

## **A Padé approximation to the scalar wavefield extrapolator for inhomogeneous media**

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### **ABSTRACT**

A seismic wavefield at depth  $z$  can be obtained by extrapolating the wavefield from zero depth. An exact analytical solution for the wavefield extrapolator does not exist for generally heterogeneous media. Relatively accurate solutions such as generalized eigenfunction transform can only be put into practice with sufficient computing power, which is generally unavailable for real seismic data processing. In this paper a Padé solution to the scalar wave extrapolator is presented, which is generally faster than the eigenvalue decomposition method without significant loss of accuracy.

### **INTRODUCTION**

Though popular in industry, Kirchhoff migration algorithms rely on high frequency ray theory and experience difficulty with handling caustics, shadow zones and extremely complex media. Wave-equation migration techniques allow energy to propagate along all possible ray paths and can give superior images. Inhomogeneous media, with strong lateral velocity contrasts, require a highly accurate wavefield extrapolator in order to map the wavefield at one depth to another. Typical wavefield extrapolation techniques such as phase-shift-plus-interpolation (PSPI) (Gazdag and Sguazzero, 1984), nonstationary phase-shift (NSPS) (Margrave and Ferguson, 1999a), split-step (phase-screen) (Stoffa et al., 1990; Freire and Stoffa, 1986) use approximate methods and small extrapolation steps to achieve efficiency and acceptable accuracy. For example, PSPI uses a set of constant reference velocities to compute reference wavefields using constant-velocity phaseshift and the real wavefield is an  $x$ - $\omega$  domain interpolation between those reference wavefields. In the extreme form when all the velocities are used to compute the extrapolator, PSPI becomes a nonstationary combination, which performs an inverse Fourier transform simultaneously with the wavefield extrapolation (Margrave and Ferguson, 1999a). This limiting form is also referred as PSPI to emphasis its relation with the Gazdag's original PSPI concept. NSPS is a complementary form of PSPI (it is an  $x$ - $\omega$  transpose of PSPI) that performs a simultaneous forward Fourier transform with the wavefield extrapolation. A natural combination of NSPS and PSPI leads to a symmetric phase-shift extrapolator (SNPS) of higher accuracy (Margrave and Ferguson, 1999b). The split-step extrapolator, also called a phase-screen method, decomposes the extrapolator into a reference term and a correction term based on perturbation theory. But it is incapable of handling strong lateral velocity variation. Jin and Wu (1998) generalized the split-step approach by using a locally transformed wavefield to achieve higher accuracy in the case of strong velocity variation. However, as oil/gas exploration activities move into regions of complex geology, such as the Alberta foothills, traditional Kirchhoff algorithms and those wave equation algorithms mentioned above may not be accurate enough.

In this paper, a direct, one-way solution to the wave equation in strongly heterogeneous media by using a Padé approximation is derived. With little loss of accuracy, the Padé approximation is generally faster than the exact eigenvalue decomposition algorithm as proposed by Yao and Margrave, 1999.

### EIGENFUNCTION SOLUTION FOR V(X) MEDIA

The 2D wave equation can be written in the Fourier domain as the Helmholtz equation

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} + \frac{\omega^2}{v^2(x)}\right)\Psi(x, z, \omega) = 0, \quad (1)$$

where  $v(x)$  is the laterally varying velocity field and  $\Psi(x, z, \omega)$  is a mono-frequency wavefield. The boundary condition for the wavefield extrapolation problem is  $\Psi(x, z, \omega)|_{z=0} = \Psi_0(x, \omega)$ , where  $\Psi_0(x, \omega)$  is assumed to consist of only up-going waves.

Yao and Margrave (1999) showed that the extrapolated wavefield at depth  $z$  can be written as

$$\underline{\underline{\Psi}}(x, z, \omega) = \underline{\underline{F}}_x^{-1} [\exp(\pm iz \sqrt{\underline{\underline{C}}^2 - \underline{\underline{K}}^2}) \underline{\underline{F}}_x \underline{\underline{\Psi}}(x, z = 0, \omega)] \quad (2)$$

Symbols with single underscore are vectors and those with double underscores are matrices.  $C^2$  is defined as

$$C^2 = \omega^2 \begin{pmatrix} s_0 & \cdot & \cdot & s_{-(n-1)} \\ \cdot & & & \\ \cdot & & & \\ s_{(n-1)} & \cdot & \cdot & s_0 \end{pmatrix}. \quad (3)$$

$s_n$  in the above matrix is defined as the forward spatial discrete Fourier transform of  $\frac{1}{v(x)^2}$

$$s_n = \frac{1}{L} \int v^{-2}(x) \exp(-in\Delta k_x x) dx \quad (4)$$

$K^2$  is the a diagonal matrix with the square of horizontal wavenumbers populating the main diagonal

$$K^2 = \begin{pmatrix} k_x^2(1) & & & \\ & \cdot & & \\ & & \cdot & \\ & & & k_x^2(n) \end{pmatrix}. \tag{5}$$

Matrix  $C^2$  is a Toeplitz matrix. For laterally varying velocity fields, it has off-diagonal components. For the constant velocity case, it is a diagonal matrix. Using eigenvalue decomposition method, equation (2) can be written as,

$$\underline{\underline{\Psi}}(x,z,\omega) = \underline{\underline{F}}_x^{-1} \left[ \underline{\underline{D}} \exp(-i\Lambda^{1/2}z) \underline{\underline{D}}^{-1} \underline{\underline{F}}_x \underline{\underline{\Psi}}(x,z=0,\omega) \right]. \tag{6}$$

$\underline{\underline{C}}^2 - \underline{\underline{K}}^2$  has been decomposed as

$$\underline{\underline{C}}^2 - \underline{\underline{K}}^2 = \underline{\underline{D}} \underline{\underline{\Lambda}} \underline{\underline{D}}^{-1}, \tag{7}$$

where  $\underline{\underline{\Lambda}}$  is a diagonal matrix with eigenvalues of  $\underline{\underline{C}}^2 - \underline{\underline{K}}^2$  populated on the main diagonal.

### PADÉ APPROXIMATION FOR MATRIX EXPONENTIAL

Matrix exponential computing techniques are well described in the math literature (for example, Golub and Van Loan, 1989). Typical approaches are the eigenvalue and Padé approximation algorithms. In present case, an approximate wavefield extrapolator, sufficiently accurate to map the wavefield to the next depth through a laterally varying velocity field, is desired rather than an exact expression. Accordingly, a Padé approximation to the wavefield extrapolator, is investigated.

#### Matrix functions and rule of substitution

Computation of a function of  $f(A)$ , where  $A$  is an n-by-n matrix, is a frequently encountered problem in many application areas. A matrix function  $f(A)$  can be defined from a scalar function  $f(z)$  by simply replacing the variable  $z$  with matrix  $A$ . For example, for the function

$$f(z) = (1+z)/(1-z), \tag{8}$$

a similar matrix function  $f(A)$  can be defined as

$$f(A) = (I + A)(I - A)^{-1}, \tag{9}$$

where the superscript  $-1$  denotes the inverse of a matrix.

Similarly, an exponential matrix function is defined as

$$f(A) = e^A. \tag{10}$$

In present case,  $A$  is the phase-shift and wavenumber mixing matrix defined in equation (2). All approximation rules to function  $e^z$ , such as Taylor series expansion, can be conveniently applied to function  $e^A$ .

### Exponential matrix function and its Padé approximation

The eigenvalue method (for example, Golub and Van, 1989) is an accurate way to compute  $e^A$ , however the speed is generally unacceptable for many applications and fast approximation techniques are desired.

Rather than the exact eigenvalue decomposition method, we consider the Padé approximation to the scalar function  $e^z$ , defined by

$$e^z \approx R_{pq}(z) = D_{pq}(z)^{-1} N_{pq}(z), \tag{11}$$

where

$$N_{pq} = \sum_{k=0}^p \frac{(p+q-k)! p!}{(p+q)! k! (p-k)!} z^k, \tag{12}$$

$$D_{pq} = \sum_{k=0}^q \frac{(p+q-k)! q!}{(p+q)! k! (q-k)!} (-z)^k, \tag{13}$$

Note that when  $q$  is zero, equation (11) becomes the  $p$ th order Taylor series expansion

$$R_{p0}(z) = 1 + z + \dots + z^p / p! \tag{14}$$

The matrix exponential can be expanded in the same way by replacing  $z$  with matrix  $A$ . A Taylor series approach to matrix exponential approximation is generally slow and inaccurate. The Padé approximation to the matrix exponential can be computed by simply replacing the  $z$  in equation (11); for example, when  $p=q=2$ , equation (11) is written as

$$R_{p=2,q=2}(z) = D_{p=2,q=2}(A)^{-1} N_{p=2,q=2}(A), \tag{15}$$

where

$$D_{p=2,q=2}(A) = I - \frac{1}{2}A + \frac{1}{12}AA, \quad (16)$$

and

$$N_{p=2,q=2}(A) = I + \frac{1}{2}A + \frac{1}{12}AA. \quad (17)$$

Padé approximation is only accurate near the origin so that the computing procedure should be altered as  $e^A = (e^{A/m})^m$ , where  $m$  is a power of 2 to achieve efficiency. It is  $e^{A/m}$  that is first computed. Error can be minimized by choosing equal  $p$  and  $q$  values (Golub and Van Loan, 1989). More details of the computing procedure of  $e^A = (e^{A/m})^m$  is given in *Matrix Computation* by Golub and Van Loan (1989).

### Padé approximation to the wavefield extrapolation matrix

Equation (2) can also be written as

$$\underline{\Psi(z, \omega)} = \underline{\underline{F_x^{-1} e^A F_x^{-1} \Psi(z=0, \omega)}}, \quad (18)$$

where  $A$  is the wavefield extrapolation matrix defined by  $A = \pm izB$  and  $B$  is defined as the square root of matrix  $C^2 - K^2$ ,

$$B^{1/2} B^{1/2} = C^2 - K^2. \quad (19)$$

The wavefield extrapolation matrix  $A$  can be computed as described in last section.

## NUMERICAL EXAMPLES

We shall show two numerical examples. The first one is the forward and inverse extrapolation through a two-block model, with the source in the low velocity media on the left. The other is extrapolation through a layer of randomly varying velocities. In both cases, we used the Padé approximation subroutine *exmpm1* in Matlab to compute the matrix exponential with default values of  $p=1$  and  $q=6$ . The  $p$  and  $q$  values can be easily changed to explore other Padé approximation.

### Zero-offset extrapolation test through a two-block model

Figure 1 shows the two-block model used to test the impulse response and its inverse. The velocity of the left side is 1500 m/s and 2500 m/s on the right. The position of the source is indicated by the on the left. The thickness of the model is 200 m. Figure 2 showed that the Padé approximation algorithm produced exactly the same result as the eigenvalue decomposition algorithm as proposed by Yao and Margrave (1999).

### Zero-offset extrapolation test through a random velocity layer

Random velocity variation is an extreme test for the accuracy of wavefield extrapolators. A random lateral velocity function is generated and then smoothed with 7-point running median to simulate extreme geology variations. Figure 3a shows the random velocity model, in which velocity varies from 1800 m/s to 3600 m/s with 80% maximum velocity contrast in adjacent blocks. Figure 3c is the exact forward extrapolation with the eigenvalue decomposition algorithm, and Figure 3d shows the inverse extrapolation with the Padé approximation. The 11 impulses are well recovered.

### FUTURE WORK

The Padé approximation algorithm with  $q=6$  and  $q=1$  is roughly about 3 times faster than the eigenvalue decomposition algorithm, however it is considerably slower than other traditional algorithms of less accuracy. Smaller  $q$  values lead to faster algorithms of less accuracy. We plan to investigate how these values can be adjusted to fit the accuracy requirement of typical imaging tasks.

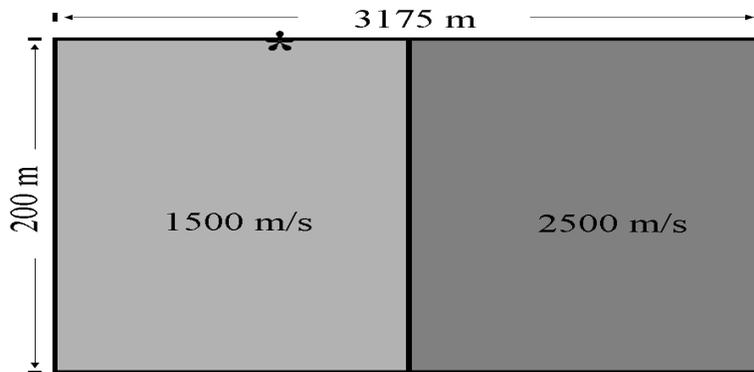
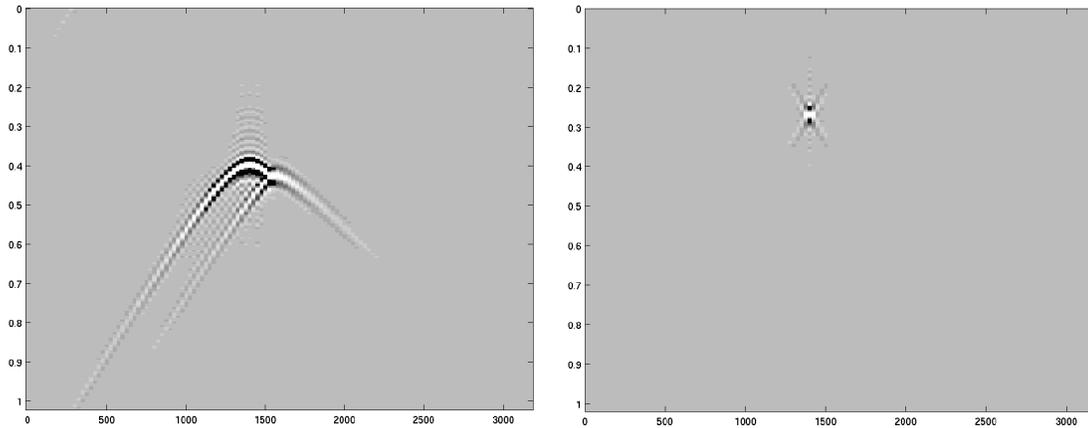
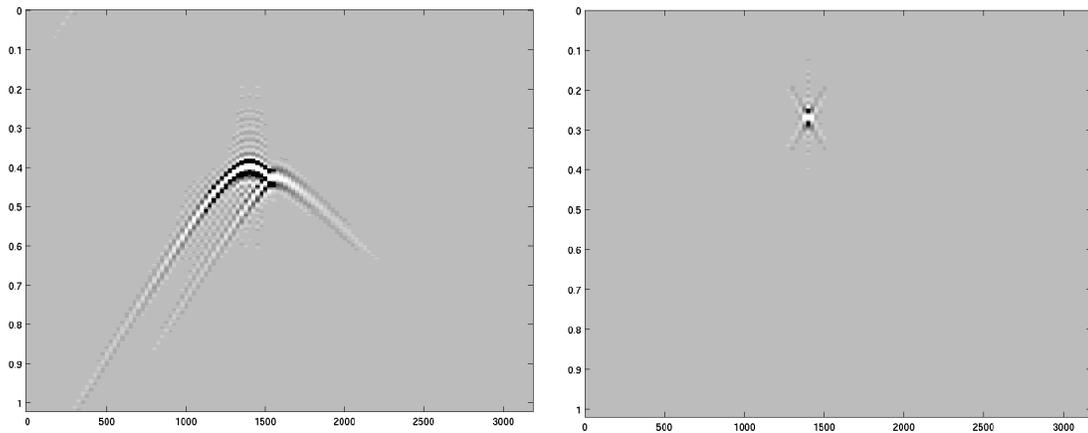


Figure 1. Two-block model to test zero-offset extrapolation with Padé approximation. The star indicates the position of the source.



(a) The forward extrapolation of a single spike computed with the Padé approximation.

(b) The inverse extrapolation of (a) computed with the Pade approximation.



(c) The forward extrapolation a single spike computed with eigenvalue decomposition.

(d) The inverse extrapolation of (c) computed with eigenvalue decomposition.

Figure 2. Forward and backward extrapolation of an impulse through a two-block model as shown in figure 1. The Padé approximation produces the same results as the eigenvalue decomposition algorithm by Yao and Margrave, 1999.

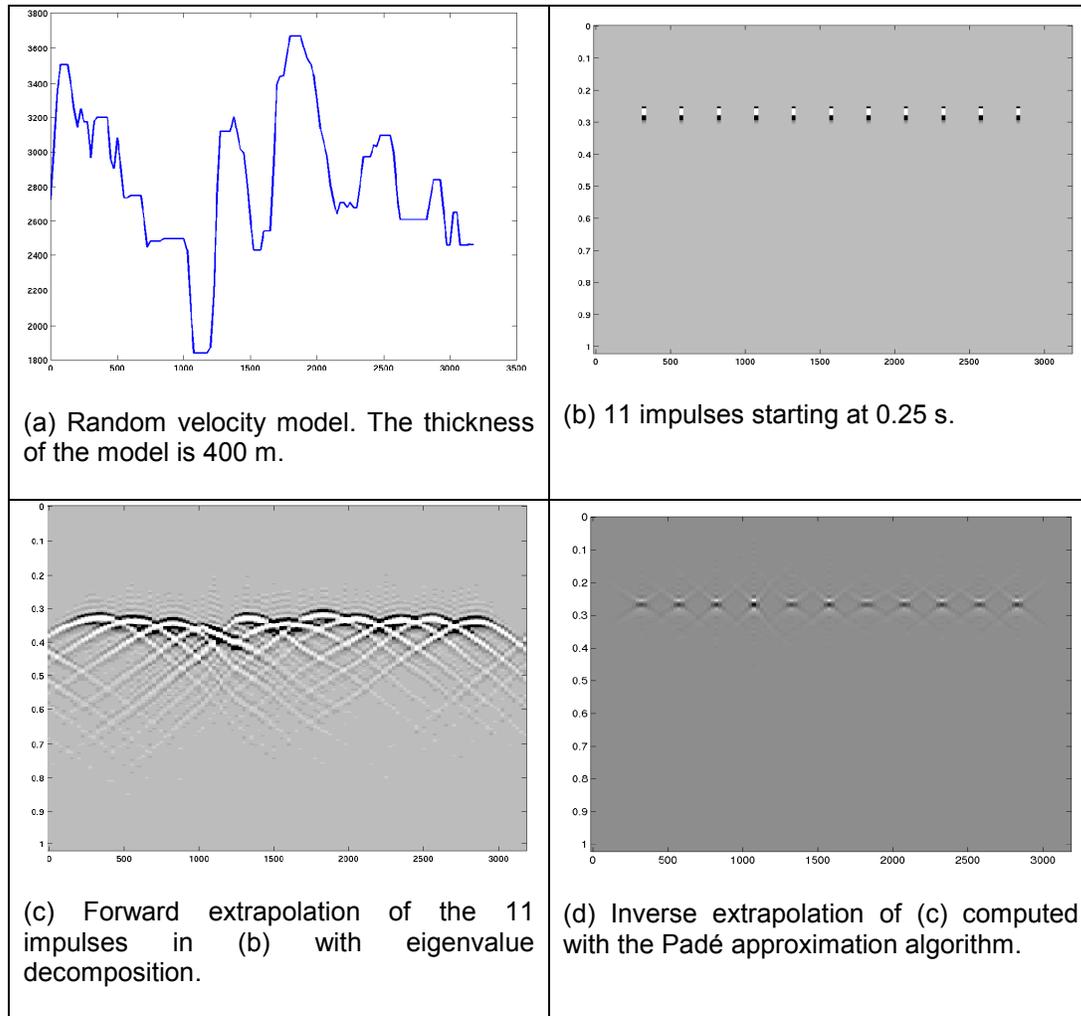


Figure 3. Zero-offset forward extrapolation with the exact eigenvalue decomposition algorithm and inverse extrapolation with the Padé approximation algorithm through a randomly varying velocity media.

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