield $\psi$ extrapolated by $L^+_P$ has an associated error that results in an approximation $\psi_P$ to $\psi$ given by

$$\psi_P(x) = \left[ L^+_P \right] \psi(m) \exp(i k \cdot y) \rho(x) \right] \times \left( 1 \right),$$

where,

$$\left[ L^+_P \psi(m) \right] \equiv \frac{1}{(2\pi)^2} \int \alpha(y, m) \psi(m) \exp(-i y \cdot m) \rho(m) dm,$$

and the symbol of this pseudo-differential operator is

$$\alpha(y, m) = \exp \left( i z \sqrt{\frac{\omega}{c(x)}} - k \cdot k \right) .$$

Depth $z$ is the extrapolation depth interval, $\omega$ is temporal frequency and $c(x)$ is the laterally variant velocity of the medium. Coordinates $x = \{x_1, x_2\}$ correspond to output space, $y = \{y_1, y_2\}$ correspond to input space, $m = \{m_1, m_2\}$ correspond to input wavenumbers, $k = \{k_1, k_2\}$ correspond to output wavenumbers. The spectrum of the wavefield is $\phi$. The sign on $z$ controls the direction of propagation (Margrave and Ferguson, 1999).
Equation (1) is the composition of two pseudo-differential operators $L_+^p$ (positive) and $L_−^p$ (negative) that results in an equivalent operator $L_p$ with the form

$$\psi_p(x) = [L_p \varphi(m)](x),$$

(4)  

In integral form, equation (4) is

$$[L_p \varphi(m)](x) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int c_p(x,m)\varphi(m)\exp(-i m \cdot x) / m,$$

(5)  

with symbol $c_p$ given by

$$c_p(x,m) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int \alpha^−(x,k)\alpha^+(y,m)\exp(-i[k-m] \cdot [x-y]) / d\alpha / d\k.$$  

(6)  

Because inversion symbol $c_p$ equation (6) results from the composition of two pseudo-differential operators it has an asymptotic formula (Stein, 1993: 237)

$$c_p(x,m) = 1 + i \nabla_m \alpha^−(x,m) \cdot \nabla_x \alpha^+(x,m) + \frac{i^2}{2} \nabla_m \nabla_m \alpha^−(x,m) \cdot \nabla_x \nabla_x \alpha^+(x,m) + \cdots,$$

(7)  

where $\nabla_m$ and $\nabla_x$ are gradient operators, and the : operator represents the contraction of two-second rank tensors. Symbol $c_p$ is unity in the first term, and all other terms represent error, with odd powers being complex valued. For $\alpha$ constant in $x$, $c_p$ is unity and $L_p$, equation (5), reduces to an inverse Fourier transform. (Inversion is exact for constant velocity.)

Inversion of $L_+^p$ applied to $\psi$ results in an approximation $\varphi_N$ to spectrum $\varphi$ given by

$$\varphi_N(m) = \left[ L_N^p \psi(x) \right] / (2\pi)^{\frac{3}{2}} \int \alpha(x,k)\psi(x)\exp(-i k \cdot y) / d\alpha / d\k,$$

(8)  

where,

$$\left[ L_N^p \psi(x) \right] = \int \alpha(x,k)\psi(x)\exp(i k \cdot x) / d\alpha / d\k.$$  

(9)  

Equation (8) is a composition of two adjoint-standard pseudo-differential operators $L_N^+$ and $L_N^-$, whose equivalent operator $L_N$ has the form

$$\varphi_N(m) = [L_N \psi(x)](m),$$

(10)  

or as an integral
\[ [L_N \psi(x)](m) = \int c_N(x, m) \psi(x) \exp(i m \cdot x) \, dx \]

with symbol \( c_N \)

\[ c_N(x, m) = \frac{1}{(2\pi)^2} \int \alpha^{-}(y, m) \alpha^{+}(x, k) \exp(-i [m - k] \cdot [x - y]) \, dy \, dk \]

Like \( c_P \), inversion symbol \( c_N \) is the composition of two pseudo-differential operators, and it too has an asymptotic formula (Appendix A)

\[ c_N(x, m) = 1 - i \nabla_m \alpha^{-}(x, m) \cdot \nabla_x \alpha^{+}(x, m) + \frac{i^2}{2} \nabla_m \nabla_m \alpha^{-}(x, m) : \nabla_x \nabla_x \alpha^{+}(x, m) - \ldots \]

Symbol \( c_N \) is similar to \( c_P \) (equation (7)); the first term is unity and the error terms are products of derivatives. However, the odd powers of derivatives differ in sign. This suggests that an average of \( c_N \) and \( c_P \) will cancel complex values and increase the order of the error of the resulting symbol. In the limit of constant velocity, equation (11) reduces to a Fourier transform.

In the space domain, the inversion of the average operator \( L_A^* \) is

\[ [L_A \psi(y)](x) = \frac{1}{2} [L_N^* L_A^* \psi(y)](x) \]

where from Margrave and Ferguson (1999)

\[ [L_N^* \psi(y)](x) = \frac{1}{2} [L_N^* \psi(y)](x) + \frac{1}{2} [L_P^* \psi(y)](x) \]

Expansion of equation (14), by replacing \( L_A^* \) and \( L_A^- \) gives

\[ [L_A \psi(y)](x) = \frac{1}{4} [L_P^* \psi(y)](x) + \frac{1}{4} [L_N^* \psi(y)](x) + \frac{1}{4} [L_P^* L_A^* \psi(y)](x) + \frac{1}{4} [L_N^* L_P^* \psi(y)](x) \]

The first two terms of \( L_A \) consist of the inversion operators \( L_P \) and \( L_N \) cast in the space domain as

\[ [L_P \psi(y)](x) = \int \psi(y) \frac{1}{(2\pi)^2} \int c_P(x, k) \exp(-i k \cdot [x - y]) \, dk \, dy \]

and
An error and stability analysis

\[ [L_y \psi(x)](y) = \int \psi(y) \frac{1}{(2\pi)^2} \int c_x(y,k) \exp(-i k \cdot [x-y]) dk dy. \]  
\[ (18) \]

\( L_P \) and \( L_N \) are transposes in this domain. Substitution of \( c_P \) equation (7) and \( c_N \) equation (1) in equations (17) and (18) gives for \( L_P \)

\[ [L_P \psi(y)](x) = \psi(x) + i \int \frac{1}{(2\pi)^2} \int \psi(y) \nabla_k \alpha(x,k) \cdot \nabla_z \alpha(x,k) \exp(-i k \cdot [x-y]) dk dy + i^2 \cdots, \]  
\[ (19) \]

and for \( L_N \)

\[ [L_N \psi(y)](x) = \psi(x) - i \int \frac{1}{(2\pi)^2} \int \psi(y) \nabla_k \alpha(y,k) \cdot \nabla_z \alpha(y,k) \exp(-i k \cdot [x-y]) dk dy + i^2 \cdots. \]  
\[ (20) \]

Operators \( L_P \) and \( L_N \) differ only in the sign of the odd orders of their derivatives (the odd orders are also complex). Their sum cancels these terms and increases the order of the error terms giving for the first two terms in equation (16)

\[ \frac{[L_P \psi(y)](x) + [L_N \psi(y)](x)}{2} = \frac{\psi(x)}{2} \]

\[ + \frac{1}{2} i^2 \int \frac{1}{(2\pi)^2} \int \psi(y) \nabla_k \nabla_k \alpha(y) \cdot \nabla_z \nabla_z \alpha(y) \exp(-i k \cdot [x-y]) dk dy + \cdots. \]  
\[ (21) \]

The third term in equation (16) corresponds to forward extrapolation by \( L_N^+ \) followed by reverse extrapolation by \( L_P^+ \)

\[ [L_P^+ L_N^+ \psi(y)](x) = \int \psi(y) \frac{1}{(2\pi)^2} \int \alpha^-(x,k) \alpha^+(y,k) \exp(-i k \cdot [x-y]) dk dy. \]  
\[ (22) \]

Substituting \( u = x - y \) in equation (22) gives

\[ [L_P^+ L_N^+ \psi(y)](x) = \int \psi(x-u) \frac{1}{(2\pi)^2} \int \alpha^-(x,k) \alpha^+(x-u,k) \exp(-i k \cdot u) dk du. \]  
\[ (23) \]

Wavefield \( \psi \) can be approximated by Taylor series

\[ \psi(x-u) = \psi(x) - u \cdot \nabla_x \psi(x) + (-u \cdot \nabla_x)^2 \psi(x) - \cdots. \]  
\[ (24) \]

Similarly, symbol \( \alpha^+ \)

\[ \alpha^+(x-u,k) = \alpha^+(x,k) - u \cdot \nabla_x \alpha^+(x,k) + (-u \cdot \nabla_x)^2 \alpha^+(x,k) - \cdots. \]  
\[ (25) \]

Replacing \( \psi \) and \( \alpha^+ \) in equation (23) with equations (24) and (25) gives
\[
\begin{align*}
[L_p L_N^+ \psi(y)](x) &= \int \left\{ \left[ 1 - u \cdot \nabla_x + \cdots \right] \psi(x) \right\} \frac{1}{(2\pi)^2} \int \alpha^-(x, k) \left( \left[ 1 - u \cdot \nabla_x + \cdots \right] \alpha^+(x, k) \right) \exp(-k \cdot u) dk du.
\end{align*}
\]

(26)

The first order terms in equation (26) are, beginning with the simplest

\[
\psi(x) \frac{1}{(2\pi)^2} \int \alpha^-(x, k) \alpha^+(x, k) \exp(-k \cdot u) dk du
= \psi(x),
\]

(27)

next,

\[
\nabla_x \psi(x) \cdot \int u \frac{1}{(2\pi)^2} \int \alpha^-(x, k) \alpha^+(x, k) \exp(-k \cdot u) dk du
= \nabla_x \psi(x) \cdot \int \frac{1}{(2\pi)^2} \int u \exp(-k \cdot u) du dk
= 0,
\]

(28)

and,

\[
\psi(x) \frac{1}{(2\pi)^2} \int \alpha^-(x, k) \nabla_x \alpha^+(x, k) \cdot \frac{1}{(2\pi)^2} \int u \exp(-k \cdot u) du dk
= \psi(x) \frac{1}{(2\pi)^2} \int \alpha^-(x, k) \nabla_x \alpha^+(x, k) \delta(k) dk
= \psi(x) \left( \nabla_k \alpha^-(x, k) \nabla_x \alpha^+(x, k) + \alpha^-(x, k) \nabla_k \alpha^+(x, k) \right)_{k=0}
= 0,
\]

(29)

where for this last term,

\[
\nabla_k \alpha^+(x, k)_{k=0} = \pm \alpha^+(x, k)_{k=0} \nabla_k \alpha^+(x, k)_{k=0}
= 0,
\]

(30)

and

\[
\nabla_k \alpha^-(x, k)_{k=0} = \left[ \frac{1}{2} \left( \frac{\omega}{c(x)} \right)^2 + k \cdot k \right]^{1/2} \nabla_k \alpha^+(x, k)_{k=0}
= 0.
\]

(31)

Assuming that second order terms (and higher) are small, equation (26) reduces to the identity
\[
[L_p L_N^+ \psi(y)](x) = \psi(x),
\]
from which we infer, to first order, \(L_p^+\) and \(L_N^+\) are approximate inverses and, therefore, the fourth term in equation (16) is
\[
[L_N L_p^+ \psi(y)](x) \approx \psi(x) .
\]
The inversion operator \(L_A\), equation (16), is now written to first order as
\[
[L_A \psi(y)](x) \approx \psi(x) .
\]
The results from the previous discussion are sufficient to derive the inversion of \(L_{PN}^+\)
\[
\left[L_{PN} \psi(y)\right](x) = \left[-\frac{1}{\lambda_p^2} L_p^{-\frac{1}{2}} L_N^{-\frac{1}{2}} L_p^{-\frac{1}{2}} L_N^{-\frac{1}{2}} \psi(y)\right](x) ,
\]
and, using the associative properties of these operators, to first order \(L_p^+\) and \(L_N^+\) are inverses, therefore
\[
\left[L_{PN} \psi(y)\right](x) \approx \psi(x) .
\]
For comparison, \(L_p\) (equation (19)) to first order, is
\[
\left[L_p \psi(y)\right](x) = \\
\psi(x) + i \int \frac{1}{(2\pi)^3} \left[ \psi(y) \nabla_\kappa \alpha(x, \kappa) \cdot \nabla_x \alpha(x, \kappa) \exp(-i \kappa \cdot [x - y]) d\kappa dy ,
\]
and for \(L_N\)
\[
\left[L_N \psi(y)\right](x) = \\
\psi(x) - i \int \frac{1}{(2\pi)^3} \left[ \psi(y) \nabla_\kappa \alpha(y, \kappa) \cdot \nabla_y \alpha(y, \kappa) \exp(-i \kappa \cdot [x - y]) d\kappa dy .
\]
Inverses \(L_{PN}\) (equation (36)) and \(L_A\) (equation (34)) have no first order error terms. Inverse operators \(L_p\) (equation (37)) and \(L_N\) (equation (38)) have first order error terms that are functions of spatial and wavenumber derivatives that are non zero for smooth variation in velocity. Thus, in this situation, extrapolators \(L_{PN}^+\) and \(L_A^+\) are more accurate than \(L_p^+\) and \(L_N^+\).

In terms of stability, again to first order, \(L_p\) and \(L_N\) have complex error terms, suggesting that \(L_p^+\) and \(L_N^+\) also generate complex values. Uncontrolled complex
values during recursive application of these extrapolators may lead to the instability observed by Margrave and Ferguson (1999).

MARMOUSI

The Marmousi synthetic data (Bourgeois et al., 1990) were acquired for use in comparing depth imaging methods based on extrapolators $L_N^+$, $L_P^+$, $L_A^+$ and $L_{PN}^+$. The prestack data were depth imaged at a depth interval of 20m. This interval was chosen as being large enough to illustrate the different stability and accuracy characteristics of the extrapolators without becoming unstable enough to preclude comparison. For a detailed description of prestack depth imaging using nonstationary extrapolators see Ferguson and Margrave (1999).

Figure 1 shows the true reflectivity computed from the density and velocity profile of the model. Figures 2 through 5 show the depth images corresponding to $L_N^+$ (Figure 2), $L_P^+$ (Figure 3), $L_A^+$ (Figure 4) and $L_{PN}^+$ (Figure 5). The depth-imaging algorithm based on $L_A^+$ gives the best image, especially in the shallower part of the model. (Arrows annotated on the figures facilitate this comparison.) The steeply dipping faults are more clearly imaged using $L_A^+$, and a large part of the section is less obscured by noise.

Comparison of the average amplitudes of the images of Figure 2 through 5 show that $L_A^+$ and $L_{PN}^+$ are more stable than $L_N^+$ and $L_P^+$. The average absolute amplitudes corresponding to $L_A^+$ (~800) and $L_{PN}^+$ (~800) are 20% less than those corresponding to $L_N^+$ (~1000) and $L_P^+$ (~1000).

CONCLUSIONS

An error and stability analysis was presented for the nonstationary wavefield extrapolators $L_N^+$, $L_P^+$, $L_A^+$ and $L_{PN}^+$ defined by Margrave and Ferguson (1999) based on analytic expressions that describe inversion of wavefields extrapolated by the four operators. The analysis supports the conclusions of Margrave and Ferguson (1999) that $L_A^+$ and $L_{PN}^+$ are more accurate and more stable than elementary extrapolators $L_N^+$ and $L_P^+$. The first order result (i.e., smooth variation of the extrapolation symbol $\alpha$ in lateral coordinate $x$) proved the error related to the inversion of $L_{PN}^+$ and $L_A^+$ is less than the inversion of $L_P^+$ and $L_N^+$. Similarly, the greater stability of $L_{PN}^+$ and $L_A^+$ was indicated.

The Marmousi model data (Bourgeois et al., 1990) were used to provide a qualitative comparison of depth imaging methods based on the different extrapolators. The best image of the model was returned by the $L_A^+$ method. Comparison of the average amplitudes of the images showed that the depth images
for all four extrapolators had grown in amplitude but that $L_{PN}^+$ and $L_{A}^+$ had grown 20% less than $L_p^+$ and $L_N^+$. 

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REFERENCES


APPENDIX A

Computing the spatial Fourier transform in $c_N$ equation (12) gives

$$c_N(x, m) = \frac{1}{(2\pi)^2} \int A^{-}(m-k, m) \alpha^+(x, k) \exp(-i[m-k] \cdot x) / k$$

Equation (A1) can also be written

$$c_N(x, m) = \frac{1}{(2\pi)^2} \int A^-(u, m) \alpha^+(x, m-u) \exp(-iu \cdot x) / u$$

Expanding $\alpha^+$ results in

$$c_N(x, m) = \frac{1}{(2\pi)^2} \int A^{-}(u, m)[\alpha^+(x, m) - u \cdot \nabla_m \alpha^+(x, m) + \cdots] \exp(-iu \cdot x) / u$$

that provides an asymptotic formula for $c_N$ that is similar to that of $c_P$ by recognizing that coordinates $u$ arise as spatial derivatives of the Fourier kernel thus

$$c_N(x, m) = 1 - i \nabla_m \alpha^-(x, m) \nabla_x \alpha^+(x, m) + \frac{i^2}{2} \nabla_m^2 \alpha^- (x, m) ; \nabla_x^2 \alpha^+(x, m) - \cdots$$

(A4)
Fig. 1. The seismic reflectivity of Marmousi computed from the density and velocity profile of the model. The arrows and ring correspond to points of comparison with Figures 2 through 5.

Fig. 2. Depth image of the Marmousi data set corresponding to $L^+_N$. The depth interval was 20m. The mean absolute amplitude of this image is $\sim$1000. The arrows indicate points of comparison on two faults in the model. The ring encloses a flatter region that seems to suffer from noise. In this image the noise corresponds to a trough followed by a peak.
Fig. 3. Depth image of the Marmousi data set corresponding to $L^+_p$. The depth interval was 20m. The mean absolute amplitude of this image is ~1000. The images of the indicated faults are less well rendered by $L^+_p$ compared to $L^+_d$ (Figure 3). The noise in the ringed area is a strong peak.

Fig. 4. Depth image of the Marmousi data set corresponding to $L^+_d$. The depth interval was 20m. The mean absolute amplitude of this image is ~800. The best focussing of the indicated faults is provided by this image. The noise in the ringed area is a strong trough.
Fig. 5. Depth image of the Marmousi data set corresponding to $L_{pn}^*$. The depth interval was 20m. The mean absolute amplitude of this image is ~800. This image has the lowest noise in the ringed area.